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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

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VOLUME 15

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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1915–JUNE, 1916.

Thursday, November 11th, 1915.

ANNUAL GENERAL MEETING.

Sir J. LARMOR, President, in the Chair.

Present nineteen members.

The Rev. A. C. Ridsdale was elected a member.

Messrs. H. R. Hassé, J. Mercer, and C. Walmsley were admitted into the Society.

The President reported the deaths of Prof. W. H. H. Hudson and of Sir Andrew Noble; and spoke briefly of their mathematical and general scientific work.

The Treasurer presented his Report. On the motion of Mr. G. N. Watson, seconded by Mr. W. H. Salmon, the Report was received.

Lt.-Col. A. Cunningham was appointed Auditor.

The Council and Officers for the ensuing Session were elected. The list is as follows:—President, Prof. Sir J. Larmor; Vice-Presidents, Mr. J. H. Grace, Prof. A. E. H. Love, and Major P. A. MacMahon; Treasurer, Dr. A. E. Western; Secretaries, Dr. T. J. I'A. Bromwich and Mr. G. T. Bennett; other members of the Council, Prof. W. Burnside, Dr. S. Chapman, Mr. A. L. Dixon, Mr. G. H. Hardy, Prof. H. Hilton, Prof. E. W. Hobson, Mr. A. E. Jolliffe, Prof. H. M. Macdonald, Mr. H. W. Richmond.

In connexion with the election of the Council and Officers, the President stated that Mr. Grace was retiring from the office of Honorary Secretary. He expressed the appreciation of the Society for the services which Mr. Grace had rendered during his tenure of the office (from

November, 1904, to November, 1915), and the hope that the Society would long be able to count on him as an active member.

Mr. G. H. Hardy read a paper "On Weierstrass's Non-Differentiable Function."

Informal communications were made by Prof. Hilton, "On Two Problems in Geometry," and by Mr. J. Hammond, "On the Distribution of Prime Numbers."

The following papers were communicated, by title, from the Chair:—

*The Second Theorem of Consistency for Summable Series: Mr. G. H. Hardy.

*The Kinetic Theory of the Motion of Ions in Gases: Mr. F. B. Pidduck.

*Some Singularities of Surfaces and their Differential Geometry: Mr. H. W. Turnbull.

Periodic Solutions of the Problem of Three Bodies, in Three Dimensions: Dr. J. W. Campbell.

*Functions of Positive Type and Related Topics in General Analysis: Mr. C. R. Dines.

Surfaces Characterized by certain Special Properties of their Directrix Congruences: Mr. C. H. Yeaton.

ABSTRACTS.

Weierstrass's Non-Differentiable Function†

Mr. G. H. HARDY.

It was proved by Weierstrass that the function

$$f(x) = \Sigma a^n \cos b^n \pi x,$$

where $0 < a < 1$ and b is a positive odd integer, has no differential coefficient for any value of x if

$$ab > 1 + \tfrac{3}{2}\pi.$$

It has been shown by Bromwich‡ that this condition may be replaced by

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* Printed in this volume.

† Printed in the *Transactions of the American Math. Soc.* (July 1916), Vol. 17, p. 301.

‡ *Infinite Series*, pp. 490, 491.

the less stringent condition

$$ab > 1+\tfrac{3}{2}\pi(1-a).$$

In this communication the question of the non-differentiability of $f(x)$, and of the function represented by the corresponding sine series, is discussed by a quite different method. The principal result obtained is that *neither function possesses a finite differential coefficient for any value of x in any case in which*

$$0 < a < 1, \quad b > 1, \quad ab \geqslant 1.$$

In this result b is not restricted to be an integer.

No such simple result holds when *infinite* differential coefficients are admitted.

The Kinetic Theory of the Motion of Ions in Gases

Mr. F. B. PIDDUCK.

The paper deals mainly with the determination of the distribution function in a number of cases. A transformation first applied by Hilbert to an equation of Boltzmann is extended to any law of force between ion and molecule, as Lunn, in an unpublished paper, has apparently done in the kinetic theory of gases. The problem of steady diffusion and mobility for elastic ions of mass equal to that of a molecule is worked out numerically, the mobility being found to be $0{\cdot}302eh/(hm)^{\frac{1}{2}}N\sigma^2$, and the coefficient of diffusion $0{\cdot}151/(hm)^{\frac{1}{2}}N\sigma^2$. These values agree closely with those derived from the formulæ of Stefan and Langevin; but the distribution of velocity is not Maxwellian, as is also clear from Lorentz's work in the electronic theory of conduction in metals. The theory of non-steady diffusion involves difficulties that do not occur otherwise. The motion of ions whose mass is either much less or much greater than that of a molecule is best treated independently from the transformed Boltzmann equation. Lorentz's fundamental equation is generalised until it is capable of rendering an account of the development of abnormal kinetic energy by light elastic ions, and the accompanying distortion of the distribution function is also determined. If k is Townsend's energy factor, v the velocity of drift, and Ω the root-mean-square value of the velocity of a gas molecule, the ratio of $k-1$ to v^2/Ω^2 is $3\pi/8$ at first, falling as the electric force increases. For large ions the distribution function satisfies a partial differential equation, and the general equation of transfer is expressed in finite form in terms of mean values. The magnetic deflexion of a stream

of ions can be treated exactly by the inverse fifth power law, while for elastic spheres the distribution of velocities can be found. Diffusion in a magnetic field is treated, and an attempt is made to explain the discrepancy between the results obtained and some equations given by Townsend. Lastly, it is shown that the effect of imperfect restitution in the collision can be allowed for by altering the mass of the ion, the external force per unit mass remaining the same.

Surfaces Characterized by certain Special Properties of their Directrix Congruences

Mr. C. H. YEATON.

The projective differential geometry* of a curved surface is equivalent to the theory of the invariants and covariants of a completely integrable system of two linear homogeneous partial differential equations of the second order in one dependent and two independent variables. When this system is reduced to a certain canonical form (A), the coefficients are seminvariants. In the first paragraph of this paper we obtain a simplified form for the integrability conditions of this system (A).

System (A) has just four linearly independent solutions, and these, interpreted as the homogeneous coordinates of a point, determine an integral surface, from which, by projection, we obtain the most general integral surface of the system.

With every point P of a non-ruled surface S, Wilczynski has associated a pair of straight lines ; one of these lines lies in the tangent plane and is called the directrix of the first kind, while the other pierces the surface at P and is called the directrix of the second kind ; these lines are denoted by d and d' respectively. Two congruences G and G' are thereby associated with the surface. The two families of developables of either congruence are determined by one and the same net of curves, the directrix curves on S. The coincidence of these two families of developables is the necessary and sufficient condition for the coincidence of the two branches of the focal surface of either congruence.

In paragraph 2 we consider the congruence G', and obtain the conditions under which one or both of its focal sheets degenerate. In the next paragraph we impose these conditions, assuming that both focal sheets

* E. J. Wilczynski, "Projective Differential Geometry of Curved Surfaces," Five Memoirs, *Transactions of the American Mathematical Society*, 1907–9.

degenerate, and further require that these focal curves be straight lines, that is, we ask if G' can be linear. Limiting the discussion to surfaces whose directrix curves are either indeterminate or conjugate, we obtain the following results.

If the directrix curves are indeterminate,* the lines d lie in a plane, while the lines d' pass through a fixed point.

If the two one-parameter families of directrix curves coincide with each other and with one family of asymptotic curves, then the focal surface of either directrix congruence consists of a single non-degenerate surface.

Surfaces whose directrix curves form a non-degenerate conjugate net may have a linear directrix congruence G'; all such surfaces are projectively equivalent to the surface S_0,

$$z = \tfrac{1}{4}xy\left(\log_e \frac{x}{y}\right)^2.$$

Any surface of this class has the following properties: The curves of the two families of asymptotic curves are projectively equivalent to each other; each lies on a quadric and is identically self-dual. The directrix curves are plane curves and one of the two families consists of conics. The focal points on the directrix d are harmonic conjugates with respect to the intersections of d and the corresponding asymptotic tangents. These focal points trace a quadric surface and a straight line which is not a generator of the quadric. This quadric is the locus of the cuspidal edges of the developables corresponding to the directrix curves of the second degree.

Paragraph 5 is devoted to a detailed discussion of the representative surface S_0 of the class. Explicit equations are obtained for the various loci mentioned in the preceding paragraph. For any point P on S_0 the directrix d' is the line through P and perpendicular to the z-axis, while d is the intersection of the tangent plane and the plane determined by the z-axis and the image of P in the xz-plane. The focal loci of the congruence G are the hyperbolic paraboloid $z+xy=0$ and the z-axis. The surface S_0 is its own point conjugate with respect to the congruence G'. The osculating ruled surfaces of S_0 have straight line directrices, which, moreover, lie in the xz- and yz-planes; that is, the directrix quadric degenerates into two distinct planes.

Finally, we construct the adjoint of sytem (A). Its integral surfaces are dualistic to those of system (A). This fact enables us to extend the re-

* Wilczynski, "Über Flächen mit unbestimmten Direktrixkurven," *Mathematische Annalen*, December, 1914.

sults of the preceding paragraphs; in particular, we obtain the conditions under which the congruence G has developable focal sheets and find that a surface can have at most one linear directrix congruence, if its directrix curves form a non-degenerate conjugate net on the surface.

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*The Apolar Generation of the Quartic Curve**

Dr. WILLIAM P. MILNE.

In Ser. 2, Vol. 9, Part 3, and Ser. 2, Vol. 10, Part 3, of the *Proceedings of the London Math. Soc.*, I found the necessary and sufficient conditions that two triads of points situated on a cubic curve must satisfy in order that every point on the curve may subtend at the triads two pencils of lines which are apolar. Since the publication of these results, I have continued these investigations for the case of the quartic curve. I have not as yet been successful in obtaining what can be regarded as a complete solution, but the following is a brief abstract of the definite results that have been obtained up to date.

The problem to be solved is:—

Given a quartic curve, to find two inscribed quadrangles PQRS and KLMN, such that if X be any point on the quartic curve, the two pencils of lines $X[P, Q, R, S]$ and $X[K, L, M, N]$ shall be apolar for all positions of X on the curve.

Before describing the exact progress towards the solution that has been made, the following definitions are necessary:—

Principal Conics.—If P, Q, R, S be any four assigned points on a quartic curve F, and if S_1 and S_2 be any two conics of the four-point system, thus defined, then p, q, r can be so determined that the quartic

$$F+pS_1^2+2qS_1S_2+rS_2^2=0 \qquad (1)$$

shall pass through the vertices of the self-polar triangle of the pencil $S_1+\lambda S_2$. We shall call the conics

$$pS_1^2+2qS_1S_2+rS_2^2=0,$$

thus obtained, the "principal conics" of the tetrad $PQRS$ with respect to

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* Read February 11th, 1915.

the quartic F. The quartic (1) we shall call the "auxiliary quartic" of the tetrad P, Q, R, S.

Generating Conic.—The auxiliary quartic passes through P, Q, R, S and the vertices of their self-polar triangle. Suppose that any conic $S_1+\lambda S_2$ of the given pencil cuts the auxiliary quartic again in four points "U." It can be proved that the conic through the "U"-tetrad apolar to $S_1+\lambda S_2$ (regarded as an envelope) cuts the auxiliary quartic again in four fixed points "L." Also, it can be shewn that there exists a unique conic S_0 of the S_1, S_2 pencil apolar to every conic through the "L" tetrad (regarded as loci). We shall call S_0 the "generating conic" of $PQRS$ with respect to the quartic F.

Self-Conjugate Conics.—If ϕ and ψ be any two tetrads of points taken at random on a plane, there exist in general *two* conics S_1 and S_2 of the ϕ-pencil such that S_1^2 and S_2^2 regarded as loci are each apolar to ψ regarded as an envelope of the fourth class. We shall call S_1 and S_2 the "self-conjugate conics" of ϕ relative to ψ.

Polo-Reciprocal Conic.—Let $S_1+\lambda S_2$ be a conic of the ϕ-tetrad, and let $\Pi_1+\lambda\Pi_2$ be its polar-conic with respect to the ψ-tetrad. If we then form the reciprocal of the conic $\Pi_1+\mu\Pi_2$ with respect to the conic $S_1+\lambda S_2$, and thereafter express the condition that the above reciprocal be apolar to the conic $S_1+\mu S_2$ (regarded as an envelope), we obtain a (2, 3)-correspondence between λ and μ. If, however, $S_1+\lambda S_2$ be one of the self-conjugate conics, we find that the three corresponding μ-conics are the particular self-conjugate conic itself taken twice over and a unique conic which we shall call the "polo-reciprocal conic."

The result that has been reached can now be described as follows:—

If ϕ and ψ be any two tetrads of points taken on a plane, their apolar locus (*i.e.*, the locus of points subtending apolar pencils at ϕ and ψ) is that unique quartic passing through each of the ϕ and ψ points and satisfying the following conditions:—

(1) The ϕ-principal-conics are the self-conjugate conics of ϕ with respect to ψ.

(2) The ψ-principal-conics are the self-conjugate conics of ψ with respect to ϕ.

(3) The ϕ-generating-conic is the polo-reciprocal conic of ϕ with respect to ψ.

(4) The ψ-generating-conic is the polo-reciprocal conic of ψ with respect to ϕ.

Thursday, December 9th, 1915.

Prof. Sir J. LARMOR, President, in the Chair.

Present fifteen members and one visitor.

Messrs. C. H. Kebby and B. M. Sen were elected members.

In the absence of the Auditor (Lt.-Col. Cunningham), his Report was presented by the President.

On the motion of Dr. W. H. Eccles, seconded by Mr. G. B. Jeffery, the Treasurer's Report was adopted, and the thanks of the Society were voted to the Treasurer and to the Auditor.

Mr. G. T. Bennett, as Secretary, reported that, during the Session 1914–15, seven new members were elected, four members resigned, six died, and the names of three were removed from the list. At the date of the Annual General Meeting (November 11th, 1915), the number of members on the list was 299, as against 305 at the beginning of the previous Session. The National Academy of Sciences (Washington, U.S.A.) had been added to the list of Societies with which this Society exchanges *Proceedings*.

The following papers were communicated:—

Diffraction by a Wedge: Mr. F. J. W. Whipple.

*Some Applications of the Two-three Birational Space Transformation: Mr. T. L. Wren.

The Circles which touch the Escribed Circles of a Triangle: Mr. T. C. Lewis.

*On the Vibrations of a Special Type of Dissipative System: Mr. H. Jeffreys.

*On Seminvariants of Linear Homogeneous Differential Equations: Prof. E. B. Stouffer.

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* Printed in this volume.

ABSTRACTS.

The Circles which touch the Escribed Circles of a Triangle

Mr. T. C. LEWIS.

1. The equations of the five circles which touch all three of the escribed circles of a triangle are obtained in the quadricircular coordinates of M. Gaston Darboux, and the following relations are established.

2. Let ABC be the triangle. Produce AB and AC to B_1 and C_1 respectively, where

$$BB_1 = \frac{as}{b} \quad \text{and} \quad CC_1 = \frac{as}{c}.$$

Similarly produce BC and BA to C_2, A_2, where

$$CC_2 = \frac{bs}{c} \quad \text{and} \quad AA_2 = \frac{bs}{a},$$

and CA and CB to A_3, B_3, where

$$AA_3 = \frac{cs}{a} \quad \text{and} \quad BB_3 = \frac{cs}{b}.$$

Then the six points B_1, C_1, C_2, A_2, A_3, B_3 are concyclic, and the circle that passes through them circumscribes the escribed circles of the triangle. If its radius be ρ,

$$4\rho = r + \frac{r_1 r_2 r_3}{r^2}.$$

3. If AB and AC be divided at B_2 and C_3 in the ratio $r_1 : r$, the circle which touches externally the circles escribed opposite B and C and internally the circle escribed opposite A will pass through B_2 and C_3, and its radius ρ_a is given by

$$\rho_a : \rho :: a : b+c :: r_1 - r : r_1 + r.$$

So for ρ_b and ρ_c, the radii of circles touching internally the escribed circles opposite B and C, respectively, and the other escribed circles externally.

4. The fifth circle is the nine-point circle; let ρ' be its radius. Then among the relations between the radii of the five circles we have

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} + \frac{1}{\rho} = \frac{1}{\rho'},$$

$$\frac{\rho_a + \rho_b + \rho_c + \rho}{\rho} = \frac{3\rho - r}{\rho + \rho'}.$$

On Seminvariants of Linear Homogeneous Differential Equations

Prof. E. B. STOUFFER.

The most general transformation of the dependent variables which leaves the system of linear homogeneous differential equations

$$y_i'' + \sum_{j=1}^{n} (2p_{ij}y_j' + q_{ij}y_j) = 0 \quad (i = 1, 2, \ldots, n),$$

unchanged in form, is given by the equations

$$y_i = \sum_{k=1}^{n} a_{ik}\bar{y}_k, \quad |a_{ik}| \neq 0 \quad (i = 1, 2, \ldots, n),$$

where a_{ik} are arbitrary functions of the independent variable x. A function of p_{ij} and q_{ij} and their derivatives which has the same value for the given system as for any system obtained from it by the above transformation is called a seminvariant. In the present paper there is obtained for each value of n a single seminvariant from which the complete system for that value of n may be obtained by successive applications of certain operators. This method of obtaining the seminvariants avoids the solution of complicated systems of partial differential equations.

Thursday, January 13th, 1916.

Prof. Sir J. LARMOR, President, in the Chair.

Present eighteen members and one visitor.

Messrs. A. S. Eddington, H. T. J. Norton, H. Todd, and A. W. Young, were elected members.

Messrs. H. Jeffreys and A. W. Young were admitted into the Society.

The President announced the death of Prof. B. Williamson, S.F.T.C.D., who had been a member of the Society since 1875, and spoke briefly of his mathematical work.

The President having called Prof. A. E. H. Love, Vice-President, to the chair, read a paper,* "The Transition from Vapour to Liquid when the Range of the Molecular Attractions is Sensible."

* Printed in this volume.

The following papers were communicated :—

(i) A Condition for the Validity of Taylor's Expansion, (ii) *A Note on the Uniform Convergence of the Series $\Sigma a_n \sin n\theta$: Mr. T. W. Chaundy.

†The Average Order of the Arithmetical Functions $P(x)$ and $\Delta(x)$: Mr. G. H. Hardy.

Green's Dyadics in the Theory of Elasticity : Mr. C. E. Weatherburn.

‡A Problem of "Analysis Situs" : Mr. G. N. Watson.

ABSTRACTS.

The Transition from Vapour to Liquid when the Range of the Molecular Attraction is Sensible

Sir J. LARMOR.

In the theory of capillarity, and of change of state, the hydrostatic pressure p is defined, in physical illustration, as the difference between two much larger quantities, the repulsion ϖ due to molecular motion and the mutual attraction P of the molecules. Its graph, in the Andrews-Thomson diagram, determines the critical point and the conditions for change of state. It is a definite quantity only where the density is uniform; thus it loses its meaning inside interfacial layers of rapid transition, though under fluid conditions it is transmitted across such layers. The instability in homogeneous fluid, and consequent separation of phases, which ensues when dp/dv becomes positive, is essentially a matter of the internal constitution of the fluid, and ought to be so deducible. It is found, however, that the homogeneous medium is unstable for variation of density when $d(p-P)/dv$ is positive : whereas instability from external stress, when the density is not disturbed, occurs within the narrower limits for which dp/dv is positive. When the range of attractions is sensible there will thus be arcs of internal instability along the isothermals above the critical point, for which, however, separation into two phases, vapour and liquid, cannot occur. It might be imagined as relieved by gradual falling away of the medium to modified states of molecular aggregation ; and, in fact,

* This note is incorporated in the paper by Messrs. T. W. Chaundy and A. E. Jolliffe, printed on p. 214 of this volume.

† Printed in this volume.

the question arises why this type of change should be regarded as excluded in the usual theory, notwithstanding the aptness of the van der Waals equation. An S-shaped convolution of the isothermal is still the condition for abrupt transition of state. Other conditions restricting the form of such a law of attraction as is compatible with the existence of a homogeneous phase are noticed.

A Condition for the Validity of Taylor's Expansion

Mr. T. W. CHAUNDY.

The result is obtained that $f(x)$ can be expanded in powers of x over exactly that region of values of x for which the expansion converges, and the function of two variables (x, n), $\left[\frac{f_n(x)}{n!}\right]^{1/n}$, is finite.

A Note on the Uniform Convergence of the Series $\Sigma a_n \sin n\theta$

Mr. T. W. CHAUNDY.

Suppose that a_n is positive and monotonic, and θ real; it is shown (1) that, if na_n does not tend to zero, the values $\theta = 2k\pi$ give non-uniform convergence; (2) that, if na_n tends steadily to zero, there is uniform convergence over any finite region.

Green's Dyadics in the Theory of Elasticity

Mr. C. E. WEATHERBURN.

The close analogy which is known to exist between the problems of Dirichlet and Neumann, and the fundamental problems of elasticity, and which in a recent paper* I have further emphasized by the introduction

* "On two Fundamental Problems in the Theory of Elasticity." Recently communicated to the *Philosophical Magazine*.

of certain dyadics constructed from Somigliana's integrals of the equations of equilibrium for an elastic isotropic body, suggests a further inquiry as to the existence of other dyadics which, in the theory of elasticity, will play a part similar to that taken by the ordinary Green's functions in the potential problems. In the present paper I show how to construct such dyadics, which I shall call the Green's dyadics for the problems of elasticity corresponding respectively to zero surface displacement and zero surface traction. The former is an ordinary Green's dyadic, and the latter a generalized. Having proved their fundamental properties I show how they may be used for the solution of various problems related to the differential equation of elastic equilibrium under no bodily forces

$$\nabla^2\bar{u}+k \text{ grad div } \bar{u} = 0.$$

In particular I discuss the problems of small vibrations of an isotropic body corresponding to the boundary conditions of zero displacement and zero surface traction; and finally, prove bilinear series for the Green's dyadics, corresponding exactly to the ordinary bilinear series for the symmetric kernel of the scalar integral equation. The paper will afford further applications of the theory of vector integral equations, which I have developed in a paper* just appearing in the *Transactions of the Cambridge Philosophical Society*; and the important role of the conjugo-symmetric kernel there indicated is here filled by the Green's dyadics to be considered.

A Problem of "Analysis Situs"

Mr. G. N. WATSON.

Jordan's theorem that a closed curve possesses an interior and an exterior is well known; more precisely this theorem is that, if Q is a fixed point not on a given regular closed curve, while P is a variable point of the curve, then, if θ be the angle between QP and the x-axis, θ changes by zero or 2π as P describes the curve. The object of this paper is to investigate a theorem of the same nature as Jordan's theorem, namely, that if the curve have a regularly turning tangent, and if the tangent at P makes an angle ψ with the x-axis, then ψ changes by 2π as P describes the curve.

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* "Vector Integral Equations and Gibbs's Dyadics."

Thursday, February 10th, 1916.

Prof. Sir J. LARMOR, President, in the Chair.

Present fourteen members and one visitor.

Mr. A. W. Veater was elected a member of the Society.

Mr. J. H. Grace read a paper "Theorems on Straight Lines Intersecting at Right Angles."

The following papers were communicated by title from the Chair :—

The Classification of Rational Approximations : Mr. J. H. Grace.

On Infinite Derivates : Mrs. G. C. Young.

The Bilinear Curvature and other Functions of Independent Direction on a Surface : Mr. E. H. Neville.

The Attraction of Equiangular Spirals : Mr. S. Brodetsky.

The following informal communications were made :—

Additions and Corrections to a former paper on "Limiting Forms of Long-Period Tides" : Mr. J. Proudman.

Certain Composite Mersenne's Numbers : Mr. R. E. Powers.

Note on a Formula connected with the Theory of Spherical Harmonics : Prof. H. F. Baker.

Note on Prof. Baker's Formula : Dr. T. J. I'A. Bromwich.

Notes on the Arithmetic of Prime Numbers : Mr. J. Hammond.

ABSTRACTS.

Theorems on Straight Lines Intersecting at Right Angles

Mr. J. H. GRACE.

The following theorem is due to Prof. Morley :—If a, b, c are three straight lines, and a', b', c' the shortest distances between them, then the shortest distances between a, a' ; b, b' ; c, c' all intersect the same line at right angles.

It is easy to see that if A, B, C are three screws, having a, b, c for axes, A' the screw on the cylindroid containing B and C that is reciprocal

to A, and so on, then the screws A', B', C' belong to one cylindroid. Then the pitches p_1, p_2, p_3 can be chosen so that the axis of A' is at right angles to a, and so must intersect a. Thus this axis is the shortest distance between a and a', and the theorem follows.

The pitches have to satisfy

$$p_1 - d_{23} \tan \theta_{23} = p_2 - d_{31} \tan \theta_{31} = p_3 - d_{12} \tan \theta_{12}.$$

Similar methods are used to prove the next and other theorems; if d is the shortest distance between two lines a, b, and g any line meeting a, b, then the shortest distance l between d and g lies on a fixed cylindroid.

There follows an immediate proof of the result, practically due to Hamilton,* that the shortest distances between a ray of a system and consecutive rays of it lie on a cylindroid; for the contiguous rays intersect two (focal) lines that cut the central one at right angles.

Morley's theorem may be stated thus:—If a figure consists originally of three lines, and there be constantly added to it the shortest distance between two non-intersecting members of it, then only ten lines are obtained in all.

A geometrical proof of this is given, and it is also shewn that if the figure consists originally of four lines the system derived from it as above is not finite.

The Classification of Rational Approximations

Mr. J. H. GRACE.

It is well known that any incommensurable number ω has an infinite number of rational approximations x/y, such that, the fraction being in its lowest terms,

$$\left| \omega - \frac{x}{y} \right| < \frac{1}{y^2}.$$

Hermite shewed that there is an infinite number, such that

$$\left| \omega - \frac{x}{y} \right| < \frac{1}{y^2\sqrt{3}},$$

and that for two consecutive approximations of this type

$$xy' - x'y = \pm 1.$$

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* See *Transactions Royal Irish Academy*, Vol. XXXII, Section A, p. 153, where Sir Robert Ball quotes Prof. Joly to this effect.

Hurwitz shewed that there is an infinite number, such that

$$\left| \omega - \frac{x}{y} \right| < \frac{1}{y^2\sqrt{5}};$$

and then the above relation between consecutive ones does not always hold, though Minkowski has remarked that for approximations such that

$$\left| \omega - \frac{x}{y} \right| < \frac{1}{2y^2}$$

it does hold.

It is proved in this paper that, if an approximation gives

$$\left| \omega - \frac{x}{y} \right| = \frac{1}{\lambda y^2},$$

where $\lambda \geqslant k$ and $\frac{3}{2} \leqslant k \leqslant 2$, then consecutive members of the class so formed satisfy

$$xy' - x'y = \pm 1.$$

For values of k outside these limits the class of approximations has not the property indicated.

There is some slight connection between this classification and Liouville's original construction of non-algebraic numbers. Liouville's transcendent numbers have, in fact, an infinite number of better approximations than any assigned algebraic number. It follows from an investigation of Markoff's that if ω has only a finite number of approximations, such that

$$\left| \omega - \frac{x}{y} \right| < \frac{1}{3y^2},$$

then ω is a quadratic surd. It seems probable that there are non-algebraic numbers having only a finite number of approximations, such that

$$\left| \omega - \frac{x}{y} \right| < \frac{1}{ky^2},$$

where $k > 3$; it is proved that this is so if k exceeds

$$1 + \sqrt{2} + \tfrac{1}{2}(\sqrt{5} - 1),$$

which is greater than 3 but less than $3\frac{1}{30}$.

The Bilinear Curvature and other Functions of Independent Direction on a Surface

Mr. E. H. NEVILLE.

The paper recommends the study in differential geometry of functions of several independent directions, and shews that this study facilitates the study of functions of one direction, and coordinates results relating to the most familiar functions of this kind. After describing the extension to functions of several directions of a method due to Darboux for the construction of functions of a single direction, and considering the most elementary properties of linear and multilinear functions of direction, the paper introduces three particular multilinear functions. The first, for which the name of bilinear curvature is proposed, is a symmetrical bilinear function of direction that reduces to normal curvature when the two directions on which it depends coincide and to geodesic torsion when these directions are at right angles; this function plays in geometry the part performed in analysis by the function denoted usually by M, the second of the fundamental magnitudes of the second order, and examples are given of its use. The second function considered is the trilinear function obtained from the bilinear curvature by the method to which allusion has been made; this function proves to be symmetrical, and the assertion of its symmetry is equivalent for any set of reference curves to the assertion of the Mainardi-Codazzi equations; moreover, the symmetry of this function implies the fundamental relation known to exist between two cubic functions of a single direction found by Laguerre and Darboux. The third function described is a bilinear function obtained by the same method, and involves an arbitrary function of position; again the function is symmetrical, a fact that alone implies the most familiar of expressions for the geodesic curvature of an orthogonal trajectory of a family of curves. The coefficients appearing in the analytical expression for the function are the modified second derivatives whose importance in the theory of invariance is well known. Moreover, since the bilinear function is defined without reference to any particular coordinates, many cases of covariance that were first discovered analytically receive an immediate explanation.

Additions and Corrections to a former Paper on "Limiting Forms of Long Period Tides"*

Mr. J. PROUDMAN.

P. 277.—The contour-lines of $b \sec \theta$ have been drawn for the North Atlantic Ocean by Mr. C. J. Rowan, of Liverpool University, from the data marked on the Admiralty Charts.† They indicate that a large portion of this ocean forms a region of type I, over which therefore the limiting form of the tide will be the equilibrium form. In the north-eastern portion there are a number of regions of type III.

P. 299.—The meanings of k_n, k_{-n} in (65) have been slightly changed from those in (63), (64). The change is obvious on comparison of (62) with (65).

P. 301.—From certain data the theory is found not to apply to the lunar fortnightly tide as recorded at the Indian ports. Two possible reasons are suggested for this, but there appears to be another. This is that the data are not sufficiently reliable for the purpose. The results of the harmonic analysis of observations taken during different years at the same port, ought to be in agreement. Now for the long period tides this is far from being the case.‡ It may be pointed out that there is a compound§ (*i.e.*, shallow water) tide of period exactly equal to that of the lunar fortnightly tide, and if the amplitudes of the two are comparable the whole application of the paper will be vitiated.

P. 304.—In the equation of continuity u, v should be replaced by u', v' respectively. This requires the omission of the terms involving u_0, v_0 in (81).

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 273–306 (1913).

† Hough states that he began to draw these lines for another purpose, *Phil. Trans. Roy. Soc.*, (A), Vol. CLXXXIX, p. 252 (1897).

‡ See Baird and Darwin, "Results of the Harmonic Analysis of Tidal Observations," *Proc. Roy. Soc.*, Vol. XXXIX, p. 135 (1885); S. Hirayama, "Results of the Harmonic Analysis of Tidal Observations made at various Ports of Japan," *Jour. Sci. Coll., Tokyo*, Vol. XXVIII, Art. 7 (1911).

§ Darwin, "Harmonic Analysis of Tidal Observations," *Brit. Assoc. Report* for 1883, p. 49 [*Sci. Papers*, Vol. 1, p. 1].

Certain Composite Mersenne's Numbers

Mr. R. E. POWERS.

The author stated that he had proved the two Mersenne's numbers $2^{103}-1$ and $2^{109}-1$ to be composite. The former was tested by means of Lucas's sequence* 3, 7, 47, ..., and the computation was completed in May, 1914; the latter, by means of Lucas's sequence* 4, 14, 194, ..., the date of completion being June, 1914.

Note on a Formula connected with the Theory of Spherical Harmonics

Prof. H. F. BAKER.

The following particular result may perhaps be worth recording in the Society's Abstracts, as well for the simplicity of the proof as for its relation with known cases.†

If U, U' be any two homogeneous integral polynomials in x, y, z of orders m, m', both even or odd, and

$$U_q = \frac{1}{[\frac{1}{2}(m-q)]!}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^{\frac{1}{2}(m-q)} U,$$

with a similar definition for U'_q, then the integral

$$\iint UU'\,dS,$$

taken over the surface of a unit sphere, is equal to

$$\frac{4\pi\,(n!)}{(2n+1)!}\sum_q 2^q U_q\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right) U'_q(x, y, z),$$

where $n = \frac{1}{2}(m+m')$, and q has all the (finite number of) positive integer values for which $\frac{1}{2}(m-q)$, $\frac{1}{2}(m'-q)$ are integral and not negative.

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* *American Journal of Mathematics*, Vol. 1, 1878; for further details see *Records of Proceedings for* 1914 (Vol. 13, p. xxxix).

† Cf. Sylvester, *Phil. Mag.*, 1876, or *Collected Papers*, Vol. III, p. 37; Hobson, *Proc. London Math. Soc.*, Vol. XXIV, 1893, p. 83; Darwin, *Scientific Papers*, Vol. III, pp. 398–422.

To prove this recall that, over the sphere, if $\mu = (u^2+v^2+w^2)^{\frac{1}{2}}$,

$$\iint e^{ux+vy+wz}\,dS = \int_0^{2\pi}\int_{-1}^{1} e^{\mu z}\,dz\,d\phi = 4\pi\left(\frac{e^{\mu}-e^{-\mu}}{2\mu}\right) = 4\pi\left\{1+\frac{\mu^2}{3!}+\frac{\mu^4}{5!}+\dots\right\},$$

while if the suffix 0 be used to denote the value at the centre of the sphere,

$$U = e^{\alpha x+\beta y+\gamma z}\,U_0, \qquad U' = e^{\alpha' x+\beta' y+\gamma' z}\,U_0',$$

where $\alpha^p\beta^q\gamma^r\,U_0$ means $\left(\dfrac{\partial^{p+q+r}\,U}{\partial x^p\,\partial y^q\,\partial z^r}\right)_0$,

and the symbols α', β', γ' are similarly used for operation on the function U' only.

Thus $$\iint UU'\,dS = 4\pi\left[1+\Sigma\frac{\mu^{2n}}{(2n+1)!}\right]U_0U_0',$$

where $\mu^{2n} = [\alpha^2+\beta^2+\gamma^2+2(\alpha\alpha'+\beta\beta'+\gamma\gamma')+\alpha'^2+\beta'^2+\gamma'^2]^n$.

As U, U' are polynomials, the only terms which do not vanish identically are those of order m in α, β, γ, and m' in α', β', γ', and so of aggregate order $m+m'$. Such terms can only enter if $m+m'$ be even, equal to $2n$, say, and then only among the terms of

$$\frac{4\pi}{(2n+1)!}[\alpha^2+\beta^2+\gamma^2+2(\alpha\alpha'+\beta\beta'+\gamma\gamma')+\alpha'^2+\beta'^2+\gamma'^2]^n\,U_0U_0'.$$

The expansion of the bracket [ ] here is like that of

$$[a\lambda^2+2h\lambda\lambda'+b\lambda'^2]^n,$$

which is $$\sum_{p,q,r}\frac{n!}{p!\,q!\,r!}(a\lambda^2)^p\,(2h\lambda\lambda')^q\,(b\lambda'^2)^r,$$

and we require only the terms of order m in λ, and m' in λ'. Thus p, q, r are to be such that

$$2p+q = m, \quad q+2r = m', \quad p+q+r = n = \tfrac{1}{2}(m+m'),$$

and the terms in question are

$$\frac{4\pi}{(2n+1)!}\sum_q\frac{n!}{q!\,QQ'}a^{\frac{1}{2}(m-q)}\,(2h)^b\,b^{\frac{1}{2}(m'-q)},$$

where $$Q = [\tfrac{1}{2}(m-q)]!, \qquad Q' = [\tfrac{1}{2}(m'-q)]!.$$

Hence, with the above given definition of U_q and U_q', the integral when

evaluated is equal to

$$\frac{4\pi\,(n!)}{(2n+1)!}\,\Sigma\,\frac{2^q}{q!}\left(\frac{\partial^2}{\partial x\,\partial x'}+\frac{\partial^2}{\partial y\,\partial y'}+\frac{\partial^2}{\partial z\,\partial z'}\right)^q U_q U_q'.$$

Now, if $$U_q=\Sigma a x^\lambda y^\mu z^\nu, \qquad U_q'=\Sigma a' x^\lambda y^\mu z^\nu,$$

where $\lambda+\mu+\nu=q$; and because, when $\lambda'+\mu'+\nu'=\lambda+\mu+\nu$,

$$\frac{\partial^{\lambda+\mu+\nu}}{\partial x^\lambda\,\partial y^\mu\,\partial z^\nu}\,x^{\lambda'}y^{\mu'}z^{\nu'}$$

vanishes, unless $\lambda'=\lambda$, $\mu'=\mu$, $\nu'=\nu$, and is then equal to $\lambda!\,\mu!\,\nu!$, we have

$$\begin{aligned}\frac{1}{q!}\left(\frac{\partial^2}{\partial x\,\partial x'}+\frac{\partial^2}{\partial y\,\partial y'}+\frac{\partial^2}{\partial z\,\partial z'}\right)^q U_q U_q' &= \Sigma\,\frac{1}{\lambda!\,\mu!\,\nu!}\,\frac{\partial^{\lambda+\mu+\nu}}{\partial x^\lambda\,\partial y^\mu\,\partial z^\nu}\,U_q\,\frac{\partial^{\lambda+\mu+\nu}}{\partial x^\lambda\,\partial y^\mu\,\partial z^\nu}\,U_q'\\ &= \Sigma\,a a'\,\lambda!\,\mu!\,\nu!\\ &= \Sigma\,a\,\frac{\partial^{\lambda+\mu+\nu}}{\partial x^\lambda\,\partial y^\mu\,\partial z^\nu}\,\Sigma\,a'\,x^\lambda y^\mu z^\nu\\ &= U_q\left(\frac{\partial}{\partial x},\,\frac{\partial}{\partial y},\,\frac{\partial}{\partial z}\right)U_q'(x,\,y,\,z),\end{aligned}$$

whereby the result is established.

Three simple examples may be added to show the meaning of the result.

1. When $U=x^{2n}$, $U'=1$, $m=2n$, $m'=0$, the only value of q for which $\frac{1}{2}(m'-q)$ is not negative is $q=0$; hence

$$U_q=\frac{1}{n!}(\nabla^2)^n\,x^{2n}=\frac{1}{n!}\,\frac{\partial^{2n}}{\partial x^{2n}}x^{2n}=\frac{(2n)!}{n!},$$

and so $$\iint x^{2n}\,dS=\frac{4\pi}{2n+1}.$$

2. When $U=ax^2+by^2+cz^2$, $U'=a'x^4+b'y^4+c'z^4$, $m=2$, $m'=4$, $n=3$, the values of q are 2 and 0. For these we have

$$U_2=U, \quad U_0=\nabla^2U=2(a+b+c), \quad U_2'=\nabla^2U'=12(a'x^2+b'y^2+c'z^2),$$

$$U_0'=\tfrac{1}{2}\nabla^4U',$$

so that $$\iint (ax^2+by^2+cz^2)(a'x^4+b'y^4+c'z^4)\,dS$$

$$= \frac{4\pi(3)!}{7!}\Big[2^2 U\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) 12(a'x^2+b'y^2+c'z^2) + 2(a+b+c)\,12(a'+b'+c')\Big]$$

$$= \frac{4\pi}{35}\{4(aa'+bb'+cc')+(a+b+c)(a'+b'+c')\}.$$

3. If U be itself a spherical surface harmonic, say Y_m, every term U_q is zero unless $q = m$. Thus, if $m' \geqslant m$, and

$$U'_q = (\nabla^2)^{\frac{1}{2}(m'-m)}\, U'/[\tfrac{1}{2}(m'-m)]!,$$

$$\iint Y_m U' dS = A Y_m \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) U'_q,$$

where $$A = 4\pi\, 2^m [\tfrac{1}{2}(m'+m)]!/(m'+m+1)!.$$

Note on the Formula given by Prof. Baker

Dr. T. J. I'A. BROMWICH.

Prof. Baker's result recalls to my mind a symbolical method which I have often used to evaluate particular cases of the formula. It is evident that, if α, β, γ are any constants, the integral over the unit sphere

$$(1) \qquad \int (\alpha x+\beta y+\gamma z)^{2n}\, d\omega = \frac{4\pi}{2n+1}(\alpha^2+\beta^2+\gamma^2)^n;$$

for by rotation of axes the integral reduces to

$$(\alpha^2+\beta^2+\gamma^2)^n \int z^{2n}\, d\omega = 2\pi(\alpha^2+\beta^2+\gamma^2)^n \int_{-1}^{+1} z^{2n}\, dz.$$

Since (1) holds for all values of α, β, γ it is an algebraic identity, corresponding terms on the two sides being equal. It is therefore legitimate to interpret (1) in a symbolic sense, using

$$(\alpha x+\beta y+\gamma z)^{2n}$$

to denote an arbitrary polynomial of degree $2n$; and $\alpha^p\beta^q\gamma^r$ to denote the appropriate coefficient in the polynomial.

To obtain the value of the integral of the product of two polynomials,

we may write in (1), $\alpha+\lambda a_1$, $\beta+\lambda\beta_1$, $\gamma+\lambda\gamma_1$ for α, β, γ respectively; and then picking out the coefficient of λ^m, we find that

$$(2) \qquad \frac{1}{l!\,m!}\int (ax+\beta y+\gamma z)^l\,(a_1x+\beta_1y+\gamma_1z)^m\,d\omega$$

is equal to the coefficient of λ^m in the expansion of

$$(3) \qquad \frac{4\pi}{(2n+1)!}\left\{(\alpha^2+\beta^2+\gamma^2)+2\lambda(\alpha a_1+\beta\beta_1+\gamma\gamma_1)+\lambda^2(a_1^2+\beta_1^2+\gamma_1^2)\right\}^n,$$

where $$2n = l+m.$$

In (2) the result may be interpreted symbolically as before, so that the brackets represent polynomials of degree l, m, with arbitrary coefficients.

In this notation, Prof. Baker's function U_q can be expressed symbolically as

$$\frac{1}{p!}\,l(l-1)\ldots(l-2p+1)(\alpha^2+\beta^2+\gamma^2)^p\,(ax+\beta y+\gamma z)^q,$$

where $l-2p=q$, and l is used in place of Prof. Baker's m. Thus

$$U_q = \frac{l!}{p!\,q!}\,(\alpha^2+\beta^2+\gamma^2)^p\,(ax+\beta y+\gamma z)^q,$$

with a similar formula for U_q'; and so we find also

$$(4) \qquad \frac{1}{l!\,m!}\left\{2^qU_q\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)U_q'(x,y,z)\right\}$$

$$= \frac{1}{p!\,q!\,r!}\,(\alpha^2+\beta^2+\gamma^2)^p\left\{2(\alpha a_1+\beta\beta_1+\gamma\gamma_1)\right\}^q(a_1^2+\beta_1^2+\gamma_1^2)^r,$$

where $$2p = l-q, \qquad 2r = m-q,$$

so that $$p+q+r = n.$$

It will now be evident that (4) when multiplied by the factor

$$4\pi\,\{n!/(2n+1)!\}$$

gives the general term in the expanded form of (3); and so Prof. Baker's formula is equivalent to the statement contained in (2) and (3).

The special case when both polynomials are solid harmonics gives a known result: but the conclusion can be written very compactly by observing that then, in the present notation,

$$\alpha^2+\beta^2+\gamma^2 = 0, \qquad a_1^2+\beta_1^2+\gamma_1^2 = 0.$$

Thus (2), (3) lead to the form

$$(5)\quad \int (ax+\beta y+\gamma z)^n (a_1x+\beta_1 y+\gamma_1 z)^n \, d\omega = \frac{4\pi (n!)^2}{(2n+1)!} 2^n (a a_1+\beta\beta_1+\gamma\gamma_1)^n,$$

where both polynomials are solid harmonics of order n.

As a further example, we may take Prof. Baker's second example in which $l = 2$, $m = 4$, and

$$(ax+\beta y+\gamma z)^2 \equiv ax^2+by^2+cz^2,$$

$$(a_1x+\beta_1 y+\gamma_1 z)^4 \equiv a'x^4+b'y^4+c'z^4.$$

Then, using (2), (3), with $n = 3$, we find, on picking out the coefficient of λ^4, that the required integral is equal to

$$4\pi \frac{2!\,4!}{7!}(12Q^2R+3PR^2) = \frac{4\pi}{35}(4Q^2R+PR^2),$$

where $$P = a^2+\beta^2+\gamma^2, \qquad Q = aa_1+\beta\beta_1+\gamma\gamma_1,$$

$$R = a_1^2+\beta_1^2+\gamma_1^2.$$

This reduces at once to the value

$$\frac{4\pi}{35}\{4(aa'+bb'+cc')+(a+b+c)(a'+b'+c')\}.$$

Notes on the Arithmetic of Prime Numbers

Mr. J. HAMMOND.

The principal results arrived at are these:

$$P(2n)-Q(2n)+R(2n) = \tfrac{1}{2}\phi(2n)+D, \qquad (1)$$

where $2n$ is any even number, $\phi(2n)$ its totient, D the number of its odd prime divisors (of course excluding 1), $P(2n)$ the number of odd primes (including 1) less than $2n$, $Q(2n)$ the number of partitions of $2n$ into two unequal primes, and $R(2n)$ the number of its partitions into two composite numbers which are prime to each other.

$$P(x)+P(x\alpha^{-1})+P(x\beta^{-1})+\ldots \; t \text{ terms} = \phi(x, m)+ct+\tfrac{1}{2}t(t-1), \qquad (2)$$

where $\alpha, \beta, \ldots$ are all the primes between $x^{\frac{1}{3}}$ and $x^{\frac{1}{2}}$, so that

$$t = P(x^{\frac{1}{2}})-P(x^{\frac{1}{3}})+1;$$

$m = 2.3.5 \ldots =$ the product of all the primes which do not exceed $x^{\frac{1}{3}}$, and c of these are odd, so that

$$c = P(x^{\frac{1}{3}}) - 1 ;$$

and $\phi(x, m)$ means the number of numbers prime to m, which do not exceed the positive limit x.

When m is any positive integer, and a any one of its prime non-divisors

$$\phi(x, am) = \phi(x, m) - \phi(xa^{-1}, m). \qquad (3)$$

One variation of (2), obtained by aid of (3), is

$$P(2n) + P(2na^{-1}) + P(2n\beta^{-1}) + \ldots \; t \text{ terms} = \phi(2n) + tD + \tfrac{1}{2}t(t-1), \quad (4)$$

when $2n$ is divisible by every prime less than its cube root, and has no prime divisor greater than its square root; $a, \beta, \ldots$ being all the prime non-divisors of $2n$ which are less than its square root.

Ex.—When $2n = 840 = 8.3.5.7,$

$D = 3$ and $a, \beta, \ldots = 11, 13, 17, 19, 23,$

$t = 6$ and $\phi(840) = 4.2.4.6 = 192,$

$m = 2.3.5.7 = 210$ and $c = 4 - 1 = 3.$

Further,

$840 \div 11 = 76+,$ $840 \div 17 = 49+,$ $840 \div 23 = 36+.$

$840 \div 13 = 64+,$ $840 \div 19 = 44+.$

Then the results (2), (4) lead to the equation

$$P(840) + P(76) + P(64) + P(49) + P(44) + P(36) = \phi(840) + 3 \times 6 + \tfrac{1}{2}(6 \times 5),$$

$$146 + 21 + 18 + 15 + 14 + 11 = 192 + 18 + 15.$$

Thursday, March 9th, 1916.

Prof. Sir J. LARMOR, President, in the Chair.

Present thirteen members and one visitor.

Miss R. E. Colomb and Mr. E. G. C. Poole were elected members of the Society.

Major MacMahon read a paper,* "Two Applications of General Theorems in Combinatory Analysis."

The following papers were communicated by title from the Chair:—

Mr. Grace's Theorem on Six Lines with a Common Transversal: Prof. H. F. Baker.

The Integrals of a certain Riccati Equation connected with Halphen's Transformation: Mr. H. E. J. Curzon.

*The Cremona Transformations of a certain Plane Sextic: Miss Hilda P. Hudson.

*The Construction of Co-apolar Triads on a Cubic Curve: Dr. W. P. Milne.

On the Dynamical Equations of the Tides: Mr. J. Proudman.

Lt.-Col. Cunningham made an informal communication in reference to numbers of the type $x.2^x+1$; all these numbers, from $x=2$ to $x=282$, have been proved composite, with the exception of $x=141$: information with reference to the exceptional case would be welcomed.

ABSTRACTS.

The Cremona Transformations of a certain Plane Sextic

Miss HILDA P. HUDSON.

A plane sextic with ten double points is the simplest rational curve that cannot be regarded as part of a degenerate homaloid, and cannot be transformed into a curve of lower degree by any Cremona transformation. An extension is obtained of Noether's theorem on the sum of the three highest multiplicities of any curve. One type of such a sextic, with six of the ten double points lying on a conic, is the projection of the intersection of a quadric and a cubic surface which touch in four points. Quadratic transformations lead to four other types of the plane sextic.

On the Dynamical Equations of the Tides

Mr. J. PROUDMAN.

This paper concerns tides in an ocean of any shape and law of depth on a rotating sphere or on a rotating plane. The essence of the method

* Printed in this volume.

used consists in working with an infinite sequence of parameters, which play the part of principal coordinates of the dynamical system formed by the ocean in question.

The ordinary differential equations of motion are formally transformed into an infinite set of linear algebraic equations, analogous to the equations of motion in principal coordinates for a gyrostatic system of finite freedom.

Light is thrown on the general question of long free periods, in connection with which a theorem on oscillations of a gyrostatic system is proved.

The validity of the method is established in the special cases of an ocean of uniform depth covering the whole sphere, and of symmetrical motion in a flat circular sea of uniform depth.

In these cases the solution of the algebraic equations can be effected by means of infinite determinants, and known results, including Hough's solution, are reproduced.

The coefficients in the algebraic equations can also be explicitly obtained for problems not yet solved, such as the cases of an ocean of uniform depth bounded by two meridians and a flat rectangular sea of uniform depth. The consideration of these cases, as well as that of the general problems of justification and solution, is reserved for future papers.

Thursday, April 27th, 1916.

Prof. Sir J. LARMOR, President, in the Chair.

Present eighteen members and four visitors.

Mr. E. G. C. Poole was admitted into the Society.

Major MacMahon read a paper, "Some Problems of Combinatory Analysis."

Dr. S. Chapman read a paper,* "On the Uniformity of Gaseous Density, according to the Kinetic Theory."

Mr. G. N. Watson read a paper, "Bessel Functions and Kapteyn Series."

Mr. T. C. Lewis made an informal communication, "Note on Four Tucker Circles."

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* Printed in this volume.

The following papers, in the absence of the authors, were communicated by title from the Chair:—

The Green's Function for the Equation $\nabla^2 u + \kappa^2 u = 0$, (II): Prof. H. S. Carslaw.

*The Nodal Points of a Plane Sextic: Mr. J. Hodgkinson.

*On the Deduction of Criteria for the Convergence of Fourier's Series from Fejér's Theorem concerning their Summability: Mr. S. Pollard.

*Note on Functions of Upper and Lower Type: Prof. W. H. Young.

*On the Derivates of a Function: Mrs. G. C. Young.

ABSTRACTS.

The Green's Function for the Equation $\nabla^2 u + \kappa^2 u = 0$, (II)

Prof. H. S. CARSLAW.

This continuation of my former paper (these *Proceedings*, Vol. 13, 1914) is suggested by Macdonald's paper on "A Class of Diffraction Problems" (these *Proceedings*, Vol. 14, 1915). Macdonald shows how the solutions of certain problems of diffraction can be deduced from those of the corresponding potential problems, his method depending upon the fact that when the solutions are expressed in series of corresponding harmonic functions, the constants in the two cases are the same.

In this paper I show that the Green's function for the equation $\nabla^2 u + \kappa^2 u = 0$ can be found as readily as the Green's function for the equation $\nabla^2 u = 0$, when the region is bounded by a sphere (or two concentric spheres), a coaxial right circular cone (or two such cones), and two axial planes; also when the boundary consists of a circular cylinder (or two coaxial circular cylinders), two axial planes, and two planes perpendicular to the axis. The method depends upon the reduction of the problem to an integral equation.

In these cases there is no need to deduce the one solution from the other. But Macdonald's method would be of special value in the case of infinite regions.

* Printed in this volume.

Note on Four Tucker Circles

Mr. T. C. LEWIS.

1. In a triangle ABC, let $B'C'$ be a line parallel to BC and meeting the sides AB, AC in the points B', C' respectively.

A Tucker circle is determined by the chord $B'C'$; it has similarly two chords parallel to the remaining sides of the triangle and terminating on sides of the triangle.

Let E be the point of contact of the in-circle with AC; draw EF' parallel to CB, to meet AB in F'; measure BB' along BA and equal to EF'; draw $B'C'$ parallel to BC and meeting AC in C'.

The Tucker circle of which $B'C'$ is a chord will touch the in-circle and also the two escribed circles opposite B and C.

2. Similarly there are two other such circles, each touching the in-circle and one of the two remaining pairs of escribed circles.

3. If E be the point of contact of the escribed circle opposite A with AC produced, and BB' be measured equal to EF' along AB produced, the corresponding Tucker circle will touch each of the three escribed circles of the triangle.

4. This last Tucker circle may otherwise be determined as follows.

The first three circles make three intercepts BB' between B and the chord $B'C'$. Measure along AB produced a length equal to the sum of those three intercepts BB'. Six points are thus determined, namely, one on each side produced in either direction. These six points lie on the same Tucker circle, already mentioned, which touches each of the three escribed circles.

Bessel Functions and Kapteyn Series

Mr. G. N. WATSON.

Series of the type $\sum_{n=0}^{\infty} a_n J_{n+\nu}\{(n+\nu)z\}$, known as Kapteyn series, naturally suggest the investigation of Bessel functions of the type $J_n(nz)$

when n is large but not necessarily integral. By means of certain contour integrals due to Debye (as well as by more elementary methods), it is possible to obtain a large number of interesting properties of $J_n(nz)$. A typical, but rather abstruse property is that, if $0 \leqslant x \leqslant 1$, $J_n(nx)/J_n(n)$ is a decreasing function of n.

The paper concludes with some simple approximate formulæ for Kapteyn series occurring in the theory of electromagnetic radiation.

Note on Functions of Upper and Lower Type

Prof. W. H. YOUNG.

In this note a new and simple proof is given of the fundamental theorem that the limit of a uniformly convergent series of functions belonging to one of the two types (upper or lower) in one of the classes into which the author distributes all mathematically definable functions, is a function of the same type and class. From this theorem the necessary and sufficient condition that a function should belong to a certain type and class, previously given by the author, is deduced.

On the Derivates of a Function

Dr. GRACE CHISHOLM YOUNG.

The primitive functions considered in this communication are for mathematical purposes practically unrestricted in character. The results are therefore much more general than any which have hitherto been obtained. Moreover, the derivates considered may be ordinary, or taken, as W. H. Young does, with respect to a monotone function $g(x)$, which is not constant in any stretch. Yet the results are the same as those obtained previously for functions of special character (functions of bounded variation, continuous functions). Three main theorems are proved, the results of which are to show that, *if we neglect a set of points whose content is certainly zero, the derivates of $f(x)$ at any point x belong to one of the three following cases :—*

(1) *there is an ordinary finite differential coefficient $f'(x)$;*

(2) *the upper derivates on each side are $+\infty$, and the lower derivates on each side are $-\infty$;*

(3) *the upper derivate on one side is* $+\infty$, *the lower derivate on the other side is* $-\infty$, *and the remaining extreme derivates are finite and equal.*

Each of these cases may occur at the points of a set of positive content, as is shown by examples, and all three cases may occur simultaneously for a single function $f(x)$, or one or more of the cases may be altogether absent. In particular we have the theorem that *a finite measurable function whose derivates are always finite, has an ordinary finite differential coefficient almost everywhere*, a result which remains true when the function and its derivates are only known to be finite almost everywhere.

We also have as a corollary to the main theorems the result that *there is no geometrical distinction of right and left with regard to the four extreme derivates,* $f^+(x), f_+(x), f^-(x), f_-(x)$, *except at a set of content zero.*

The methods used are simple, and do not involve integration, nor the use of Cantor's numbers.

Thursday, May 11th, 1916.

Prof. Sir J. LARMOR, President, in the Chair.

Present nineteen members.

Messrs. A. H. Amor and D. M. Y. Sommerville were elected members of the Society.

Prof. A. S. Eddington was admitted into the Society.

The President announced the deaths of two members, the Rev. M. M. U. Wilkinson (member since 1873), and the Rev. James White (member since 1869); he gave a brief account of their work and of their services to mathematics.

The President also announced that donations to the Library had been received from the Misses Hudson and from Sir James Stirling.

Prof. H. M. Macdonald read a paper, "A Note on Electrostatics Problems."

Mr. G. B. Jeffery read a paper, "The Relations between Spherical, Cylindrical, and Spheroidal Harmonics."

Mr. G. T. Bennett (Hon. Sec.) gave a brief account of a paper by Mr. E. K. Wakeford,* "The Double Six."

In the absence of the authors the following papers were communicated by title from the Chair :—

Theorems on Conformal Transformation: Mr. J. G. Leathem.

A Problem in the Analytic Theory of Numbers: Messrs. G. H. Hardy and S. Ramanujan.

ABSTRACT.

Theorems on Conformal Transformation

Mr. J. G. LEATHEM.

A function $\mathfrak{C}(\zeta)$ of a complex variable $\zeta = \xi + i\eta$ is such that the relation $dz = \mathfrak{C}(\zeta)\,d\zeta$ specifies a conformal representation of the half-plane $\eta > 0$ on some definite region in the z plane, and, when $\eta = 0$, $\mathfrak{C}(\xi) \equiv q \exp(i\chi)$; for $|\zeta|$ great, $\mathfrak{C}(\zeta) \to K\zeta^{\nu}$, and on the real axis $\mathfrak{C}(\xi)$ has zeros or infinities typified by the limit form $A(\xi - a)^{\mu}$. Such a function is a general curve-factor, and for it the following theorems are proved :—

$$\mathfrak{C}(\zeta_0) = K\Pi(\zeta_0 - a)^{\mu} \operatorname{Lim} \Pi(\zeta_0 - \xi)^{-\delta\chi/\pi}, \tag{1}$$

$$\mathfrak{C}(\zeta_0) = Ke^{\frac{1}{2}\nu i\pi} \operatorname*{Lim}_{R\to\infty} \left\{ R^{\nu} \exp i \int_{\xi=-R}^{\xi=R} \frac{1}{\pi} \log(\zeta_0 - \xi)\, d\log q \right\}, \tag{2}$$

the product corresponding to values of ξ ranging from $-\infty$ to ∞, and the integral having its Cauchy principal value.

Formula (1) exhibits the most general curve-factor as the limit of a product of Schwarzian factors, and from it are deduced certain properties of curve-factors. It can also be used to generate types of curve-factor of assigned linear and angular ranges.

Formula (2) has an interpretation of some interest in hydrodynamics.

Corresponding theorems are proved for periodic curve-factors and corner-factors.

* Printed in this volume.

Thursday, June 8th, 1916.

Prof. Sir J. LARMOR, President, in the Chair.

Present twenty-three members and one visitor.

Miss R. E. Colomb was admitted into the Society.

The President announced that a Special General Meeting of the Society would be held in July to consider the transfer of the meeting-place of the Society to the rooms of the Royal Astronomical Society, and the consequent alterations of the By-laws.

The President also announced that a letter had been received from Prof. Mittag-Leffler declaring the intention of himself and of Madame Mittag-Leffler to endow a Mathematical Institute and Library in their house near Stockholm.

The President further announced the deaths of two members, Prof. Barrell (member since 1885), and Mr. John Griffiths (member since 1871), and spoke briefly of their services to mathematics.

Prof. M. J. M. Hill read a paper, "On the Classification of the Integrals of a Linear Partial Differential Equation of the First Order."

Mr. T. C. Lewis made a communication, "On Three Groups of Four Circles related to a Triangle."

Prof. A. E. H. Love made an informal communication "On Series of Legendre Functions derived from the Potential of a Uniform Circular Wire."

In the absence of the authors, the following papers were communicated by title from the Chair :—

(1) On Non-Absolutely Convergent, not necessarily Continuous, Integrals; (2) On the Convergence of Fourier Series and of their Derived Series: Prof. W. H. Young.

On the General Linear Differential Equation: Dr. S. Brodetsky.

A Note on the Series $\Sigma a_n \sin n\theta$ and $\Sigma a_n \cos n\theta$, where (a_n) is a Sequence of Positive Numbers, tending steadily to Zero: Mr. A. E. Jolliffe.

A Symmetrical Relation between Legendre's Functions with Parameters $\cosh \alpha$ and $\coth \alpha$: Mr. F. J. W. Whipple.

A Problem in Diophantine Approximation: Mr. H. T. J. Norton.

ABSTRACTS.

On the Classification of the Integrals of Linear Partial Differential Equations of the First Order

Prof. M. J. M. HILL.

With the usual notation, taking the equation as

$$Pp+Qq=R,$$

it is known that besides the ordinary integrals u, v which satisfy *identically* the equations

$$P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z}=0,$$

$$P\frac{\partial v}{\partial x}+Q\frac{\partial v}{\partial y}+R\frac{\partial v}{\partial z}=0,$$

there exist in some cases other integrals w, which are such that $P\frac{\partial w}{\partial x}+Q\frac{\partial w}{\partial y}+R\frac{\partial w}{\partial z}$ does not vanish identically, but only when x, y, z are connected by the relation $w=0$.

It is shown that the most general form of an equation which has $w=0$ as an integral of the kind alluded to is

$$P\left(\frac{\partial w}{\partial x}+p\frac{\partial w}{\partial z}\right)+Q\left(\frac{\partial w}{\partial y}+q\frac{\partial w}{\partial z}\right)=W,$$

where W is any function of x, y, z which vanishes with w, and it is supposed that P, Q, W have no common factor.

In the next place it is shown that the integral $w=0$, if it exist, can be obtained by applying a suitable method of integration to the subsidiary Lagrangian system of simultaneous equations,

$$dx/P=dy/Q=dz/R.$$

The definition I have given covers all the known forms of integrals which are not included in the Lagrangian ordinary integrals. I think that I shall be following Prof. Forsyth in calling them special integrals.

It may happen that $w=0$ makes one or both of the ordinary integrals infinite. In this case the ordinary integral is supposed to be replaced by its reciprocal or other function of itself which is not infinite when $w=0$.

If then J be the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$, the special integrals fall into two classes.

(1) If J do not vanish with w, then w is called a special integral of the first class.

Some of these integrals of the first class are given by equations which if rationalised would satisfy the conditions for a singular solution. They are associated with the focal surface of the congruence of curves which represent the integrals of the subsidiary Lagrangian system. If the signs of the radicals be fixed, then each curve in the congruence is bounded by its focal points, the part on one side of a focal point belonging to the differential equation corresponding to one sign of the radical, the part on the other side of the focal point to the other sign of the radical.

There are also special integrals of the first class given by equations in which the coefficients are transcendental.

(2) If J vanish with w, it is proved that an ordinary integral exists which vanishes with w.

The ordinary integral, vanishing with w, is formed by equating to an arbitrary constant a product of factors or a quotient. Each factor of such a product, or the numerator or denominator of such a quotient, if it contain the dependent variable, and if when equated to zero it gives finite values for the dependent variable in terms of the independent variables, satisfies the above definition for a special integral. I regard such special integrals as isolated portions of ordinary integrals, in which they are included, but since in these cases w is not itself a function of u and v, they are called special integrals of the second class.

I go on to refer to two cases in which Cauchy's theorem is not applicable to the solution of the subsidiary Lagrangian system of equations.

In the first case the coefficients are many valued. These generally give the special integrals of the first class mentioned above. In connection with this I show that the question "If the points on an integral are critical points of the coefficients of the equation, can the integral ever be a particular form of an ordinary integral of the equation?" must be answered in the affirmative.

In the second case the subsidiary Lagrangian system of equations belongs to the class in which the only values which Cauchy's theorem furnishes for some of the dependent variables within a certain domain of the independent variable are constants.

Of these I give some examples which fall under the second class that I have referred to above.

On Three Groups of Four Circles related to a Triangle

Mr. T. C. LEWIS.

A note on four Tucker circles, read on April 27th, 1916, gave an easy geometrical determination of the intersections of the sides of a triangle with four of the seventeen tangent circles of any three among the inscribed and escribed circles, the nine-point being a fifth such circle.

It is now found that precisely the same intercepts, measured from other points and along other sides, suffice to determine intersections of the remaining twelve circles, thus making the tale complete.

Of the seventeen circles, nine have with the triangle a common chord parallel to each side. Five of the nine, viz., the nine-point and the four Tucker circles, are common to each of these three groups of nine. The remaining four, being different for the three groups, yield the remaining twelve tangent circles, all told. They are the subject of the present note.

Let F be the foot of the perpendicular from C on AB.

Take any one of the four Tucker circles, having with the triangle a common chord $B'C'$ parallel to BC. CC' is the intercept on AC, measured from C towards A.

From F measure along AB a length equal to CC', viz.,

(1) in the direction FA in the two cases in which the Tucker circle touches both the escribed circles opposite to B and C,

(2) in the direction FB in the two other cases, when the Tucker circle touches both the inscribed circle and the one escribed opposite to A.

The point so determined is a point on a circle which touches the same triad as the Tucker circle, and therefore similarly conforms to the conditions satisfied by it under (1) or (2). An intersection on AC is similarly determined. But such a circle has with the triangle a common chord $b'c'$ parallel to BC. And $Ab' : b'B$ is

r_1/r when the Tucker circle touches all escribed circles,

r/r_1 ,, ,, ,, inscribed and escribed B, C,

$-r_2/r_3$,, ,, ,, ,, ,, ,, A, B,

$-r_3/r_2$,, ,, ,, ,, ,, ,, A, C.

Moreover it follows at once that the points b' for the first two circles, as also for the other two, are equidistant from the mid-point of AB.

Thus there is an easy geometrical determination of four points on each of these (three times) four circles.

On Non-Absolutely Convergent, not necessarily Continuous, Integrals

Prof. W. H. YOUNG.

The present communication deals with a generalisation of Harnack's method of forming non-absolutely convergent integrals. In so doing, certain properties of ordinary integrals are preserved, and certain others have to be sacrificed. The property of the indefinite integral function of being a continuous function having already been tacitly abandoned in the theory of absolutely convergent integrals with respect to discontinuous functions of bounded variation, is not one whose retention seems vital. In abandoning it, however, the whole theory of non-absolutely convergent integration becomes at once much simpler and far more varied. Moreover, the subclass of continuous integrals is found to include all integrals hitherto defined, and to be more extensive than any till now suggested.

The principles which govern the new definition are the following.—

(I) *Fundamental is the distinction between intervals in which $f(x)$ is summable and points of non-summability*, that is points such that $f(x)$ is not summable in any interval containing one of them. When the summability is with respect to the continuum, these points are called *Harnack points*, and the set of them the Harnack set H. This set, being closed, is the sum of a countable aggregate of countable sets and a perfect nucleus H_N. If $f(x)$ is not summable with respect to H_N, the process can be repeated, this perfect set taking the place of the continuum, and so on *ad infinitum*. The process leads after a countable series of steps to a final set *the nucleolus of non-summability*, contained in the Harnack set, and in all the closed sets obtained by our three processes of derivation, deduction, and non-summability. This nucleolus may be a finite set, but, if not, it is a perfect set with respect to which $f(x)$ is either (i) summable, or (ii) totally non-summable [that is, there is no interval containing a point of the set, such that $f(x)$ is summable over it with respect to the set]. In this last case no definition is given of the integral of $f(x)$. Otherwise the integral is determined by the following conventions.

(*a*) *The integral over any closed interval containing no Harnack point is the Lebesgue integral.*

(*b*) *If* (a, b) *is an interval in which the nucleolus of non-summability consists of the points* a *and* b *alone, and* $\int_{a'}^{b'} f(x)dx$ *has been defined for every pair of its internal points,* $(a < a' < b' < b)$, *then, if as* a' *approaches* a *and* b' *approaches* b *along suitable sequences,* $\int_{a'}^{b'} f(x)\,dx$ *has a unique limit, this limit is defined to be* $\int_a^b f(x)dx$, *and we write*

$$\int_a^b f(x)\,dx = \underset{(a'\to a,\ b'\to b)}{{}^*\mathrm{Lt}} \int_{a'}^{b'} f(x)dx,$$

the suitable sequences, indicated by the starred limit, being those determined by the analysis previously made of the Harnack set. If, for instance, a *is a point of the first derived set of* H, *the approach is along* H. *If* a *is a point of the first deduced set* H_ω, *the approach is along a sequence containing a point from each of the derived sets of* H. *Only if* a *is an isolated Harnack point is* a' *unrestricted in its approach to* a.

(*b*) *The integral of* $f(x)$ *from* a *to* b *is the sum of its integrals over the sub-intervals* (a, c) *and* (c, b), *if these have already been defined.*

(*c*) *If* E *is any perfect set, and* (a_1, b_1), (a_2, b_2), ... *its black intervals, and* $f(x)$ *is such that it is summable over* E, *and its integral over each black interval has already been defined, and if these integrals form an unconditionally convergent series, then*

$$\int_a^b f(x)dx = \int_E f(x)\,dx + \sum_{n=1} \int_{a_n}^{b_n} f(x)\,dx.$$

We now introduce a new principle, founded on no single foreknown concept like that of summability, but on the notion of integral so far defined, or subsequently to be defined, stage by stage. This is the following.

(II) *If* E *is a perfect set, having the properties enumerated in* (*c*), *except that the series does not converge unconditionally, we can analyse* E *in a countable series of steps into distinct sub-sets having the missing property, plus a final set.* If this also has the missing property, the Y-integral can be defined by the conventions already made.

The general definite Y-integral having now been defined, and in con-

sequence the indefinite integral, or Y-integral function, $F(x)$ determined, it is shown that $F(x)$ has the following properties :—

(1) *$F(x)$ is the limit of a continuous function, whose value at any point is one of the limits of its values in the neighbourhood on either side : consequently it assumes all values between its upper and lower bounds.*

(2) *If E is the final set in our analysis of the Harnack set, and is perfect, $F(x)$ is continuous with respect to E.*

(3) *If $f(x)$ is any function having a Y-integral $F(x)$, and $g(x)$ is a function which is either summable, or has a Y-integral $G(x)$ with only a countable set of Harnack points, then $f(x)+g(x)$ possesses a Y-integral given by the sum of $F(x)$ and $G(x)$.*

(4) *If $\int_a^x f(x)dx = F(x)$ exists as a Y-integral, so does $\int_a^x f(x)\,g(x)dx$, where $g(x)$ is any monotone increasing or decreasing function, and is given by the formula for integration by parts, namely,*

$$\int_a^x f(x)\,g(x)dx = \Big[F(x)\,g(x)\Big]_a^x - \int_a^x F(x)\,dg(x).$$

(5) *The Second Theorem of the Mean holds in its two familiar forms, namely,*

$$\int_a^b f(x)\,g(x)dx = g(a)\int_a^x f(x)dx + g(b)\int_x^b f(x)dx,$$

where $g(x)$ is a monotone increasing (decreasing) function, and x some point of (a, b), and

$$\int_a^b f(x)\,g(x)dx = g(a)\int_a^x f(x)\,dx,$$

when $g(x)$ is monotone decreasing.

The class of Y-integrals then falls into four classes, each embracing the next after it :—

(α) *Those which have infinite discontinuities.*

(β) *Those which are bounded without being necessarily continuous.*

(γ) *Those which are continuous.*

(δ) *Those which possess the Harnack-Moore-Denjoy property, and are accordingly also continuous.*

The criteria are given for an integral of class (γ), and examples are given, and it is shown that, if we demand that $f(x)+g(x)$ should *always* have $F(x)+G(x)$ for integral, we are bound to restrict the name of integral to those of class (δ). On the other hand, if we are only concerned with two functions $f(x)$ and $g(x)$, this result holds generally, under conditions which are here determined.

Application is then made of bounded Y-integrals to the theory of trigonometrical series. The term Fourier series is used in the usual sense, except that Y-integrals take the place of ordinary integrals, and the following theorems are obtained :—

If $f(x)$ possesses a bounded Y-integral $F(x)$, then the integrated series of the Fourier series of $f(x)$ is the Fourier series of $F(x)$, and accordingly converges to $F(x)$, when summed in the Cesàro manner, at all points not belonging to at most a certain set of content zero.

If $f(x)$ possesses a bounded Y-integral, with at most a countable set of discontinuities, and $g(x)$ is a function of bounded variation, then $\int_a^b f(x)\,g(x)\,dx$ is equal to the Cesàro sum of the series obtained by integrating term-by-term the product of $g(x)$ and the Fourier series of $f(x)$, provided only that, if $b-a > 2\pi$, $f(x)$ has the period 2π.

We may also here replace the Fourier series of $f(x)$ by that of $g(x)$, and multiply by $f(x)$, if, when $b-a > 2\pi$, $g(x)$ has the period 2π instead of $f(x)$.

On the General Linear Differential Equation

Dr. S. BRODETSKY.

The solution of the equation

$$y_n+p_1(x)\,y_{n-1}+\ldots+p_{n-1}(x)\,y_1(x)+p_n(x)=0$$

is shown to be a linear function of quantities

$$\{f_0(\xi, x)\}_{\xi=0}, \quad \{f_1(\xi, x)\}_{\xi=0}, \quad \ldots, \quad \{f_{n-1}(\xi, x)\}_{\xi=0},$$

which are all obtainable from a single function by the use of the differential operator

$$\Delta \equiv \frac{\partial}{\partial x}-\frac{\partial}{\partial \xi}.$$

The solution can also be expressed in terms of the exponential functions

$$\{e^{L_1 x}\}_{\xi=0}, \quad \{e^{L_2 x}\}_{\xi=0}, \quad \ldots, \quad \{e^{L_n x}\}_{\xi=0},$$

where $L_1, L_2, \ldots, L_n$ are n operators of the form

$$\frac{d}{d\xi}+\lambda(\xi),$$

the operators satisfying the symbolical equation

$$L^n+p_1(\xi)L^{n-1}+\ldots+p_{n-1}(\xi)L+p_n(\xi)=0.$$

Where L is not followed by a function of ξ, it operates on unity.

If the differential equation is

$$y_n+p_1(x)y_{n-1}+\ldots+p_{n-1}(x)\,y_1+p_n(x)\,y=p(x),$$

the particular integral is a function

$$\{f(\xi, x)\}_{\xi=0},$$

which has the property that all the functions $f_0, f_1, \ldots, f_{n-1}$ can be derived from it by application of the operator Δ.

Addition and subtraction theorems for the functions $f, f_0, \ldots, f_{n-1}$ are also obtained.

A Note on the Series $\Sigma a_n \sin n\theta$ and $\Sigma a_n \cos n\theta$, where (a_n) is a Sequence of Positive Numbers tending steadily to Zero

Mr. A. E. JOLLIFFE.

It has been shown by Mr. T. W. Chaundy and myself that the necessary and sufficient condition for the series $\Sigma a_n \sin n\theta$ to be uniformly convergent in any interval, θ being real, is $na_n \to 0$. Uniform convergence is sufficient for continuity, though not necessary. I have been able to show that the condition $na_n \to 0$ is necessary for continuity. It is evidently sufficient.

The series $\Sigma a_n \cos n\theta$ will not be convergent at $\theta = 0$, unless Σa_n is convergent, in which case the series will be uniformly convergent everywhere by the Weierstrass test. I have been able to show that when Σa_n is divergent, but na_n is always less than some fixed number, then, if $\Sigma a_n \cos n\theta$ converges to $f(\theta)$ for values of θ other than zero or a multiple of 2π, $f(\theta)\Big/\int^{\pi/\theta} \phi(\theta)\,d\theta$, where a_n is $\phi(n)$, and the lower limit is any fixed number, tends to unity as θ tends to zero in any manner.

SPECIAL GENERAL MEETING.

Thursday, July 13th, 1916.

Prof. Sir J. LARMOR, President, in the Chair.

Present nineteen members.

The President moved and the Treasurer (Dr. A. E. Western) seconded the four resolutions of which notice had been given, namely:—

1. That the Society approves the arrangement negotiated by the Council with the Royal Astronomical Society whereby the meetings of the Society shall be held in the rooms of the Royal Astronomical Society, and the Registered Office of the Society shall be situated there.

2. That the Society approves the arrangement negotiated by the Council with the Authorities of the Science Museum at South Kensington for the deposit of the Society's Library on loan in the Science Museum Library, under regulations which shall preserve the rights of Members of the Society.

3. That in By-law IV 1 "Of the Annual General Meeting" the words "on the second Thursday in November at half-past 5 o'clock in the evening" be omitted, and the following words substituted: "on such day and time in November as the Council may from time to time determine."

4. That in By-law IX 3, relating to the publication of papers, the word "one" shall be substituted for the word "two," and the words "referee or referees" substituted for the word "referees." And there shall be added at the end of this By-law the following words:—"But the Council may give authority to the Secretaries to forward any paper for printing and publication during the summer vacation, at their discretion, if the reports of the referee or referees are favourable."

The resolutions were carried unanimously.

By the courtesy of the Royal Astronomical Society, the Special General Meeting was held in their rooms, Burlington House, London, W.

LIBRARY

Presents.

BETWEEN August, 1915, and December, 1916, the following presents were made to the Library :—

By Miss H. P. Hudson :—

Cambridge Mathematical Tracts, nos. 1–17.
Cambridge Philosophical Society, Proceedings, vols. 3–18.
Cambridge Philosophical Society, Transactions, vols. 13–18.
Messenger of Mathematics, vols. 1–42.
Mathematics from the "Educational Times," old series, and new series, vols. 1–27.
Nature (complete).
Quarterly Journal of Mathematics, vols. 1–45.

By Sir James Stirling :—

Collected Works of Cauchy, Dirichlet, and Gauss.

By their respective authors or publishers :—

Cashmore.—"Fermat's Last Theorem."
Darboux, G.—"Leçons sur la théorie générale des surfaces," vol. 2.

Calcutta : Indian Engineering, vol. 58, nos. 1–9.
Cape Town : South African Journal of Science, vol. 11, no. 10 ; vol. 12, nos. 1–11.
London : Educational Times, nos. 657–660.
London : Educational Times, Math. Questions and Solutions, new series, vols. 27–29.
London : Mathematical Gazette, vol. 8, nos. 118–120.
London : Nautical Almanac for 1918.
Madras : Indian Mathematical Society, Journal, vols. 7, 8, nos. 1–5.
Paris : L'enseignement math., ann. 17, nos. 5, 6 ; ann. 18, nos. 1–4.
Sendai : Tôhoku University, Science Reports, vol. 5, no. 4.
Sendai : Tôhoku Mathematical Journal, vol. 7, nos. 3, 4 ; vol. 8, nos. 1–4.
Tokyo : Sūgaku-Butsurigaku, vol. 8, nos. 1–19.

Exchanges.

Between August, 1915, and July, 1916, the following exchanges were received :—

American Journal of Mathematics, vol. 37, nos. 3, 4 ; vol. 38, no. 1.
American Mathematical Society, Transactions, vols. 16, 17.

American Mathematical Society, Bulletin, vol. 21, no. 10; vol. 22, nos. 1–10; vol. 23, nos. 1–3.
American Philosophical Society, Proceedings, vol. 53, nos. 216–219; vol. 55, nos. 1, 2, 6, 7.
Amsterdam: Nieuw Archiev, deel 11, stuk 2–4.
Amsterdam: Wiskundige Opgaven, deel 12, stuk 1, 2.
Cambridge Philosophical Society, Proceedings, vol. 18, pts. 5, 6.
Cambridge Philosophical Society, Transactions, vol. 22, pts. 8, 9.
Cambridge, Mass.: Annals of Mathematics, vol. 17.
Dublin: Royal Irish Academy, Proceedings, vol. 32, nos. 2–4; vol. 33, nos. 1–3.
Edinburgh: Mathematical Society, Proceedings, vols. 32, 33.
Edinburgh: Royal Society, Proceedings, vols. 33 (no. 4), 34, 35.
Firenze: Biblioteca Nazionale Centrale di Firenze, Bollettino, nos. 87, 88, 90.
France: Société Mathématique, Bulletin, vol. 43, nos. 1–4.
Institute of Actuaries, Journal, vols. 48 (nos. 2–4), 49, 50 (no. 1).
Institute of Naval Architects, Transactions, 1914, 1915.
København: Kgl. Danske Vidensk. Selsk., Forhandlinger, Oversigt, 1916, nos. 1, 2.
Livorno: Periodico di Matematica, anno 30, nos. 5, 6; anno 31, nos. 1–6.
Livorno: Periodico di Matematica, Supplemento, anno 18, nos. 8, 9; anno 19, nos. 1–9; anno 20, no. 1.
London: Nature, vols. 97, 98.
London: Physical Society, Proceedings, vol. 27, nos. 3, 5; vol. 28, nos. 1–5.
London: Royal Society, Proceedings, Series A, nos. 630–646.
London: Royal Society, Transactions, Series A, vols. 215, 216.
Manchester: Literary and Philosophical Society, Memoirs and Proceedings, vols. 58, 59, 60 (nos. 1, 2).
Palermo: Rendiconti del Circolo Matematico, vols. 38–40; vol. 41, fasc. 1.
Paris: Bulletin des sciences mathématiques, 1915; Août.–Déc. 1916.
Paris: Journal de l'Ecole Polyt., 1914.
Porto: Academi Polytechnica de Porto, vol. 10, nos. 2, 3.
Roma: Reale Accademia dei Lincei, Rendiconti, vols. 24, 25.
Torino: R. Accademia delle Scienze, Atti, vol. 50, nos. 1–15; vol. 51, nos. 1–9.
Toulouse: Faculté des Sciences, Annales, vol. 5.
Washington: Proceedings Nat. Academy of Sciences, vols. 1, 2.
Washington: Naval Observations, 1915.
Zurich: Vierteljahrsschrift, 1916, Hefte 1, 2.

OBITUARY NOTICES

BENJAMIN WILLIAMSON.

Dr. Williamson was born in Cork in 1827; he lived to the age of eighty-nine years, and had become the oldest member of his University. Going up to Trinity College, Dublin, from Kilkenny College, one of a noted group of Irish schools which had their opportunity in those early days before travelling facilities to England had been provided, he won a Fellowship in 1852. At that time official college promotion was blocked, owing to the abolition of the restrictions to celibacy and to Orders in the Church, which had previously operated by establishing a steady flow of successions to the valuable church preferments that were connected with the University of Dublin. In the years of waiting he built up a reputation as a teacher, such as can be readily appreciated from his published treatises. The writer can just remember the welcome reception in Ireland of his *Differential Calculus* in 1872, and his *Integral Calculus* of two years later. Their aim may be said to have been to impart facility in the practical use of the subject as a "calculus" for the advancement of the quantitative sciences, rather than to develop and fix a logical and philosophical groundwork for the fundamental ideas on which the calculus must repose. This limitation of aim made the books more useful and attractive to the average student, who desired to acquire mathematical facility for the sake of applying it, than were the more technical but still logically incomplete English books of that period. The range of illustrations and applications, mainly in infinitesimal geometry, revealed the elegance of the author's mind, and also the trend of the Irish school to which he belonged. Their success was indicated by the continued appearance of new editions; and they were followed by books, not so widely known, on *Dynamics* (jointly with Dr. Tarleton), and on *Stress and Strain*, the result of their author's period of tenure of the Chair of Natural Philosophy.

He became a member of the London Mathematical Society in 1875, and was elected a Fellow of the Royal Society in 1879. He was not much

given to publishing separate investigations, but important work of this kind appeared from time to time as appendices in his books.

His charming personal and social qualities found scope in the bright society of Trinity College and its intimate association with the intellectual life of the Irish metropolis: for a long period his college rooms formed a focus of Irish academic society. For reasons above mentioned it was not until 1897 that he came into a position, by succession to a Senior Fellowship, to affect the policy of the University, where his influence was exerted with the party of advance who have contributed so much to the development of the newer and more expensive studies. In time he became Vice-Provost, and it is only two years since increasing bodily weakness compelled him to withdraw from active duties.

J. L.

WILLIAM ESSON.

WILLIAM ESSON was born at Dundee in 1838, was educated at Inverness Royal Academy and at Cheltenham Grammar School. In 1855 he came to Oxford as a Bible clerk of St. John's College. He obtained first classes in Mathematical Moderations (1856), Mathematical Final Schools (1859) —besides taking two second classes in Classics—and won the Junior Mathematical Scholarship in 1857, and the Senior in 1860. In 1860, he was elected to a Fellowship at Merton, and became mathematical tutor. In 1884 he was appointed Estates Bursar of Merton College, and continued to hold that office till his death. His services both to the College and to the University, as administrator and man of business, were very great. He served the University for many years as Curator of the Chest, and was a visitor of the University Observatory.

He became a member of the London Mathematical Society in 1866, and as early as 1869 he was elected a Fellow of the Royal Society, after the publication in the *Transactions* (1864, 1866) of his pioneer work in mathematical chemistry, which was done in collaboration with Mr. A. G. Vernon Harcourt. These papers on "The Laws of Connexion between the Conditions of Chemical Change and its Amount" are of first-rate importance, and are well known to all students of chemistry as "Harcourt and Esson."

In the last three years of Prof. Sylvester's life, when failing health made him give up lecturing, Esson was appointed to act as his deputy, and on Sylvester's death in 1897, Esson was appointed to the Chair of Savilian Professor of Geometry. His lectures were chiefly on synthetic geometry, particularly on the loci of the intersections of projective pencils of curves, and on a systematic account of the characteristic numbers connected with curves of given order and class. Two papers of his, "Notes on Synthetic Geometry," in our *Proceedings* (Vol. XXVIII), and "On the Characters of Plane Curves" in the *Proceedings* of the Fifth International Mathematical Congress at Cambridge, give a good idea of the substance of his lectures. The second contains the typical paradox that a straight line considered as a curve of degree one and class zero has four double tangents and minus three inflexions.

Oxford owes a real debt of gratitude to Esson for his successful work

as a leader in college mathematical teaching, especially in his earlier life when he was tutor or lecturer to some half a dozen colleges. As one of his pupils has said, "In the 'sixties and 'seventies there were two classes of mathematical students in Oxford—those who blessed the Providence which had put them under Esson, and those who envied the others."

Until the spring of 1916, advancing years seemed to have but little effect on his vigour and health, but his only surviving son, Major William Esson, went down in H.M.S. "Russell," and the shock may be said to have killed him. He died on Friday, August 25th, 1916.

A. L. D.

PROFESSOR W. H. H. HUDSON.

WILLIAM HENRY HOAR HUDSON was born in London in 1838, went to St. John's College, Cambridge, from King's College, London, in 1857, and graduated as third Wrangler in January 1861, becoming a Fellow of St. John's College in May, 1862, a preferment which he vacated by marriage in 1875. He was Mathematical Lecturer of St. John's College from 1869 to 1881, when he became Professor of Mathematics at King's College, London, serving in that office until 1903. His only son, Ronald William Henry Turnbull Hudson, was Senior Wrangler in 1898, and also became a Fellow of St. John's College: his early scientific promise, illustrated by a brilliant treatise on *Kummer's Quartic Surface*, came to a tragic end through a mountaineering accident on Glydr Fawr, North Wales, in September, 1904. His daughters are also known as mathematicians of high attainment and original achievement.

Professor Hudson became a member of the London Mathematical Society in 1868; he was a member of the Council from 1894 to 1898, and took an active part in the business of the Society.

He was a refined and elegant geometer, as was shown in his lecturing work at Cambridge, so far as the rather narrow grooves of that period allowed, and especially in his papers set as Examiner in the Mathematical Tripos.

Later his official position in London gave him the opportunity of throwing himself into the important public problems of the improvement of mathematical education in schools and colleges, to which he made many published contributions. This interest brought him much into touch with educational and scientific organisations: in particular he was usually present at the annual meetings of the British Association, and his companionship will be well remembered on the expeditions to the important meetings held in the British Dominions over seas.

J. L.

SIR ANDREW NOBLE, BART., K.C.B., F.R.S.

Although Sir Andrew Noble made no pretension to being an expert mathematician, it is very fitting that his interest in the science should be recorded in our *Proceedings*. Trained as an Officer of the Royal Artillery, a service in which precise calculation is vital to progress, he early recognised the importance of keeping ballistic theory in touch with the improvement of guns and propellants. His opportunity came when, as a Captain, he served as Secretary of the Committees on Gunnery in 1858 and subsequent years. His insight and business qualities in that position attracted the attention of the late Lord Armstrong, and led to an invitation, in 1860, to join the directorate of the engineering and armaments firm, which was then in the course of rapid development, and was becoming of first-rate national importance; of this great corporation, extended by the absorption of the sister pioneer firm of Whitworths, he has more recently been the head for many years. Throughout his career he was pre-eminent in the development of gun construction and the improvement of projectiles, both by his own work and by his faculty of co-ordinating the abilities of other workers. His unique position, with this combination of theory and practice, has been marked by the award of a Royal Medal by the Royal Society in 1880, and the Albert Medal of the Royal Society of Arts in 1909; and one of the occupations of his last years was the collection in book form of the numerous memoirs on this and cognate subjects which he had published in the *Philosophical Transactions*, and elsewhere. He had been a member of the Mathematical Society since 1865. He became a Fellow of the Royal Society in 1870, and was a Vice-President for the Sessions of 1899 and 1909. He was an honorary Doctor of Science of Oxford and of Cambridge, and held many decorations from foreign countries for services in his profession.

Nowhere was he happier than in scientific coteries and in the society of scientific men; and they fully reciprocated his regard. Many of them mathematicians, physicists, geologists, have enjoyed spells of holiday in his country homes in Northumbria, and in Scotland—Prof. Sylvester not least, over whom he assumed almost paternal care. After enduring with resignation some years of partial inactivity necessitated by failing health, he has now been lost to us in his eighty-third year. J. L.

ERRATA.

1. In Prof. W. H. Young's paper "On Integrals and Derivates with respect to a Function," the footnotes on pp. 61, 62 should be interchanged.

2. In Sir Joseph Larmor's paper "On the Transition from Vapour to Liquid when the Range of the Molecular Attraction is sensible," p. 191, line 9, *should read:*

"For example, the critical temperature of water is given as 365 C.; thus the equation would indicate that water could not sustain negative pressure at temperatures above 538 absolute or 265 C."

PAPERS

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ON DIRICHLET'S DIVISOR PROBLEM

By G. H. Hardy.

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I.

Introduction.

1.1. It was proved by Dirichlet* that, if $d(n)$ denotes the number of divisors of n, unity and n itself included, then

$$D(x) = \sum_1^x d(n) = x \log x + (2\gamma - 1)\, x + O(\sqrt{x}), \tag{1.11}$$

where γ is Euler's constant. We may therefore write

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x), \tag{1.12}$$

where

$$\Delta(x) = O(\sqrt{x}). \tag{1.13}$$

The problem which I call "Dirichlet's Divisor Problem" is that of determining as precisely as possible the maximum order of the error term

* "Über die Bestimmung der mittleren Werthe in der Zahlentheorie", *Berliner Sitzungsberichte*, 1849, pp. 69–83; *Werke*, Vol. 2, pp. 49–66.

$\Delta(x)$. That the relation (1.13) could be replaced by some more precise relation was asserted by Dirichlet himself* ; but there is nothing to show what were the results at which he had arrived.

It was proved by Voronoï† in 1903 that

$$\Delta(x) = O(x^{\frac{1}{3}} \log x), \tag{1.14}$$

and this result is still the best that is known. It is remarkable that Voronoï's proof, though difficult and long, is "elementary": it does not, that is to say, involve any appeal to the theory of functions of a complex variable, as do all the principal results in the theory of the distribution of primes. In fact no one has as yet succeeded in deducing from the theory of the Zeta-function a result even as good as (1.14), though Landau‡ has developed a powerful and important method which not only leads to the slightly less precise result

$$\Delta(x) = O(x^{\frac{1}{3}+\epsilon}),\S \tag{1.15}$$

where ϵ is any positive number, but is capable of application to a whole class of problems of which Dirichlet's problem is one.

1.2. My object in this paper is to attack the problem from the other end, and to prove, not an "O" result, but what Mr. Littlewood and I‖

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* "Seit unserm neulichen Gespräch . . . . ist es mir gelungen, die Summe $\sum_{s=1}^{n} \left[\frac{n}{s}\right]$, wo [ ] nach Gauss dass grösste Ganze bezeichnet und die ich bisher nur mit einem Fehler der Ordnung $\sqrt{n}$ angeben konnte, bedeutend in die Enge zu treiben. Die Auffindung des hiezu dienenden Mittels, welches aller Wahrscheinlichkeit nach auch auf die folgenden Fälle anwendbar seyn wird, macht mir zwar grosses Vergnügen, kommt mir aber in sofern zu ungelegener Zeit als ich . . . ." (Dirichlet to Kronecker, 23 July 1858, *Werke*, Vol. 2, p. 407).

† "Sur un problème du calcul des fonctions asymptotiques", *Journal für Math.*, Vol. 126, pp. 241–282.

‡ "Über die Anzahl der Gitterpunkte in gewissen Bereichen", *Göttinger Nachrichten*, 1912, pp. 687–771.

§ This formula had been given by Pfeiffer ("Über die Periodicität in der Teilbarkeit der Zahlen und über die Verteilung der Klassen positiver quadratischer Formen auf ihre Determinanten", *Jahresbericht der Pfeiffer'schen Lehr- und Erziehungs-Anstalt zu Jena*, 1886, pp. 1–21) long before the appearance of Voronoï's memoir. The proof given by Pfeiffer was not entirely satisfactory, but Landau has shown that the ideas on which Pfeiffer bases his proof are sound and lead to a proof not only of (1.15) but of Voronoï's more precise formula (1.14). See Landau, "Die Bedeutung der Pfeiffer'schen Methode für die analytische Zahlentheorie", *Wiener Sitzungsberichte*, Vol. 121, Part 2*a*, pp. 2195–2332.

‖ "Some Problems of Diophantine Approximation (II)", *Acta Mathematica*, Vol. 37, pp. 193–238 (p. 225).

have called an "Ω" result; viz., that

$$\Delta(x) = \Omega_R(x^{\frac{1}{4}}), \qquad \Delta(x) = \Omega_L(x^{\frac{1}{4}}), \tag{1.16}$$

or, in other words, that a constant K exists such that each of the inequalities

$$\Delta(x) > Kx^{\frac{1}{4}}, \qquad \Delta(x) < -Kx^{\frac{1}{4}} \tag{1.161}$$

holds for an infinity of values of x surpassing all limit.* It follows that the function $x^{\frac{1}{4}}\log x$ of Voronoï's relation (1.14) certainly cannot be replaced by any function of lower order than $x^{\frac{1}{4}}$. So far as I am aware, nothing is known in this direction except the relation

$$\Delta(x) = \Omega\left\{2^{(1-\epsilon)\frac{\log x}{\log\log x}}\right\}, \tag{1.17}$$

which follows at once from the fact that

$$d(n) > 2^{(1-\epsilon)\frac{\log n}{\log\log n}}, \tag{1.171}$$

for an infinity of values of n.†

1.3. The theorem expressed by the relations (1.16) or (1.161) is analogous to a well known theorem of Schmidt. It was proved by Schmidt‡ that

$$\begin{cases} \Pi(x) - Li\,x + \frac{1}{2}Li\sqrt{x} = \Omega_R\left(\dfrac{\sqrt{x}}{\log x}\right), \\ \Pi(x) - Li\,x + \frac{1}{2}Li\sqrt{x} = \Omega_L\left(\dfrac{\sqrt{x}}{\log x}\right), \end{cases} \tag{1.31}$$

where $\Pi(x)$ is the number of primes not exceeding x. A simplified

* I have already published this result, with summary indications of the proof, under the title "Sur le problème des diviseurs de Dirichlet" (*Comptes Rendus*, 10 May 1915).

† Wigert, "Sur l'ordre de grandeur du nombre des diviseurs d'un entier", *Arkiv för Math.*, Vol. 3, No. 3, pp. 1–18; see also Landau, *Handbuch*, p. 219. A number of more precise results concerning the maximum order of $d(n)$ will be found in Mr. Ramanujan's paper "On Highly Composite Numbers", *Proc. London Math. Soc.*, Ser. 2, Vol. 14, pp. 347–409. It should be observed that the function $2^{\log n/\log\log n}$ is of higher order than any power of $\log n$, but of lower order than any power of n.

‡ "Über die Anzahl der Primzahlen unter gegebener Grenze", *Math. Annalen*, Vol. 57, pp. 195–204.

form of Schmidt's proof will be found in Landau's *Handbuch**. My proof of (1.16) does not differ in principle from Landau's proof of (1.31); its difficulty lies not in its principle but in the construction of the new formulæ required for the application of the principle to the divisor problem. It will therefore probably make the argument clearer if I recapitulate shortly the proof, not exactly of the relations (1.31), but of the substantially equivalent† relations

$$\psi(x)-x=\Omega_R(\sqrt{x}), \qquad \psi(x)-x=\Omega_L(\sqrt{x}), \tag{1.311}$$

where $\psi(x)$ is Tschebyschef's function

$$\psi(x)=\sum_{p^m\leqslant x}\log p.$$

1.4. In proving (1.311) we may assume the Riemann hypothesis‡. It is then easy to show that the series

$$\Sigma\frac{\psi(n)-n}{n^{1+s}}$$

is convergent for $\sigma>\frac{1}{2}$, and represents a function

$$F(s)=-\frac{\zeta'(s)}{s\zeta(s)}-\zeta(s)+g(s),$$

where $g(s)$ is regular for $\sigma>0$, so that $F(s)$ is regular for $\sigma>0$ except for simple poles at the zeros of $\zeta(s)$. It follows that, if $s=\frac{1}{2}+i\gamma$ is a zero of $\zeta(s)$, then

$$\Sigma\frac{\psi(n)-n}{n^{\frac{3}{2}+i\gamma+\delta}}\sim-\frac{\kappa}{(\frac{1}{2}+i\gamma)\,\delta},$$

as $\delta\to0$, where κ is the multiplicity of the zero.

Suppose now that (*e.g.*)

$$\psi(n)-n<K\sqrt{n} \tag{1.41}$$

for all sufficiently large values of n. Then

$$\Sigma\frac{\psi(n)-n-K\sqrt{n}}{n^{\frac{3}{2}+i\gamma+\delta}}\sim-\frac{\kappa}{(\frac{1}{2}+i\gamma)\,\delta}, \tag{1.42}$$

and

$$\Sigma\frac{\psi(n)-n-K\sqrt{n}}{n^{\frac{3}{2}+\delta}}\sim-\frac{K}{\delta}, \tag{1.43}$$

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* Pp. 711 *et seq.*; see also Landau, "Über einen Satz von Tschebyschef", *Math. Annalen*, Vol. 61, pp. 527–550. It has been proved recently by Littlewood that more is true than is stated by the relations (1.31), in fact that

$$\pi(x)-Li\,x=\Omega_R\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right), \quad \pi(x)-Li\,x=\Omega_L\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right).$$

See Littlewood, "Sur la distribution des nombres premiers", *Comptes Rendus*, 22 June 1914; Hardy and Littlewood, "Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes", *Acta Mathematica* (as yet unpublished).

† Cf. Hardy and Littlewood, *l.c. supra.*

‡ If this is false then *more* is true. See Landau, *Handbuch*, p. 713.

since $F(s)$ is regular for $s = \frac{1}{2}$, and $\zeta(s)$ is regular for $s = 1+i\gamma$ but has a simple pole for $s = 1$. But $\psi(n)-n-K\sqrt{n}$ is ultimately of constant sign, and $|n^{-i\gamma}| = 1$; so that these results are contradictory if

$$K < \frac{\kappa}{\sqrt{(\frac{1}{4}+\gamma^2)}}.$$

Thus (1.41) is certainly false for sufficiently small values of K and an infinity of values of n.

II.

The Function $F(s) = \Sigma \frac{d(n)}{\sqrt{n}} e^{-s\sqrt{n}}$.

2.1. I write

$$s = \sigma+it, \qquad z = x+iy. \tag{2.11}$$

From the formulæ

$$e^{-s} = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} s^{-z}\,\Gamma(z)\,dz \quad (\sigma > 0,\ \kappa > 0),* \tag{2.12}$$

$$\{\zeta(z)\}^2 = \Sigma \frac{d(n)}{n^z} \quad (x > 1), \tag{2.13}$$

we deduce

$$F(s) = \Sigma \frac{d(n)}{\sqrt{n}} e^{-s\sqrt{n}} = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} s^{-z}\,\Gamma(z)\,\{\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)\}^2\,dz, \tag{2.14}$$

where now $\sigma > 0$, $\kappa > 1$. From (2.14) it follows, by a simple application of Cauchy's theorem,† that

$$F(s) = -\frac{4\log s}{s} + \sum_0^p \frac{(-s)^n}{n!}\{\zeta(\tfrac{1}{2}-\tfrac{1}{2}n)\}^2 + J_p, \tag{2.15}$$

where p is any positive integer, and

$$J_p = \frac{1}{2\pi i}\int_{-p-\frac{1}{2}-i\infty}^{-p-\frac{1}{2}+i\infty} s^{-z}\Gamma(z)\,\{\zeta(\tfrac{1}{2}+\tfrac{1}{2}z)\}^2\,dz. \tag{2.16}$$

Let us suppose for the moment that s is real and $0 \leqslant s < 4\pi$. Then

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* Mellin, "Om definiter integraler", *Acta Soc. Fennicae*, Vol. 20, No. 7, pp. 1–39 (p. 6), and "Abriss einer einheitlichen Theorie der Gamma- und hypergeometrischen Funktionen", *Math. Annalen*, Vol. 68, pp. 305–337.

† We have $|\Gamma(z)| = O(|y|^K e^{-\frac{1}{2}\pi|y|})$, $|\zeta(\frac{1}{2}+\frac{1}{2}z)| = O(|y|^K)$,

where K is independent of y, uniformly for $-p-\frac{1}{2} \leqslant x \leqslant \kappa$. There is a double pole at $z = 1$, and simple poles at $z = 0$, $z = -1$, ..., $z = -p$.

$J_p \to 0$ as $p \to \infty$. For, if

$$z = -p-\tfrac{1}{2}+iy,$$

we have

$$|s^{-z}| = s^{p+\frac{1}{2}},$$

$$|\Gamma(z)| = |\Gamma(-p-\tfrac{1}{2}+iy)| = \frac{\pi}{\cosh \pi y\, |\Gamma(p+\tfrac{3}{2}-iy)|},$$

$$\begin{aligned}|\xi(\tfrac{1}{2}+\tfrac{1}{2}z)|^2 &= |\xi(-\tfrac{1}{2}p+\tfrac{1}{4}+\tfrac{1}{2}iy)|^2 \\ &= |2(2\pi)^{-\frac{1}{2}p-\frac{3}{4}+\frac{1}{2}iy} \cos(\tfrac{1}{4}p+\tfrac{3}{8}-\tfrac{1}{4}iy)\pi\, \Gamma(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy)\, \xi(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy)|^2 \\ &< K(2\pi)^{-p}\, e^{\frac{1}{2}\pi|y|}\, |\Gamma(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy)|^2;\end{aligned}$$

and so

$$\begin{aligned}|s^{-z}\Gamma(z)\{\xi(\tfrac{1}{2}+\tfrac{1}{2}z)\}|^2 &< K\left(\frac{s}{2\pi}\right)^p e^{-\frac{1}{2}\pi|y|} \left|\frac{\{\Gamma(\tfrac{1}{2}p+\tfrac{3}{4}-\tfrac{1}{2}iy)\}^2}{\Gamma(p+\tfrac{3}{2}-iy)}\right| \\ &= K\left(\frac{s}{2\pi}\right)^p e^{-\frac{1}{2}\pi|y|} \left|\int_0^1 \{w(1-w)\}^{\frac{1}{2}p-\frac{1}{4}-\frac{1}{2}iy}\, dw\right| \\ &< K\left(\frac{s}{2\pi}\right)^p e^{-\frac{1}{2}\pi|y|} \int_0^1 \{w(1-w)\}^{\frac{1}{2}p-\frac{1}{2}}\, dw \\ &< K\left(\frac{s}{4\pi}\right)^p e^{-\frac{1}{2}\pi|y|}.*\end{aligned}$$

Hence

$$|J_p| < K\left(\frac{s}{4\pi}\right)^p \int_{-\infty}^{\infty} e^{-\frac{1}{2}\pi|y|}\, dy < K\left(\frac{s}{4\pi}\right)^p \to 0. \tag{2.17}$$

2.2. We have therefore

$$F(s) = -\frac{4\log s}{s} + \sum_0^\infty \frac{(-s)^n}{n!}\{\xi(\tfrac{1}{2}-\tfrac{1}{2}n)\}^2, \tag{2.21}$$

if $0 < s < 4\pi$, or

$$F(s) = -\frac{4\log s}{s} + G(s), \tag{2.22}$$

say. The power series $G(s)$ is convergent for $|s| < 4\pi$, and represents a branch of an analytic function regular at the origin; and the equation (2.22), considered as a relation between analytic functions, holds through-

* Because $0 \leqslant w(1-w) \leqslant \frac{1}{4}$. In this series of equations and inequalities the K's are of course independent both of p and of y.

out their region of existence. The study of the singularities of $F(s)$ is therefore reduced to the study of those of $G(s)$.*

III.

The Function $G(s) = \Sigma \frac{(-s)^n}{n!} \{\xi(\frac{1}{2}-\frac{1}{2}n)\}^2$. *The Singularities of* $G(s)$ *and* $F(s)$.

3.1. We proceed now to the discussion of the singularities of the function

$$G(s) = \sum_0^\infty \frac{(-s)^n}{n!} \{\xi(\tfrac{1}{2}-\tfrac{1}{2}n)\}^2. \tag{3.11}$$

Suppose first that $|s| < 4\pi$, so that the power series is convergent. We have

$$\xi(\tfrac{1}{2}-\tfrac{1}{2}n) = 2(2\pi)^{-\frac{1}{2}n-\frac{1}{2}} \cos\{\tfrac{1}{4}(n+1)\pi\}\, \Gamma(\tfrac{1}{2}n+\tfrac{1}{2})\, \xi(\tfrac{1}{2}n+\tfrac{1}{2}),$$

$$\cos^2\{\tfrac{1}{4}(n+1)\pi\} = \tfrac{1}{2}+\tfrac{1}{2}i^{n+1}+\tfrac{1}{2}(-i)^{n+1},$$

and so

$$G(s) = A+Bs+\frac{1}{\pi} M\left(-\frac{s}{2\pi}\right)+\frac{i}{2\pi} M\left(-\frac{is}{2\pi}\right)-\frac{i}{2\pi} M\left(\frac{is}{2\pi}\right), \tag{3.12}$$

where

$$A = \{\xi(\tfrac{1}{2})\}^2, \qquad B = -\{\xi(0)\}^2 = -\tfrac{1}{4},$$

$$M(s) = \sum_2^\infty \frac{\{\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}{\Gamma(n+1)} \{\xi(\tfrac{1}{2}n+\tfrac{1}{2})\}^2 s^n \tag{3.13}$$

$$= \sum_2^\infty c_n \{\xi(\tfrac{1}{2}n+\tfrac{1}{2})\}^2 s^n,$$

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* It can be shown by similar reasoning that the function

$$F_{\alpha,\beta}(s) = \Sigma \frac{d(n)}{n^\alpha} e^{-sn^\beta} \quad (0 < \beta < \tfrac{1}{2})$$

is the sum of a certain elementary algebraic or algebraico-logarithmic function and an *integral* function $G_{\alpha,\beta}(s)$. This has been observed already by Mellin, "Die Dirichletschen Reihen, die zahlentheoretischen Funktionen und die unendlichen Produkte von endlichen Geschlecht", *Acta Societatis Fennicae*, Vol. 31, No. 2, pp. 1–48 (p. 12), and *Acta Mathematica*, Vol. 28, pp. 37–64. This memoir contains a large variety of formulæ of the highest interest for possible applications in the Analytic Theory of Numbers; but Mellin does not seem to have considered the case in which $\beta = \frac{1}{2}$, which is plainly the critical case in the theory of $d(n)$.

say. Also

$$M(s) = \sum_2^\infty c_n s^n \sum_1^\infty \frac{d(q)}{q^{\frac{1}{2}n+\frac{1}{2}}} \tag{3.14}$$

$$= \sum_1^\infty \frac{d(q)}{\sqrt{q}} \sum_2^\infty c_n \left(\frac{s}{\sqrt{q}}\right)^n = \sum_1^\infty \frac{d(q)}{\sqrt{q}} L\left(\frac{s}{\sqrt{q}}\right),$$

where

$$L(s) = \sum_2^\infty \frac{\{\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}{\Gamma(n+1)} s^n. \tag{3.15}$$

3.2. The function $L(s)$ is an elementary function. For suppose s real and $0 < s < 2$. Then

$$L(s) = -\pi - s + \sum_0^\infty \frac{\{\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}{\Gamma(n+1)} s^n, \tag{3.21}$$

and

$$\sum_0^\infty \frac{\{\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}{\Gamma(n+1)} s^n = \sum_0^\infty s^n \int_0^1 \{w(1-w)\}^{\frac{1}{2}n-\frac{1}{2}} dw \tag{3.22}$$

$$= 2\sum_0^\infty s^n \int_0^{\frac{1}{2}\pi} (\cos\theta \sin\theta)^n d\theta = 2\int_0^{\frac{1}{2}\pi} \frac{d\theta}{1 - s\cos\theta\sin\theta}$$

$$= \int_0^\pi \frac{d\phi}{1 - \frac{1}{2}s\sin\phi} = \frac{8}{\sqrt{(4-s^2)}} \arctan\sqrt{\left(\frac{2+s}{2-s}\right)},$$

where the many valued functions have their principal values, so that $L(s)$ is positive for $0 < s < 2$ and

$$L(s) \sim \frac{2\pi}{\sqrt{(2-s)}}$$

as $s \to 2$. We have thus proved that if $|s| < 2$ then

$$M(s) = \sum_1^\infty \frac{d(q)}{\sqrt{q}} L\left(\frac{s}{\sqrt{q}}\right), \tag{3.23}$$

where

$$L(s) = -\pi - s + \frac{8}{\sqrt{(4-s^2)}} \arctan\sqrt{\left(\frac{2+s}{2-s}\right)}. \tag{3.24}$$

The principal branch of $L(s)$ has, as its sole singularity, an algebraic infinity of order $\frac{1}{2}$ at the point $s = 2$,* the principal part of $L(s)$ near

* Other branches have a singularity at $s = -2$. Compare Hardy, "On the Singularities of Functions defined by Taylor's Series", *Proc. London Math. Soc.*, Ser. 2, Vol. 5, pp. 197–205.

$s = 2$, being

$$\frac{2\pi}{\sqrt{(2-s)}}.$$

3.3. It is easy to deduce from the preceding analysis that the singularities of $M(s)$ are those of

$$L\left(\frac{s}{\sqrt{q}}\right) \quad (q = 1, 2, 3, \ldots).$$

We have in fact, if p is any positive integer,

$$(3.31)\quad M(s) - \sum_1^{p-1} \frac{d(q)}{\sqrt{q}} L\left(\frac{s}{\sqrt{q}}\right) = \sum_2^{\infty} c_n s^n \left[\left\{ \zeta\left(\tfrac{1}{2}n+\tfrac{1}{2}\right) \right\}^2 - \sum_1^{p-1} \frac{d(q)}{q^{\frac{1}{2}n+\frac{1}{2}}} \right]$$

$$= \sum_2^{\infty} c_n s^n \sum_p^{\infty} \frac{d(q)}{q^{\frac{1}{2}n+\frac{1}{2}}} = M_p(s),$$

where $M_p(s)$ is regular for $|s| < 2\sqrt{p}$. We are thus enabled to specify completely the singularities of $G(s)$, and so of $F(s)$. In particular we see that $F(s)$ *has algebraical infinities of order* $\frac{1}{2}$ *at each of the points*

$$(3.32)\qquad s = 4\pi i\sqrt{q}, \quad -4\pi i\sqrt{q} \quad (q = 1, 2, 3, \ldots),$$

and that

$$(3.33)\qquad F(s) \sim \frac{i}{2\pi} M\left(-\frac{is}{2\pi}\right) \sim \frac{i}{2\pi} \frac{d(q)}{\sqrt{q}} L\left(-\frac{is}{2\pi\sqrt{q}}\right)$$

$$\sim \frac{e^{\frac{1}{4}\pi i}\, d(q)}{q^{\frac{1}{4}}} \sqrt{\left(\frac{2\pi}{s - 4\pi i\sqrt{q}}\right)},$$

when $s = \sigma + 4\pi i\sqrt{q}$ *and* $\sigma \to 0$.

IV.

Application to the Divisor Problem.

4.1. Let

$$(4.11)\qquad f(s) = \Sigma d(n)\, e^{-s\sqrt{n}} = -F'(s),$$

when $\sigma > 0$. Then it follows from what precedes

(i) that $f(s)$ is regular at all points of the imaginary axis except the origin and the points (3.32);

(ii) that

$$f(s)+\frac{4\log s}{s^2}-\frac{4}{s^2} \tag{4.12}$$

is regular at the origin ;

(iii) that $f(s)$ has an algebraical infinity of order $\frac{3}{2}$ at each of the points (3.32) ; and

(iv) that

$$f(s)\sim\frac{e^{\frac{1}{4}\pi i}d(q)}{2q^{\frac{1}{4}}}\frac{\sqrt{(2\pi)}}{\sigma^{\frac{3}{2}}} \tag{4.13}$$

when $s=\sigma+4\pi i\sqrt{q}$ and $\sigma\to 0$. We shall also require the corresponding results for the simpler functions

$$\phi(s)=(\log n+2\gamma)\,e^{-s\sqrt{n}}, \tag{4.141}$$

$$\psi(s)=\Sigma n^{-\frac{1}{4}}e^{-s\sqrt{n}}. \tag{4.142}$$

We have, in the first place,

$$\phi(s)=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}s^{-z}\,\Gamma(z)\,Z(z)\,dz, \tag{4.15}$$

if $\sigma>0$, $\kappa>2$, and

$$Z(z)=-\zeta'(\tfrac{1}{2}z)+2\gamma\zeta(\tfrac{1}{2}z). \tag{4.151}$$

Applying to (4.15) a transformation similar to that of 2.1,* we find that

$$\phi(s)=-\frac{4\log s}{s^2}+\frac{4}{s^2}+g(s), \tag{4.152}$$

where $g(s)$ is an integral function of s.

In the second place we have

$$\psi(s)=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}s^{-z}\,\Gamma(z)\,\zeta(\tfrac{1}{4}+\tfrac{1}{2}z)\,dz, \tag{4.161}$$

if $\sigma>0$ and $\kappa>\frac{3}{2}$; from which, by a repetition of our arguments, we deduce that

$$\psi(s)=\sqrt{\pi}\,s^{-\frac{3}{2}}+h(s), \tag{4.162}$$

where $h(s)$ is an integral function of s.

---

* The details of the analysis are now much simpler.

4.2. From the results of the last paragraph we deduce that the function $\mathfrak{F}(s)$ defined, when $\sigma > 0$, by the equation

$$\mathfrak{F}(s) = \Sigma d_n e^{-s\sqrt{n}}, \tag{4.21}$$

where

$$d_n = d(n) - \log n - 2\gamma, \tag{4.211}$$

is regular at all points of the imaginary axis except the points (3.32), at each of which it has the same singular part as $f(s)$.

Now let

$$D_n = \sum_1^n d_\nu, \tag{4.22}$$

and

$$\mathfrak{G}(s) = \sum_1^\infty \frac{D_n}{\sqrt{n}} e^{-s\sqrt{n}}; \tag{4.23}$$

and let us consider first the behaviour of $\mathfrak{G}(s)$ when $s = \sigma + 4\pi i\sqrt{q}$ and $\sigma \to 0$. We have

$$e^{-s\sqrt{n}} - e^{-s\sqrt{(n+1)}} = \frac{se^{-s\sqrt{n}}}{2\sqrt{n}} + O\left(\frac{e^{-\sigma\sqrt{n}}}{n}\right), \tag{4.24}$$

and so

$$\mathfrak{F}(s) = \sum_1^\infty d_n e^{-s\sqrt{n}} = \sum_1^\infty D_n \{e^{-s\sqrt{n}} - e^{-s\sqrt{(n+1)}}\} \tag{4.25}$$

$$= \tfrac{1}{2}s\mathfrak{G}(s) + O\sum_1^\infty \frac{e^{-\sigma\sqrt{n}}}{\sqrt{n}} * = \tfrac{1}{2}s\mathfrak{G}(s) + O(1/\sigma).$$

Hence

$$\mathfrak{G}(s) = \frac{2}{s}\mathfrak{F}(s) + O\left(\frac{1}{\sigma}\right) \sim \frac{(1+i)\,d(q)}{4i\sqrt{\pi}\,q^{\frac{3}{4}}\sigma^{\frac{3}{4}}}. \tag{4.26}$$

We must also consider how $\mathfrak{G}(s)$ behaves when $s \to 0$ by positive values. We have now

$$e^{-s\sqrt{n}} - e^{-s\sqrt{(n+1)}} = \frac{se^{-s\sqrt{n}}}{2\sqrt{n}} + O\left(\frac{s^2e^{-s\sqrt{n}}}{n}\right) + O\left(\frac{se^{-s\sqrt{n}}}{n\sqrt{n}}\right), \tag{4.27}$$

where the constants implied by the O's are independent of both s and n. Hence, by (4.25), we have

$$\mathfrak{F}(s) = \tfrac{1}{2}s\mathfrak{G}(s) + O\left(s^2\Sigma\frac{e^{-s\sqrt{n}}}{n}\right) + O\left(s\Sigma\frac{e^{-s\sqrt{n}}}{n}\right) = \tfrac{1}{2}s\mathfrak{G}(s) + O(1),$$

---

* Here we use Dirichlet's result $\Delta(n) = O(\sqrt{n})$, which is sufficient for our purpose.

so that

$$\mathfrak{G}(s) = O(1/s).* \tag{4.28}$$

4.3. The relations (4.162), (4.26), and (4.28) enable us to complete the proof of our main theorem. We have, in fact,

$$\Sigma \frac{D_n}{\sqrt{n}} e^{-(4\pi i\sqrt{q}+\sigma)\sqrt{n}} \sim \frac{(1+i)\,d(q)}{4i\sqrt{\pi}\,q^{\frac{3}{4}}\sigma^{\frac{3}{2}}}, \tag{4.311}$$

$$\Sigma \frac{D_n}{\sqrt{n}} e^{-\sigma\sqrt{n}} = o\left(\frac{1}{\sigma^{\frac{3}{2}}}\right), \tag{4.312}$$

$$\Sigma n^{-\frac{1}{4}} e^{-(4\pi i\sqrt{q}+\sigma)\sqrt{n}} = o\left(\frac{1}{\sigma^{\frac{3}{2}}}\right), \tag{4.313}$$

$$\Sigma n^{-\frac{1}{4}} e^{-\sigma\sqrt{n}} \sim \frac{\sqrt{\pi}}{\sigma^{\frac{3}{2}}}, \tag{4.314}$$

when $\sigma \to 0$. It follows that

$$\Sigma \frac{D_n - Kn^{\frac{1}{4}}}{\sqrt{n}} e^{-(4\pi i\sqrt{q}+\sigma)\sqrt{n}} \sim \frac{(1+i)\,d(q)}{4i\sqrt{\pi}\,q^{\frac{3}{4}}\sigma^{\frac{3}{2}}}, \tag{4.321}$$

$$\Sigma \frac{D_n - Kn^{\frac{1}{4}}}{\sqrt{n}} e^{-\sigma\sqrt{n}} \sim -\frac{K\sqrt{\pi}}{\sigma^{\frac{3}{2}}}. \tag{4.322}$$

Suppose now that

$$D_n - Kn^{\frac{1}{4}} \leqslant 0, \tag{4.33}$$

for all sufficiently large values of n. Then

$$\Sigma \frac{|D_n - Kn^{\frac{1}{4}}|}{\sqrt{n}} e^{-(4\pi i\sqrt{q}+\sigma)\sqrt{n}} \sim -\frac{(1+i)\,d(q)}{4i\sqrt{\pi}\,q^{\frac{3}{4}}\sigma^{\frac{3}{2}}},$$

$$\Sigma \frac{|D_n - Kn^{\frac{1}{4}}|}{\sqrt{n}} e^{-\sigma\sqrt{n}} \sim \frac{K\sqrt{\pi}}{\sigma^{\frac{3}{2}}}.$$

Since $|e^{-4\pi i\sqrt{q}}| = 1$, these results are contradictory if

$$K < \frac{1}{2\pi\sqrt{2}} \frac{d(q)}{q^{\frac{3}{4}}}. \tag{4.34}$$

Similarly we can show that it is impossible that

$$D_n + Kn^{\frac{1}{4}} \geqslant 0, \tag{4.35}$$

for all sufficiently large values of n.

* It is easy to obtain a much more precise equation, but this very crude one is sufficient.

Since

$$D_n = \sum_1^n \{d(\nu) - \log \nu - 2\gamma\} = \Delta(n) + O(\log n), \tag{4.36}$$

it follows that *neither of the inequalities*

$$\Delta(n) - Kn^{\frac{1}{4}} \leqslant 0, \qquad \Delta(n) + Kn^{\frac{1}{4}} \geqslant 0 \tag{4.37}$$

can hold for all sufficiently large values of n. We have therefore proved our main theorem.

The value of q which gives the best value of K is $q = 2$. We have then

$$\frac{1}{2\pi\sqrt{2}} \frac{d(2)}{2^{\frac{3}{4}}} = \frac{1}{2^{\frac{1}{2}}\pi} = .13\ldots > \tfrac{1}{8};$$

so that we may take

$$K = \tfrac{1}{8}. \tag{4.38}$$

The actual value of K is however of little importance.*

V.

Applications of the Method to other Problems. The Problems of Piltz and Sierpinski.

5.1. Dirichlet's problem was generalised by Piltz†, who considered instead of $d(n)$ the number $d_k(n)$ of decompositions of n into k factors‡. Piltz proved that

$$D_k(n) = \sum_1^n d_k(n) = x\{a_{k,1}(\log x)^{k-1} + \ldots + a_{k,k}\} + \Delta_k(x), \tag{5.11}$$

---

* In the first place, $\frac{1}{4}$ may not be the best possible index; there may be a number α such that $\frac{1}{4} < \alpha \leqslant \frac{1}{3}$ and

$$\Delta(x) = \Omega_R(x^\alpha), \qquad \Delta(x) = \Omega_L(x^\alpha).$$

And even if $\frac{1}{4}$ is the best possible index, it is, in the light of Mr. Littlewood's result concerning $\Pi(x)$, highly probable that $x^{\frac{1}{4}}$ can be replaced by some such function as $x^{\frac{1}{4}}(\log x)^\beta$, where $\beta > 0$. See the additional note at the end of the paper (p. 23).

† "Über das Gesetz, nach welchem die mittlere Darstellbarkeit der natürlichen Zahlen als Produkte einer gegebenen Anzahl Faktoren mit der Grösse der Zahlen wächst", *Dissertation*, Berlin, 1881.

‡ Thus $d_3(1) = 1$, $d_3(2) = 3$, $d_3(4) = 6$: the decompositions of 4 being 1.1.4, 1.4.1, 4.1.1, 1.2.2, 2.1.2, 2.2.1.

where the a's are constants, and

$$\Delta_k(x) = O\{x^{\frac{k-1}{k}}(\log x)^{k-2}\}. \tag{5.111}$$

Landau* has shown that

$$\Delta_k(x) = O(x^{\frac{k-1}{k+1}+\epsilon}), \tag{5.12}$$

for all positive values of ϵ; and this is the best result known. Landau† has, however, pointed out that it follows from a theorem of Littlewood that, *if the Riemann hypothesis concerning the roots of the Zeta-function is true*‡, then

$$\Delta_k(x) = O(x^{\frac{1}{2}+\epsilon}), \tag{5.121}$$

for every k and every positive ϵ.

The application of the method of this paper to Piltz's problem presents no fresh difficulty of principle. I find that

$$\Delta_k(x) = \Omega_R(x^{\frac{k-1}{2k}}), \qquad \Delta_k(x) = \Omega_L(x^{\frac{k-1}{2k}}). \tag{5.13}$$

It should be observed that Landau's O-index $\frac{k-1}{k+1}$ takes for $k = 2, 3, 4, \ldots$ the values

$$\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{7}{9}, \ldots,$$

tending to 1, while my Ω-index $\frac{k-1}{2k}$ takes the values

$$\tfrac{1}{4}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{5}{12}, \tfrac{3}{7}, \tfrac{7}{16}, \ldots,$$

tending to $\frac{1}{2}$. It is rash to hazard a conjecture on such a subject, but the fact that Landau's indices exceed $\frac{1}{2}$ from $k = 4$ onwards suggest that the real order of $\Delta_k(x)$ may be nearer to my result than to his.

5.2. Let $r(n)$ denote the number of representations of n as the sum of two squares, *i.e.*, the number of solutions of the equation

$$u^2+v^2 = n, \tag{5.21}$$

---

* *L.c. supra* (p. 2, footnote ‡), p. 723.

† *Ibid.*, p. 728. See also Littlewood, "Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans le demi-plan $\sigma > \frac{1}{2}$", *Comptes Rendus*, 29 Jan. 1912.

‡ Indeed if only it be true that

$$\zeta(\tfrac{1}{2}+ti) = O(|t|^{\epsilon})$$

for every positive ϵ.

u and v being integers, positive, negative, or zero. It is well known that

$$r(n) = 4\{d_1(n) - d_3(n)\},$$

where $d_1(n)$ and $d_3(n)$ denote the numbers of the divisors of n of the forms $4k+1$ and $4k+3$ respectively. It is also well known that

$$R(x) = \sum_1^x r(n) = \pi x + \mathrm{P}(x),^* \tag{5.22}$$

where

$$\mathrm{P}(x) = O(\sqrt{x}). \tag{5.221}$$

The problems which arise in connection with $\mathrm{P}(x)$ are wholly analogous to those connected with $\Delta(x)$, and the theory is in much the same state, the best known result as to the order of $\mathrm{P}(x)$ being

$$\mathrm{P}(x) = O(x^{\frac{1}{3}}).^\dagger \tag{5.222}$$

The question arises as to whether the method of this paper is applicable to this problem also; and the answer is in the affirmative. It shows that a constant K exists such that each of the inequalities

$$\mathrm{P}(x) > Kx^{\frac{1}{4}}, \qquad \mathrm{P}(x) < -Kx^{\frac{1}{4}} \tag{5.23}$$

is satisfied for values of x which surpass all limit. The true maximum order of $\mathrm{P}(x)$ therefore lies somewhere between $x^{\frac{1}{3}}$ and $x^{\frac{1}{4}}$.

The proof of this result is in principle entirely similar to that of the corresponding result for $\Delta(n)$, and is formally a little simpler. It starts from the equation

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} e^{-s\sqrt{(m^2+n^2)}} = 2\pi s \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} \frac{1}{\{s^2+4\pi^2(m^2+n^2)\}^{\frac{3}{4}}}, \tag{5.24}$$

which holds when $\sigma > 0$. I proved this formula in 1908‡, by a method

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* $R(x)$ is the number of points with integral coordinates which lie inside or on the circle

$$u^2+v^2 = x.$$

† This result is due to Sierpinski, "O pewnęm zagadnieniu z rachunku funkcyj asymptotycznych", *Prace matematyczno fizyczne*, Vol. 17, pp. 77–118. Another proof has been given by Landau, "Über die Zerlegung der Zahlen in zwei Quadrate", *Annali di Matematica*, Vol. 20, p. 28. See also Landau, "Über einen Satz des Herrn Sierpinski", *Giornale di Matematiche di Battaglini*, Vol. 51, pp. 73-81, where it is shown that Landau's method alluded to on p. 2 leads to the slightly less precise relation

$$\mathrm{P}(n) = O(n^{\frac{1}{3}+\epsilon}).$$

‡ "Some multiple integrals", *Quarterly Journal*, Vol. 39, pp. 357–375 (p. 373). At the time when this paper was written I had no thought of any application of its results in the analytic theory of numbers.

depending on the theory of the linear transformation of the double Theta-functions: it may also be proved by the methods of Sections 2 and 3. The equation (5 . 24) may be written in the form

$$(5\,.\,241) \qquad 1+\sum_1^\infty r(n)e^{-s\sqrt{n}} = \frac{2\pi}{s^2}+2\pi s\sum_1^\infty \frac{r(n)}{(s^2+4\pi^2 n)^{\frac{1}{2}}};$$

and reveals at once the existence of the algebraical infinities of the function

$$\Sigma r(n)e^{-s\sqrt{n}},$$

on the imaginary axis, which play the chief part in the argument.

I also proved the more general formula

$$(5\,.\,25)\quad \Sigma\Sigma e^{-s\sqrt{(\alpha m^2+\beta mn+\gamma n^2)}} = \frac{4\pi s}{\Delta}\Sigma\Sigma\left\{s^2+\frac{16\pi^2}{\Delta^2}(\alpha m^2+\beta mn+\gamma n^2)\right\}^{-\frac{3}{4}},$$

where α, β, γ are integers, and α and $\Delta^2 = 4\alpha\gamma-\beta^2$ positive. This formula enables us to obtain a corresponding result for the function $\mathrm{E}(x)$ which bears to the ellipse

$$\alpha u^2+\beta uv+\gamma u^2 = x$$

the same relation that $\mathrm{P}(x)$ bears to the circle $u^2+v^2 = x$.*

VI.

The Series $\Sigma \frac{d(n)}{\sqrt{n}} e^{-it\sqrt{n}}$.

6 . 1. The function $L(s)$ of 3 . 2 is, near the singular point $x = 2$, of the form

$$\frac{2\pi}{\sqrt{(2-x)}}+P\{\sqrt{(2-x)}\},$$

where P denotes a power series convergent for sufficiently small values of

* If $E(x)$ is the number of points with integral coordinates which lie in or on the ellipse, then

$$E(x) = \frac{2\pi x}{\Delta}+\mathrm{E}(x).$$

Landau has proved that $\mathrm{E}(x) = O(x^{\frac{1}{3}})$:

see his memoirs already quoted.

A full proof of the results stated in 5 . 2 will be found in my paper "On the representation of a number as the sum of two squares", *Quarterly Journal*, Vol. 46, pp. 263–283. This paper was written after the present one. See also the additional note at the end.

the variable. It follows that $F(s)$ is regular for $s = it$, or of the form

$$\frac{e^{\frac{1}{4}\pi i}d(q)}{q^{\frac{1}{4}}}\sqrt{\left(\frac{2\pi}{s-it}\right)}+P\{\sqrt{(s-it)}\}, \tag{6.11}$$

according as t is not or is of the form $\pm 4\pi\sqrt{q}$, where q is an integer.

It has been proved by Dr. Marcel Riesz* that, *if*

$$a_1+a_2+\ldots+a_n = o(e^{\lambda_n c}) \quad (c>0), \tag{6.12}$$

then the Dirichlet's series $\Sigma a_n e^{-\lambda_n s}$, *obviously convergent for* $\sigma > c$, *is convergent at every regular point of the line* $\sigma = c$.

The condition (6.12) is certainly satisfied if

$$a_n = o\{(\lambda_n-\lambda_{n-1})\,e^{\lambda_{n-1}c}\}, \tag{6.121}$$

and if $\lambda_n-\lambda_{n-1} = O(1)$, we may take $c = 0$.† If, in particular we take $\lambda_n = \sqrt{n}$, we obtain the theorem: *if*

$$a_n = o\left(\frac{1}{\sqrt{n}}\right) \tag{6.122}$$

then the series $\Sigma a_n e^{-s\sqrt{n}}$ *is convergent at every regular point of the line* $\sigma = 0$. The actual theorem which I wish to use is however not quite this, but a slightly less precise theorem, which can be proved in substantially the same way, viz. *if*

$$a_n = o(n^{-\frac{1}{2}+\delta}), \tag{6.123}$$

for every positive δ, *then*

$$a_1e^{-s\sqrt{1}}+a_2e^{-s\sqrt{2}}+\ldots+a_ne^{-s\sqrt{n}} = o(n^{\delta}) \tag{6.124}$$

at all regular points on the imaginary axis.

Observing that

$$d(n) = o(n^{\delta}), \tag{6.13}$$

* "Sur les séries de Dirichlet et les séries entières", *Comptes Rendus*, 22 Nov. 1909. This theorem is Theorem 37 of our joint tract "The general theory of Dirichlet's series". Detailed proofs have been published only in the special cases in which $\lambda_n = n$ (Riesz, "Über einen Satz des Herrn Fatou", *Journal für Math.*, Vol. 140, pp. 89–99) and $\lambda_n = \log n$ (Landau, "Über die Bedeutung einiger neuen Grenzwertsätze der Herrn Hardy und Axer", *Prace matematyczno fizyczne*, Vol. 21, pp. 97–177, and in particular pp. 151–167).

† Cf. Hardy and Riesz, *loc. cit.* The theorem then reduces, when $\lambda_n = n$, to a well known theorem of Fatou.

we see that

$$\sum_1^n \frac{d(\nu)}{\sqrt{\nu}} e^{-it\sqrt{\nu}} = o(n^\delta) \tag{6.14}$$

*for all values of t which are not of the form $\pm 4\pi\sqrt{q}$.** It follows that *the series*

$$\Sigma \frac{d(n)}{n^a} e^{-it\sqrt{n}}, \tag{6.15}$$

where $t \neq \pm 4\pi\sqrt{q}$, q being an integer, is convergent for $a > \frac{1}{2}$.

Riesz's theorems all hold uniformly in any interval of regularity, and the series (6.15) is, for any value of a greater than $\frac{1}{2}$, uniformly convergent in any interval free from exceptional points.

6.2. Suppose now that $t = 4\pi\sqrt{q}$. Then $F(s)$ is, in the neighbourhood of the point

$$s = s_0 = 4\pi i\sqrt{q},$$

of the form

$$\frac{H}{\sqrt{(s-s_0)}} + P\{\sqrt{(s-s_0)}\},$$

where H is a constant whose value is given by (3.33). Also the function defined, for $\sigma > 0$, by the series

$$\Sigma n^{-\frac{3}{4}} e^{-s\sqrt{n}},$$

is, near $s = 0$, of the form

$$2\sqrt{\left(\frac{\pi}{s}\right)} + P(s).\dagger$$

It follows that the function $\Phi(s)$, defined when $\sigma > 0$ by the series

$$\Sigma \left\{ \frac{d(n)}{\sqrt{n}} e^{-4\pi i\sqrt{(qn)}} - \frac{H}{2\sqrt{\pi}} n^{-\frac{3}{4}} \right\} e^{-s\sqrt{n}}, \tag{6.21}$$

is, in the neighbourhood of the origin, of the form

$$P(\sqrt{s}). \tag{6.22}$$

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* We can, in fact, replace n^δ by a definite function of increase less than that of any power of n, viz.,

$$2^{(1+\epsilon)\frac{\log n}{\log\log n}},$$

where ϵ is any positive number.

† The function is substantially the integral of the function $\psi(s)$ of 4.1.

We cannot apply Riesz's theorem, as we have stated it, to the series (6.21), since $P(\sqrt{s})$ is not regular for $s = 0$. But Riesz has shown that the hypothesis of regularity in his theorem may be replaced by much more general hypotheses. In particular all the results stated in 6.1 hold, for $s = 0$, whenever the function represented by the series satisfies the condition

$$|f(s)| < K|s|^{\alpha} \quad (\alpha > 0)$$

in a region $$0 < \sigma \leqslant \sigma_1, \quad -t_1 \leqslant t \leqslant t_1;$$

a condition obviously satisfied in this case. It follows that

$$(6.23) \qquad \sum_1^{\nu} \left(\frac{d(\nu)}{\sqrt{\nu}} e^{-4\pi i\sqrt{(q\nu)}} - \frac{H}{2\sqrt{\pi}} \nu^{-\frac{1}{4}} \right) = o(n^{\delta}),$$

and so that

$$(6.24) \qquad \sum_1^{n} \frac{d(\nu)}{\sqrt{\nu}} e^{-4\pi i\sqrt{(q\nu)}} = \frac{2H}{\sqrt{\pi}} n^{\frac{1}{4}} + o(n^{\delta}) = \frac{2(1+i)\,d(q)}{q^{\frac{1}{4}}} n^{\frac{1}{4}} + o(n^{\delta}).$$

We have thus proved the following theorem: *if*

$$S_n = \sum_1^{n} \frac{d(\nu)}{\sqrt{\nu}} e^{-it\sqrt{\nu}} \qquad (t > 0),$$

then $$S = o(n^{\delta})$$

or $$S = \frac{2(1+i)\,d(q)}{q^{\frac{1}{4}}} n^{\frac{1}{4}} + o(n^{\delta}),$$

according as t is not or is of the form $4\pi\sqrt{q}$, where q is an integer. The series

$$\Sigma \frac{d(\nu)}{\nu^{\alpha}} e^{-it\sqrt{\nu}}$$

is convergent for $\alpha > \frac{1}{2}$, or for $\alpha > \frac{3}{4}$, accordingly.

The series $$\Sigma \frac{\sigma(\nu)}{\nu^{\alpha}} e^{-it\sqrt{\nu}},$$

where $\nu\sigma(\nu)$ is the sum of the divisors of ν, behaves more simply: it is *always* convergent for $\alpha > \frac{1}{2}$. This has been shown by Wigert* by means of general theorems belonging to the arithmetic theory of series. The corresponding result for $d(n)$, given above, seems to lie rather deeper.

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* "Sur quelques fonctions arithmétiques", *Acta Mathematica*, Vol. 37, pp. 113–140.

6.3. The series

$$\chi(t) = \Sigma \frac{d(\nu)}{\nu^{\frac{3}{4}}} e^{-it\sqrt{\nu}} \tag{6.31}$$

may be discussed in the same way. It is convergent except for

$$t = \pm 4\pi\sqrt{q},$$

where q is an integer, and uniformly convergent throughout any interval of values of t free from these exceptional values.

The series

$$\frac{\lambda^{\frac{1}{4}}}{\pi\sqrt{2}} \Sigma \frac{d(\nu)}{\nu^{\frac{3}{4}}} \cos\{4\pi\sqrt{(\lambda\nu)} - \tfrac{1}{4}\pi\} = \frac{\lambda^{\frac{1}{4}}}{\pi\sqrt{2}} \Re\{e^{\frac{1}{4}\pi i} \chi(4\pi\sqrt{\lambda})\} \tag{6.32}$$

is of quite exceptional interest. It is convergent for all real values of λ, and uniformly convergent throughout any interval of values of λ which includes no integral value. At the point $\lambda = n$ it has a finite discontinuity equal to

$$d(n).$$

This circumstance at once suggests that the series must play a fundamental part in the problem of the determination of an explicit formula for

$$d(1) + d(2) + \ldots + d(n),$$

analogous to Riemann's formula for the number of primes less than x. Such a formula was first found by Voronoï*. The formula is†

$$\begin{aligned} \Delta(\lambda) - \tfrac{1}{4} &= \sum_{n \leqslant \lambda} d(n) - \lambda \log \lambda - (2\gamma - 1)\lambda - \tfrac{1}{4} \\ &= \sqrt{\lambda} \sum_{1}^{\infty} \frac{d(\nu)}{\sqrt{\nu}} [H_1\{4\pi\sqrt{(\lambda\nu)}\} - Y_1\{4\pi\sqrt{(\lambda\nu)}\}], \end{aligned} \tag{6.33}$$

where $Y_1(x)$ is the ordinary second solution of Bessel's equation, and

$$H_1(x) = \frac{2}{\pi} \int_1^{\infty} \frac{we^{-xw}dw}{\sqrt{(w^2-1)}}$$

is one of Hankel's cylinder-functions. If we make use of the known asymptotic expansion of $Y_1(x)$, and observe that $H_1(x)$ tends exponentially

* "Sur une fonction transcendante et ses applications à la sommation de quelques séries", *Annales de l'École Normale*, Ser. 3, Vol. 21, pp. 207-268, and 459-534. See the note at the end of the paper.

† I have altered Voronoï's notation.

to zero as $x \to \infty$, we find at once that

$$\Delta(\lambda) = \frac{\lambda^{\frac{1}{2}}}{\pi\sqrt{2}} \Sigma \frac{d(\nu)}{\nu^{\frac{3}{4}}} \cos\{4\pi\sqrt{(\lambda\nu)} - \tfrac{1}{4}\pi\} + R(\lambda), \tag{6.34}$$

where $R(\lambda)$ is a series absolutely and uniformly convergent for all positive values of λ. The "non-trivial" part of $\Delta(\lambda)$ is therefore precisely the series (6.32).

There are analogous results for the function $P(\lambda)$ of 5.2.*

Additional Note (23 *Aug.* 1915).

It was only while this paper was being printed that I discovered that the formula (6.33) had been proved already by Voronoï. It is owing to this that Mr. Ramanujan, in the introduction to his paper "Highly composite numbers"† attributes the formula to me. I owe my knowledge of Voronoï's memoir, quoted on p. 20, to a reference in a paper by Landau‡ of which I received a copy about a month ago.

Voronoï's proof of the formula is exceedingly elaborate and difficult, and I cannot yet claim to have mastered all its details. I have myself constructed two independent proofs. One of these is not much less difficult than Voronoï's, which it resembles in its general character. Voronoï bases his proof on a study of the function

$$g(x) = -\tfrac{1}{4}\log x - \tfrac{1}{2}\gamma - \frac{\log(4\pi^2 x)}{4\pi^2 x} + \frac{1}{2\pi^2} \sum_1^\infty d(n) \log\frac{x}{n} \left(\frac{1}{x-n} + \frac{1}{x+n}\right),$$

and I base mine on a study of the function

$$h(x) = x \sum_1^\infty \frac{d(n)}{n(\sqrt{n} - x)}.$$

If my proof is shorter than his, it is because I have at my disposal a number of theorems in the general theory of Dirichlet's series proved only since he wrote, and in particular Riesz's generalisations of Fatou's theorem, and am so able to prove the convergence of the critical series (6.32) in a comparatively simple manner.

* The proofs of these are given in my paper in the *Quarterly Journal* referred to on p. 16.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 14, 1915, pp. 347-409.

‡ "Zur analytischen Zahlentheorie der definiten quadratischen Formen (Über die Gitterpunkte in einen mehrdimensionalen Ellipsoid)", *Berliner Sitzungsberichte*, 20 May 1915, pp. 458-476.

My second proof is much simpler, and is in all essentials similar to that of the corresponding formula in the "Sierpinski" problem (5.2), a proof given in full in my paper in the *Quarterly Journal* referred to on p. 16. A formal deduction of this latter formula is given in another paper of Voronoï*, which also has come to my knowledge through a reference by Landau. Voronoï states explicitly that his proof is not rigorous, and mine is, so far as I can discover, the first that is.

It is evident in the light of these facts that my statement† that it was Wigert "who was the first to recognise the importance in the Analytic Theory of Numbers of series of Bessel's functions" is inaccurate. This in no way diminishes my own indebtedness to Wigert's work.

I begin my second proof of (6.33) by proving the formula

$$f(s) = \Sigma d(n)\, e^{-s\sqrt{n}} = -\frac{4}{s^2}(\log s - 1) + \tfrac{1}{4}$$

$$+ \frac{1}{2\pi^2}\Sigma\frac{d(q)}{q}\left\{ l\left(-\frac{s}{2\pi\sqrt{q}}\right) - \tfrac{1}{2}l\left(-\frac{is}{2\pi\sqrt{q}}\right) - \tfrac{1}{2}l\left(\frac{is}{2\pi\sqrt{q}}\right)\right\},$$

where

$$l(x) = \int_0^{\frac{1}{2}\pi} \frac{\sin\theta\, d\theta}{(1-\frac{1}{2}x\sin\theta)^2}.$$

This formula, which is substantially equivalent to (2.21) and (3.12) of this paper, then assumes the *rôle* of (5.241), or (2.12) of my paper in the *Quarterly Journal;* and the rest of the argument differs from that of Section III of the latter paper only in formal detail, the kernel of the proof consisting, in the one case as in the other, in an application of Riesz's theorems.

So much as regards the explicit formulæ: I must also add a few remarks concerning the "Ω" results. I find from another paper of Landau's‡ which I have also received recently that he has attacked this side of the "two square" problem simultaneously with myself. My result

* "Sur le développement, à l'aide des fonctions cylindriques, des sommes doubles $\Sigma f(pm^2+2qmn+rn^2)$, où $pm^2+2qmn+rn^2$ est une forme positive à coefficients entiers", *Verhandlungen des dritten Internationalen Math.-Kongresses in Heidelberg*, 1904, pp. 241–245. Further reference to the formulæ given by Voronoï is made by Sierspinski in a paper "O sumowaniu szeregu $\Sigma\tau(n)f(n)$, gdzie $\tau(n)$ oznacza liczbę rozkładów liczby n na sumę kwadratów dwóch liczb całkowitych", *Prace matematyczno fizyczne*, Vol. 18, pp. 1–59: but it appears to me (so far as I can judge from the Polish) that Sierpinski merely quotes Voronoï's work.

† *Loc. cit.*, p. 266.

‡ "Über die Gitterpunkte in einem Kreise (Zweite Mitteilung)", *Göttinger Nachrichten*, 5 June 1915.

states a little more than his, viz.,

$$P(x) = \Omega_R(x^{\frac{1}{4}}), \qquad P(x) = \Omega_L(x^{\frac{1}{4}}),$$

instead of $$P(x) = \Omega(x^{\frac{1}{4}-\epsilon})$$

for every positive ϵ. But the essential part of the result, viz. that Sierpinski's index $\frac{1}{3}$ cannot be replaced by any index less than $\frac{1}{4}$, is precisely the same. I have no evidence that Landau has considered the Ω-problem for $\Delta(n)$, which is the main subject of this paper.

In the footnote at the end of Section IV of this paper I implied that the exact value of the constant K furnished by the argument was unimportant, because $x^{\frac{1}{4}}$ could in all probability be replaced by some function of slightly more rapid increase. I have since succeeded in proving that this conjecture is correct. I find in fact

$$\Delta(x) = \Omega_R\{(x\log x)^{\frac{1}{4}}\log\log x\}, \qquad \Delta(x) = \Omega_L\{(x\log x)^{\frac{1}{4}}\log\log x\};$$

and similarly

$$P(x) = \Omega_R\{(x\log x)^{\frac{1}{4}}\}, \qquad P(x) = \Omega_L\{(x\log x)^{\frac{1}{4}}\}.$$

The proofs are based on the explicit formulæ and an application of the theory of Diophantine approximation after the manner of Bohr and Landau* and Littlewood.†

To prove, for example, that

(1) $$\Delta(x) = \Omega_R\{(x\log x)^{\frac{1}{4}}\log\log x\}$$

is equivalent to proving that

$$\Sigma\frac{d(n)}{n^{\frac{3}{4}}}\cos\{4\pi\sqrt{(n\lambda)}-\tfrac{1}{4}\pi\} = \Omega_R\{(\log\lambda)^{\frac{1}{4}}\log\log\lambda\},$$

or that

(2) $$\Re F(it) = \Omega_R\{(\log t)^{\frac{1}{4}}\log\log t\},$$

where $$F(s) = e^{\frac{1}{4}\pi i}\Sigma\frac{d(n)}{n^{\frac{3}{4}}}e^{-s\sqrt{n}}.$$

I prove first that

(3) $$\Re F(s) = \Omega_R\{(\log t)^{\frac{1}{4}}\log\log t\}$$

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* "Über das Verhalten von $\zeta(s)$ und $\zeta_\kappa(s)$ in der Nähe der Geraden $\sigma = 1$", *Göttinger Nachrichten*, 1910, pp. 303–320.

† *L.c.* p. 4.

in the quarter-plane $\sigma > 0$, $t > 1$, *i.e.* that a positive K exists such that the inequality

$$\Re F(s) > K(\log t)^{\frac{1}{4}} \log\log t \tag{4}$$

has solutions in every quarter-plane $\sigma > 0$, $t > T \geqslant 1$.

I take $\sigma = 1/g$ and $N = (Hg)^2$, where g is a large positive number and H a constant to be fixed later, and I write

$$\Re F(s) = \Sigma \frac{d(n)}{n^{\frac{3}{4}}} e^{-\sigma\sqrt{n}} \cos(t\sqrt{n} - \tfrac{1}{4}\pi) = \sum_1^N + \sum_{N+1}^\infty = S_1 + S_2. \tag{5}$$

Suppose now that η is an arbitrarily small positive number. Then it follows from Dirichlet's fundamental theorem on Diophantine approximation* that a t exists such that

$$1 < t < (1/\eta)^N,$$

and

$$|\{t\sqrt{n}\}| < \eta \quad (n = 1, 2, \ldots, N),$$

where $\{x\}$ denotes the difference of x from the nearest integer. Choosing an appropriate η, and a corresponding value of t, we have

$$S_1 > K \sum_1^N \frac{d(n)}{n^{\frac{3}{4}}} e^{-\sqrt{n}/g}.$$

Transforming the right-hand side by partial summation, and using the formula

$$D(n) = d(1) + d(2) + \ldots + d(n) \sim n \log n,$$

we find that

$$S_1 > K\sqrt{g} \log g. \tag{6}$$

In these inequalities *the K's are independent of H.*

On the other hand

$$|S_2| < \sum_{N+1}^\infty \frac{d(n)}{n^{\frac{3}{4}}} e^{-\sqrt{n}/g},$$

and similar transformations, applied to the series on the right-hand side, show that

$$|S_2| < \kappa_H \sqrt{g} \log g, \tag{7}$$

where κ_H is a number which may be made as small as we please by in-

* See, *e.g.*, Hardy and Littlewood, "Some problems of Diophantine approximation (I)", *Acta Mathematica*, Vol. 37, pp. 155–190 (p. 159).

creasing H sufficiently. From (5), (6), and (7) it follows that

$$\Re F(s) > K\sqrt{g}\log g. \tag{8}$$

This inequality holds for $\sigma = 1/g$ and some corresponding value of t. But it is not difficult to show that $\Re F(s)$ is bounded in any strip such as $\sigma > 0$, $1 \leqslant t \leqslant T$, and so that the values of t for which (8) holds must tend to infinity with g. Since $N = (Hg)^2$ and $t < (1/\eta)^N$, we obtain (4) and so (3).

The deduction of (1) from (3) depends on the same modification of a theorem of Lindelöf that was used by Littlewood in his investigations quoted on p. 4 (footnote *). This theorem will be proved in full in our forthcoming memoir in the *Acta Mathematica*.

The corresponding result for $\Delta_k(x)$ is

$$\Delta_k(x) = \Omega_R\{(t\log t)^{\frac{k-1}{2k}}(\log\log t)^{k-1}\}.* \tag{9}$$

It is highly probable, in the light of all that is now known, that

$$\Delta(x) = O(x^{\frac{1}{4}+\epsilon}), \qquad \Delta_k(x) = O(x^{\frac{k-1}{2k}+\epsilon}), \tag{10}$$

for every positive ϵ. All that I am able to prove in this direction, however, is that

$$\{\Delta(1)\}^2+\{\Delta(2)\}^2+\ldots+\{\Delta(n)\}^2 = O(n^{\frac{3}{2}+\epsilon}) \tag{11}$$

and so

$$|\Delta(1)|+|\Delta(2)|+\ldots+|\Delta(n)| = O(n^{\frac{3}{4}+\epsilon}). \tag{12}$$

If I could prove the analogous result for every k, I could deduce that

$$\zeta(\tfrac{1}{2}+ti) = O(t^{\epsilon}):$$

in particular this would follow from (10).

* I have not yet constructed a complete proof of this result.

ON PERIODIC IRROTATIONAL WAVES AT THE SURFACE OF DEEP WATER

By W. BURNSIDE.

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No exact solution of the problem of permanent irrotational waves in deep water, where the water surface merely changes its position in space without changing its form, has hitherto been discovered. Indeed it is not known whether any exact solution exists.

In the ordinary treatment of the problem, to a first approximation, it is assumed that all terms containing the square of the velocity function are negligible. On this assumption it is shewn that if the ratio of the height (vertical distance from trough to crest) to the length of the waves is small enough a sinusoidal wave can be propagated without sensible change. Exactly what "small enough" means the theory as ordinarily presented gives no means of determining.

For a wave of given shape, height and length, it should be possible to calculate the extreme variation of pressure over the free surface, and the ratio of this to, say, the hydrostatic difference of pressure between trough and crest gives some idea of the permanence of the motion. In the absence of an exact theory it is difficult to see that more than this can be done.

Of the different calculations carrying the algebraic approximation to a high order of small quantities, the most complete is that given by Stokes (*Collected Papers*, Vol. I, "Supplement to a Paper on the Theory of Oscillatory Waves"). The approximation is carried to the fifth power of the parameter "b," and to extend it is merely a matter of taking trouble. The complete result would be to express the coordinates x and y in terms of ϕ and ψ in the form

$$x = -\phi + be^{\psi}\sin\phi + \sum_{2}^{\infty} b^n P_n(b) e^{n\psi}\sin n\phi,$$

$$y = -\psi + be^{\psi}\cos\phi + \sum_{2}^{\infty} b^n Q_n(b) e^{n\psi}\cos n\phi,$$

where $P_n(b)$ and $Q_n(b)$ are power-series in b. These results have a meaning and can be used for actual approximate calculation only, if $P_n(b)$ and $Q_n(b)$ are convergent power-series when b does not exceed some value, say b_0, while for suitable values of b and for real negative values of ψ, the series for x and y are convergent.

Until the form of the power series $P_n(b)$ and $Q_n(b)$ have been determined it is impossible to deal with their convergence. Assuming that they are convergent, it is clear from physical considerations that there must be an upper limit b_1 for b, in order that the series for x and y may be convergent for negative values of ψ, and there are no means of determining b_1.

It follows that Stokes's approximation to the fifth order given in equation (17) and (18) of the memoir cannot be used to determine the extreme variation of pressure over the surface of a wave of the type given by equation (19), and of given height and length; because it is not known whether the corresponding value of b is greater or less than the above b_1. Indeed no more can safely be said than that, *if b is small enough*, a wave of type given by equation (19) is more permanent than a wave of type

$$y = b \sin x.$$

My conclusion is that these successive approximations cannot be used for purposes of numerical calculation, and that if numerical results are to be obtained, it must be by direct calculation for each type, when the ratio of height to length is given.

2. The stream function for a stationary wave of length 2π, which is symmetric about the origin, is necessarily of the form

$$\psi = V(y - a_1 e^{-y} \cos x - a_2 e^{-2y} \cos 2x - \ldots),$$

and without loss of generality the free surface may be taken to be $\psi = 0$. By a given type of wave is implied the using of a finite number of terms of the series.

Assuming the series expressing z or $x+iy$ in terms of w or $\phi+i\psi$ to be convergent, so that it may be reversed, Stokes's approximation to the fifth power of b, becomes on reversion

$$w = -z + ibe^{iz} - \tfrac{1}{2} ib^4 e^{2iz} + \tfrac{1}{1\,2} ib^5 e^{3iz},$$

giving $$\psi = -y + be^{-y} \cos x - \tfrac{1}{2} b^4 e^{-2y} \cos 2x + \tfrac{1}{1\,2} b^5 e^{-3y} \cos 3x.$$

On the assumption then that permanent irrotational waves exist, the coefficients $a_2, a_3, \ldots$ in the above form for ψ should be proportional to $a_1^4, a_1^5, \ldots$.

3. For waves of the first type

$$\psi = V(y - ae^{-y}\cos x).$$

If y_1 and $-y_2$ are the ordinates of trough and crest, so that

$$y_1 = ae^{-y_1}, \qquad y_2 = ae^{y_2},$$

the ratio of height to length is $(y_1+y_2)/2\pi$. The pressure equation is

$$\frac{p}{\rho} + C = gy - \tfrac{1}{2}V^2(1 + 2ae^{-y}\cos x + a^2e^{-2y}),$$

and at the surface, where

$$y = ae^{-y}\cos x,$$

$$p - p_0 = \rho V^2\left[y\left(\frac{g}{V^2} - 1 + a^2\right) - \tfrac{1}{2}a^2(e^{-2y} - 1 + 2y)\right].$$

If $$\frac{g}{V^2} - 1 + a^2 = \tfrac{1}{2}a^2k, \qquad e^{-2y} - 1 + 2y = f(y),$$

the variation of pressure over the surface is given by the variation between y_1 and $-y_2$ of

$$\tfrac{1}{2}\rho a^2V^2[ky - f(y)].$$

The function $f(y)$ is essentially positive with $y = 0$ for a unique minimum. It increases more rapidly for negative values of y than for positive, while y_2 is clearly greater than y_1. If, as in the ordinary theory, k is assumed to be zero, the extreme variation of pressure over the surface is

$$\tfrac{1}{2}\rho a^2V^2 f(-y_2).$$

Unless a is very small this may be materially diminished by taking k so as to make the pressures at trough and crest equal. The variation of pressure so obtained is not the actual minimum, but it is certainly not far from it. To go further it is necessary to take particular cases.

When $$a = \cdot 25, \quad y_1 = \cdot 204, \quad y_2 = \cdot 357,$$

$$f(y_1) = \cdot 074, \quad f(-y_2) = \cdot 331.$$

The ratio of height to length is

$$\frac{\cdot 561}{2\pi} = \cdot 089.$$

When $k = 0$, $$V^2 = \frac{16g}{15},$$

and the extreme variation of pressure is $\cdot 011g\rho$.

When k is taken so as to make the pressure at trough and crest equal,

$$\cdot 204k - \cdot 074 = -\cdot 357k - \cdot 331,$$

or

$$k = -\cdot 458.$$

At trough and crest the value of $ky - f(y)$ is $-\cdot 167$. Its maximum value occurs when $y = -\cdot 101$, and is $\cdot 019$. With this value of k,

$$V^2 = \frac{16g}{14\cdot 771},$$

and the extreme variation of pressure is

$$\cdot 006g\rho.$$

Hence for a wave in which the ratio of the height h to the length l is $\cdot 089$, travelling with a velocity given by

$$V^2 = 1\cdot 08\,\frac{gl}{2\pi},$$

the extreme variation of pressure over the surface is $\cdot 0106g\rho h$. This is about $8\frac{1}{2}$ times the difference of the atmospheric pressures at trough and crest.

It is sufficient to state the results in two other cases with smaller values of a. When $a = \cdot 166$, $h/l = \cdot 055$, and the velocity may be chosen so that the extreme variation of pressure over the surface is less than twice the difference of the atmospheric pressures at trough and crest. When $a = \cdot 125$, $h/l = \cdot 04$, and with a suitable velocity the extreme variation of pressure is less than $\frac{3}{4}$ of the difference of atmospheric pressures at trough and crest. In these latter two cases the variation of pressure over the surface is so small that the first type of wave may be regarded as giving a practical solution when the ratio of height to length does not exceed $\cdot 055$.

4. For a wave of the second type, in which

$$\psi = V(y - a_1 e^{-y}\cos x - a_2 e^{-2y}\cos 2x),$$

the function determining the pressure at the surface is a much more complicated one than in the previous case. When a_1 and a_2 are given, the problem is to determine V so that the extreme variation of the pressure over the surface shall be as small as possible.

This has been replaced by the more feasible problem of determining V so that the extreme variation of pressure at nine points on the surface

dividing the distance between trough and crest into eight equal segments (horizontally) shall be as small as possible. The actual variation of pressure over the surface will be greater than that determined from the nine points, but probably only slightly greater.

Taking $a_1 = \cdot 25$, the value of a_2 given by Stokes's approximation, which should be the best, is $a_2 = -\cdot 002$. These values give ·09 for the ratio of height to length. The extreme variation of pressure at the nine points cannot in this case, by choosing V, be made less than $\frac{4}{9}$ of the variation given by the first type for the same ratio of height to length.

If, however, the value $a_2 = -\cdot 003$ be used, the extreme variation at the nine points can be reduced to $\frac{5}{18}$ of that given by the first type. No other value that has been tried for a_2 gives so good a result as this last one.

Mr. Mitchell has shewn that if permanent irrotational waves exist at all, the extreme ratio of length to height for them is about ·14. The ratio ·09 here considered is well within this limit, and therefore the value $-\cdot 002$ given by Stokes's formula for a_2, ought, assuming the existence of permanent waves, to give the best result for the second type. The fact that it does not may be regarded as suggesting the non-existence of permanent irrotational waves.

PROOF OF THE COMPLEMENTARY THEOREM

By J. C. FIELDS.

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IN a paper* recently published in the *Proceedings*, the writer obtained the expression

$$\lambda_\kappa = ni_\kappa + \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} - \tfrac{1}{2} \sum_{s=1}^{r_\kappa} \left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right) \nu_s^{(\kappa)} \tag{1}$$

for the number of the conditions imposed by a set of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ on the constants in the general rational function of (z, u) which can be represented in the form

$$(z-a_\kappa)^{-i_\kappa}((z-a_\kappa, u)), \tag{2}$$

the integer i_κ being supposed to be taken sufficiently great. This expression for λ_κ, deduced with reference to an arbitrary fundamental equation of degree n in u, we shall utilize in proving the complementary theorem. It may furthermore be noted that the proof here given is characterized by the use made of the inequalities (12) and (13).

The orders of coincidence $\tau_s^{(\kappa)}$ furnished by the basis (τ) are connected with the orders of coincidence $\bar{\tau}_s^{(\kappa)}$ furnished by the complementary basis $(\bar{\tau})$ by the relations

$$\tau_s^{(\kappa)} + \bar{\tau}_s^{(\kappa)} = \mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}} \quad (s = 1, 2, \ldots, r_\kappa), \tag{3}$$

for finite values $z = a_\kappa$, and by the relations

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}} \quad (s = 1, 2, \ldots, r_\infty), \tag{4}$$

for the value $z = \infty$. By the aid of these relations we can replace (1) by

* "Proofs of certain General Theorems relating to Orders of Coincidence," *Proc. London Math. Soc.*, Ser. 2, Vol. 12. pp. 318–335, formula (24).

the equality

$$\lambda_\kappa = n i_\kappa + \tfrac{1}{2} \sum_{s=1}^{r_\kappa} (\tau_s^{(\kappa)} - \bar{\tau}_s^{(\kappa)}) \nu_s^{(\kappa)}, \tag{5}$$

for finite values of the variable z, and by

$$\lambda_\infty = n(i_\infty + 1) + \tfrac{1}{2} \sum_{s=1}^{r_\infty} (\tau_s^{(\infty)} - \bar{\tau}_s^{(\infty)}) \nu_s^{(\infty)}, \tag{6}$$

for the value $z = \infty$. We shall find it convenient to separate the basis (τ) into the partial bases $(\tau)'$ and $(\tau)^{(\infty)}$ corresponding to the finite values of z, and to the value $z = \infty$ respectively. In like manner we shall separate the basis $(\bar{\tau})$ into the corresponding partial bases $(\bar{\tau})'$ and $(\bar{\tau})^{(\infty)}$.

Writing

$$Q(z) = \prod_\kappa (z - a_\kappa)^{i_\kappa}, \tag{7}$$

where the product is extended to a finite number of values $z = a_\kappa$, we shall assume that the exponents i_κ have been taken sufficiently great to suit our purpose in what follows. We are at liberty also, if we will, to give to all these exponents the same value i. Let us now consider the function

$$H(z, u) = \frac{N(z, u)}{Q(z)} + P(z, u), \tag{8}$$

where $N(z, u)$ and $P(z, u)$ are polynomials in (z, u) of degree $n-1$ in u, the degree in z of $N(z, u)$ being less by 1 than the degree of $Q(z)$, and the degree in z of $P(z, u)$ being chosen sufficiently great and indicated by i_∞. The number of the conditions imposed on the general function $H(z, u)$ by the set of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ is evidently λ_κ. Also the conditions imposed on the general function of the form (8) by the several sets of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ included in the partial basis $(\tau)'$ are plainly independent of one another. The total number of the conditions imposed on the general function of the form (8) by the partial basis $(\tau)'$ will then be given by the sum

$$\sum_\kappa{}' \lambda_\kappa = n \sum_\kappa{}' i_\kappa + \tfrac{1}{2} \sum_\kappa{}' \sum_{s=1}^{r_\kappa} (\tau_s^{(\kappa)} - \bar{\tau}_s^{(\kappa)}) \nu_s^{(\kappa)}. \tag{9}$$

The number of the conditions imposed on the general function of the form (8) by the partial basis $(\tau)^{(\infty)}$ is given by the expression for λ_∞ in (6).

Let us now construct the product

$$H(z, u)\, \bar{H}(z, u), \tag{10}$$

in which we shall first suppose $\bar{H}(z, u)$ to be some specific function. In this product consider the principal residue relative to the value $z = \infty$. It is evidently not identically zero when the function $H(z, u)$ has been taken sufficiently general. On equating to zero the principal residue relative to the value $z = \infty$ in the product, we then impose a condition on the constant coefficients in the function $H(z, u)$.

We shall now suppose $\bar{H}(z, u)$ to be the general rational function built on the basis $(\bar{\tau})$. It then involves a certain number $N_{\bar{\tau}}$ of arbitrary constants. On equating to zero the principal residue relative to the value $z = \infty$ in the product (10), we then plainly impose on the constant coefficients in the function $H(z, u)$ precisely $N_{\bar{\tau}}$ conditions, the function $H(z, u)$ being supposed to be taken sufficiently general to begin with. These $N_{\bar{\tau}}$ conditions are evidently included under the λ_∞ conditions imposed on the function $H(z, u)$ by the partial basis $(\tau)^{(\infty)}$, for these λ_∞ conditions are obtained on equating to zero the principal residue relative to the value $z = \infty$ in the product of $H(z, u)$ by the general rational function conditioned by the partial basis $(\bar{\tau})^{(\infty)}$. The $N_{\bar{\tau}}$ conditions here in question are however also included under the conditions imposed on the general function $H(z, u)$ by the partial basis $(\tau)'$. For these conditions include the vanishing of the principal residues relative to the finite values $z = a_\kappa$ in the product (10), and therewith the vanishing of the principal residue relative to the value $z = \infty$ in this product.

We see then that at least $N_{\bar{\tau}}$ conditions are common to the λ_∞ conditions imposed by the partial basis $(\tau)^{(\infty)}$ on the general function $H(z, u)$, and to the $\sum_\kappa' \lambda_\kappa$ conditions imposed on this function by the partial basis $(\tau)'$. The total number of the conditions imposed on the general function $H(z, u)$ by the basis (τ) will then evidently be $\leqslant \sum_\kappa \lambda_\kappa - N_{\bar{\tau}}$. The total number of these conditions is, however, plainly $n\sum_\kappa \iota_\kappa + n - N_\tau$, obtained on subtracting N_τ from the total number of the arbitrary constants involved in the general function of the form of $H(z, u)$. We therefore have

$$n\Sigma\iota_\kappa + n - N_\tau \leqslant \Sigma\lambda_\kappa - N_\tau. \tag{11}$$

Substituting in this inequality for the numbers λ_κ the expressions given in formulæ (5) and (6), we obtain

$$N' - N_\tau \leqslant \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{\nu_\kappa}{}' (\tau_s^{(\kappa)} - \tau_s^{(\kappa)}) \nu_s^{(\kappa)}. \tag{12}$$

Interchanging (τ) and $(\bar{\tau})$ we arrive in like manner at the inequality

$$N_{\bar{\tau}} - N_{\bar{\tau}} \leqslant \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{\nu_\kappa}{}' (\bar{\tau}_s^{(\kappa)} - \tau^{(\kappa)}) \nu^{(\kappa)}. \tag{13}$$

Addition of the corresponding sides of the inequalities in (12) and (13) shows at once the inadmissibility of the inequality sign in both formulæ. Retaining therefore in both cases the sign of equality and subtracting corresponding sides of the equalities, we arrive at the complementary theorem which can immediately be stated in the form

$$N_\tau + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{\tau_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} = N_{\bar\tau} + \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{\bar\tau_\kappa} \bar\tau_s^{(\kappa)} \nu_s^{(\kappa)}. \tag{14}$$

It is evident that the theorem can also be stated in the equivalent form

$$M_\tau - \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{\tau_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} = M_{\bar\tau} - \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{\bar\tau_\kappa} \bar\tau_s^{(\kappa)} \nu_s^{(\kappa)}, \tag{15}$$

where M_τ and $M_{\bar\tau}$ designate the numbers of the conditions imposed by the bases (τ) and $(\bar\tau)$ respectively on a rational function of (z, u) of sufficiently general form. A sufficiently general function would evidently be any rational function which includes both the general rational function built on the basis (τ), and the general rational function built on the basis $(\bar\tau)$.

ON INTEGRALS AND DERIVATES WITH RESPECT TO A FUNCTION

By W. H. Young.

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1. From the fact that the derivates of the integral of a summable function differ from one another and from the summable function itself only at a set of content zero, it follows that any such function has the property that a *lu*- and an *ul*-function can be found which agree with it except at a set of content zero. This result is then a consequence of Lebesgue's celebrated theorem.

In my paper on "A New Method in the Theory of Integration," presented to the Society in May 1910,* I pointed out the possibility of reconstructing the whole theory of Lebesgue integration by fixing the attention primarily on *lu*- and *ul*-functions connected in this manner. More recently in a communication made to the Royal Society,† and in a paper published in these *Proceedings*,‡ I showed how, in point of fact, without employing Lebesgue's theorem, the *lu*- and *ul*-functions in question could be chosen to have the given function between them. The proof was only given for the case of a bounded function, but, if the hypothesis is made that the function is summable, the same argument shows that the property is possessed by any unbounded summable function.§ This is pointed out in the present paper, where, as in that last mentioned, the more general case of integration with respect to a function of bounded variation is considered.

The argument employed, of a very simple character, is closely con-

* *Proc. London Math. Soc.*, Ser. 2, Vol. 9, 1910, pp. 15–50.

† "On the New Theory of Integration," 1912, *Roy. Soc. Proc.*, A, Vol. 88, pp. 170–178.

‡ "On Integration with respect to a Function of Bounded Variation," 1913, *Proc. London Math. Soc.*, Ser. 2, Vol. 13, pp. 109–150.

§ The hypothesis that the function is summable is already unnecessarily wide, as is shown by the work of Egoroff and Lusin. I propose to return to this question shortly in connection with the integration of non-summable functions.

nected with one of my earlier definitions of integration. In this definition the ideas of Darboux are fully developed and given a form which shows the inevitableness of the modern theory of integration, while at the same time it renders evident the place where Riemann's considerations were inadequate.

The question naturally presents itself as to whether Lebesgue's important theorems as to relations between the derivates of a function and their primitive cannot be directly deduced from the new definition, without the use of Lebesgue's machinery; and the further question closely connected with the former, as to whether Lebesgue's theorem, which reveals to us the existence of the associated *lu*- and *ul*-functions, cannot be deduced from the existence of these functions, when this has been first proved independently.

Hitherto in the discussion which I have given of Lebesgue's theorems above referred to, and the extension of them which I have obtained, I have proceeded to a considerable extent on the lines marked out by Lebesgue himself, and not developed this branch of the general theory on the basis of my own point of view. There was, however, this difference that, unlike Lebesgue, I have eliminated the use of transfinite numbers, remaining in fact in the region of the finite, while I have at the same time made no use of the Theory of Sets of Points.*

A remarkably elegant treatment of the Lebesgue theorems has been published by De la Vallée Poussin.† In it the writer starts with a result due to himself, proving the existence of functions which he calls "majorant" and "minorant" to an integral, namely, such continuous functions as have their derivates respectively all greater and all less than the integrand, and approach the integral as near as we please. De la Vallée Poussin's proof avoids Transfinite Numbers, but employs reasoning in the Theory of Sets of Points analogous in character to that adopted by Lebesgue in his proofs of his theorems.

In the present communication I point out that this theorem of De la Vallée Poussin's is so closely connected with the account that I have given of the Theory of Integration that it must naturally be incorporated in it. Occasion is also taken to show that Lebesgue's theorem in the general form in which I have enunciated it, where the points at which the function is infinite may be any countable set, follows by the same reasoning, and that further the extension to the case in which the independent variable

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* "On Derivates and their Primitive Functions," 1912, *Proc. London Math. Soc.*, Ser. 2. Vol. 12, pp. 207–217, *et a. l.*

† *Loc. cit.*, *infra* (§ 15).

is any monotone increasing function holds equally. We thus obtain, in particular, an extremely concise and simple proof of the following theorem :—

If $f(x)$ is a derivate (right-hand or left-hand, extreme or intermediate) of a continuous function $F(x)$ with respect to a monotone increasing function $g(x)$, and if $f(x)$ is known to be summable with respect to $g(x)$ and finite, except at a countable set of points, then

$$F(x)-F(a) = \int_a^x f(x)\, dg(x).$$

More generally we have a pair of theorems of which one is the following :—

If $F(x)$ is a function which is

lower semi-continuous on the right,

and upper semi-continuous on the left,

and has a right-hand (left-hand) derivate $f(x)$ with respect to a monotone increasing function $g(x)$, which is summable with respect to $g(x)$ over the set of points at which $f(x) < 0$, then either

(i) *$f(x) = -\infty$ at a set of points of potency c :*

or (ii) *$F(x)$ is an upper semi-integral with respect to $g(x)$, $F(x)-F(a)$ exceeding the integral of $f(x)$ with respect to $g(x)$ over the set of points at which $f(x) < 0$ by a positive monotone increasing function.*

It is now some years since I first had occasion to lay stress on the importance of the concepts of upper and lower semi-continuity. Since then I have illustrated this importance in a variety of connexions, but the mathematical world at large does not seem yet to have realised the fruitfulness of these concepts, except in quite isolated instances. It will be seen in the present paper how the definition which I have given of generalised integration, based on the consideration of semi-continuous functions, leads at once to all the fundamental and now classical theorems of integration in the modern sense, and this not only when the integration is with respect to an ordinary variable, but also when this variable is replaced by any monotone increasing function $g(x)$, continuous or not. In the development of the proofs of these theorems, certain results are required with respect to the derivates of any semi-continuous function. One such theorem I have already had occasion to state and prove, in a form

somewhat less general, however, than is required for the present purpose. Besides this theorem there are others closely connected with them, but not directly required in the theory of integration. I have accordingly in this present communication given an account of all these theorems in as simple a form as possible. It will be seen that they constitute interesting generalisations of known theorems with regard to continuous functions, and serve of themselves to throw great light on the fundamental principles of the Differential Calculus. In the Theory of Sets of Points they only involve the notion of limiting point.

I may in particular call attention to the quadruplet of theorems of § 9, typified by the following :—

If $f(x)$ is upper semi-continuous on the right, the upper bound of any left-hand derivate is the same as that of the incrementary ratio, the derivates and incrementary ratio being ordinary or taken with respect to any monotone increasing function $g(x)$ without constant stretches : from which the generalised form of Dini's theorem follows as an immediate corollary.

2. We proceed first to prove the theorem about the *lu*- and *ul*-functions, without assuming that the function $f(x)$ is bounded.

THEOREM.—*If $f(x)$ is a summable function, we can find a lu-function $a(x) \leqslant f(x)$ and an ul-function $b(x) \geqslant f(x)$, both having the same integral.*

When $f(x)$ is a bounded function, this has been already proved, we assume therefore that $f(x)$ is unbounded. And first let $f(x)$ be a positive function. Let f_n denote the function which has the same value as $f(x)$, except where the latter is greater than n, where $f_n(x) = n$. Then

$$f_1 \leqslant f_2 \leqslant \ldots \to f$$

is a monotone ascending sequence of bounded functions, whose limit is f, and the limit of their integrals I is the integral of f, supposed summable.

Let a_n and b_n' denote respectively a positive u-function $\leqslant f_n$ and a positive l-function $\leqslant n$ and $\geqslant f_n$, the difference between the integrals of a_n and b_n' being less than $2^{-n}e$, where e is as small as we please. Then

$$\int_a^b a_n(x)\,dx, \quad \int_a^b f_n(x)\,dx, \quad \text{and} \quad \int_a^b b_n'(x)\,dx,$$

all have the same limit, namely I.

Now the function $a_n(x)$ is already a u-function $\leqslant f(x)$, whose integral, if n is sufficiently large, differs from that of f by as little as we please.

If, as n increases, these functions do not increase monotonely, we make them do so in the usual way, namely by making a_2 take the value of a_1 wherever the latter is the greater. We thus have at once a monotone ascending sequence of u-functions each $\leqslant f(x)$, the limit of whose integrals is I. *The limiting function $a(x)$ is therefore such a lu-function as is required.*

We have to modify the function $b'_n(x)$ rather more in order to obtain the second requirement of the theorem. If the succession of functions b'_n as n increases is not monotone ascending, we again modify it so as to make it so. For this purpose we increase the value of $b'_2(x)$ to that of $b'_1(x)$ wherever the latter is the greater, and similarly for the successive functions. Let the new functions, which now form a monotone ascending sequence, be denoted by

$$b_1 \leqslant b_2 \leqslant \ldots \to b_e(x),$$

then the limiting function $b_e(x)$ is an l-function, and is $\geqslant f(x)$, since each $b_n \geqslant f_n$, and f_n has f for limit.

Again, if S_r denote the set of points at which b'_r is the greatest of the functions $b'_1, b'_2, \ldots, b'_n$, we have $\int_a^b = \sum_{r=1}^{n} \int_{S_r}$; and therefore

$$\int_a^b \{b_n(x) - f_n(x)\}\, dn \leqslant \sum_{r=1}^{n} \int_a^b \{b'_r - f_n(x)\}\, dx \leqslant 2^{-1}e + \ldots + 2^{-n}e \leqslant e.$$

Hence $$\operatorname*{Lt}_{n\to\infty} \int_a^b b_n(x)dx \leqslant \operatorname*{Lt}_{n\to\infty} \int_a^b f_n(x)dx + e,$$

and therefore $$\leqslant \int_a^b f(x)dx + e,$$

whence $\int_a^b b_e(x)dx$ is greater than $\int_a^b f(x)dx$ by a quantity not greater than e. Since e is as small as we please, this proves that we have found an l-function $b_e(x) \geqslant f(x)$, whose integral differs from that of f by as little as we please.

Since e is at our disposal, we can, by giving it in succession the values of any monotone decreasing sequence with zero as limit, and modifying the corresponding functions in the usual way, obtain a monotone descending sequence of l-functions, each $\geqslant f(x)$, the limit of whose integrals is I. *The limiting function $b(x)$ is therefore such an ul-function as is required.*

This proves the theorem when $f(x)$ is a positive function.

3. In the general case we write as usual

$$f(x) = f_1(x) - f_2(x),$$

where $f_1(x)$ is zero wherever $f(x)$ is negative, and is equal to $f(x)$ elsewhere. Then, since f_1 and f_2 are both positive and summable, we can, by the preceding article, find a *lu*-function $g_1(x) \leqslant f_1(x)$, and having the same integral, and an *ul*-function $g_2(x) \geqslant f_2(x)$, and having the same integral. The difference $g_1 - g_2$ being a *lu*, and $\leqslant f(x)$, constitutes the first of the two functions sought. Similarly we obtain the other. Thus the theorem is true for any summable function $f(x)$.

4. THEOREM.*—*If a function $f(x)$ is upper semi-continuous on the right, the set of points at which*

$$f(x) \geqslant k$$

is closed on the left, and has therefore, in particular, an extreme left-hand end-point.

In fact, if $x_1, x_2, \ldots$, is a sequence of points on the right of its limiting point x, and such that, for all values of n,

$$f(x_n) \geqslant k,$$

one of the limits of f at the point x will be $\geqslant k$, and therefore, since each such limit is $\leqslant f(x)$, f being upper semi-continuous on the right,

$$f(x) \geqslant k.$$

This proves the theorem.

5. THEOREM.—*If $f(x)$ is upper semi-continuous on the right throughout an interval (a, b), and its value at the left-hand end-point a is less than or equal to that at the right-hand end-point b, that is*

$$f(a) \leqslant f(b),$$

then there is a point at which all the left-hand derivates of $f(x)$ are $\geqslant 0$ in the half-open interval $(a < x \leqslant b)$; these derivates being ordinary, or taken with respect to any monotone increasing function $g(x)$ without constant stretches.

If $f(x)$ is constant in any part of (a, b), the result is evident. If not, and if $f(a) = f(b)$, we can find a point c at which $f(c)$ has a different value, and in one of the two intervals (a, c) and (c, b) we shall have then the

* W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity," 1907, *Quarterly Journal of Mathematics*, Vol. 39, p. 72.

value at the left-hand end-point less than that at the right-hand end-point. It is therefore only necessary to prove the theorem under the hypothesis that

$$f(a) < f(b).$$

In this case, if k is any value greater than a and less than b, the set of points at which

$$f(x) \geqslant k$$

includes the point b, and does not include the point a; but it has an extreme left-hand end-point X, which accordingly lies in the half-open interval $(a < x \leqslant b)$. In the half-open interval $(a \leqslant x < X)$, therefore $f(x) < k$. Therefore

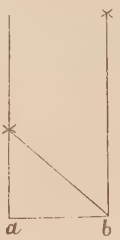
$$\{f(X)-f(X-h)\}/\{g(X)-g(X-h)\} \geqslant 0,$$

for sufficiently small positive values of h. Hence proceeding to the limit $h \to 0$, the derivates on the left at X are seen to be $\geqslant 0$. This proves the theorem.

6. It is easy to give an example of an upper semi-continuous function $f(x)$, such that the point X found in the preceding construction is always the point b itself. This example is also of importance in showing that the theorem states the utmost possible under the circumstances, whereas we know* that, if $f(x)$ is also lower semi-continuous on the left, the potency of the points mentioned in the enunciation, of which X is one is c.

Ex.—Let $f(a) = 1$, $f(b) = 2$, and in the completely open interval (a, b), let $f(x)$ be any upper semi-continuous function, all of whose values are $\leqslant 1$, for instance

$$f(x) = (b-x)/(b-a) \quad (a \leqslant x < b).$$



7. Theorem. — *If $f(x)$ is upper semi-continuous on the right, throughout the interval (a, b), and*

$$m = \{f(b)-f(a)\}/\{g(b)-g(a)\}$$

is the incrementary ratio of the interval with respect to a monotone increasing function $g(x)$ without constant stretches, then a point can be

* W. H. Young, "Term-by-Term Integration of Oscillating Series," 1909, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 104. See below, § 13.

found in the half-open interval $(a < x \leqslant b)$ *at which the left-hand derivates are* $\geqslant m$, *provided either* (i) $g(x)$ *is continuous on the right, or* (ii) $f(b) \geqslant f(a)$.

This results, in fact, from the preceding theorem, using instead of $f(x)$ the function

$$h(x) = f(x) - f(a) - m\{g(x) - g(a)\},$$

which is zero at a and at b, and is also upper semi-continuous on the right, under either of the hypotheses (i) and (ii). Indeed, in case (i), $g(x)$ is continuous on the right, so that $-mg(x)$ is also continuous on the right; while in case (ii), m being positive, $-mg(x)$ is upper semi-continuous on the right; thus in both cases $h(x)$ is the sum of two functions, each of which is upper semi-continuous on the right. The derivates of $h(x)$ with respect to $g(x)$ being the limits of

$$\{f(x+h) - f(x)\} / \{g(x+h) - g(x)\} - m,$$

differ only from the derivates of $f(x)$ by m. Hence there is a point at which the derivates of $f(x)$ on the left are $\geqslant m$.

8. From the preceding theorem, with its two cases, we get the following:—

THEOREM.—*If* $f(x)$ *is upper semi-continuous on the right throughout the interval* (a, b), *the upper bounds of the upper and lower left-hand derivates, or any intermediate left-hand derivate, are the same, and coincide with the upper bound of the incrementary ratio, the derivates being ordinary, or taken with respect to a monotone increasing function* $g(x)$, *provided either* (i) $g(x)$ *is continuous on the right, or* (ii) $f(x)$ *is not monotone decreasing throughout the interval.*

Let U be the upper bound of the incrementary ratio, so that

$$\{f(x+h) - f(x)\} / \{g(x+h) - g(x)\} \leqslant U,$$

when the points x and $x+h$ belong to the closed interval (a, b). Then, proceeding to the limit $h \to 0$, we see that

$$\text{any derivate of } f(x) \leqslant U. \tag{1}$$

Hence

$$\text{upper bound of any derivate of } f(x) \text{ with respect to } g(x) \leqslant U. \tag{2}$$

Now, in case (i), whatever points x and $x-h$ we take, we can, by the

case (i) of the preceding theorem, find a point t, where

$$x-h \leqslant t \leqslant x, \ \ldots, \ (h \geqslant 0),$$

such that

all the left-hand derivates of $f(t) \geqslant \{f(x)-f(x-h)\}/\{g(x)-g(x-h)\}$.

In case (ii) the same is true, provided $f(x-h) \leqslant f(x)$, it being certain that such points x and $x-h$ exist, since $f(x)$ is not monotone decreasing throughout the interval. Thus, in case (ii), U is certainly not negative, and there are lower left-hand derivates of $f(x)$, which are not less than any chosen *positive* incrementary ratio. We may therefore in either of the cases (i) and (ii) take the upper bounds of each side of the preceding inequality, and the same inequality will hold. Thus

$$\text{upper bound of any left-hand derivate of } f(x) \geqslant U. \tag{3}$$

From (2) and (3) we see that

$$\text{upper bound of any left-hand derivate of } f(x) = U,$$

which proves the theorem.

N.B.—It may be noticed that, although the theorem is stated for the closed interval (a, b), it holds equally when we omit one or both end-points a and b. Moreover, the upper bound is $+\infty$, unless $f(x)$ is lower semi-continuous on the left.

9. Corresponding to each of the theorems of §§ 4–8, we shall have three correlative theorems, relating to functions which are asymmetrically semi-continuous in the other possible modes. Without stating these theorems at length for the first three of the theorems, we give the enunciation of the complete fourfold theorem of § 8, in order to draw certain corollaries.

Theorems. — *If* $f(x)$ *is* $\left\{\begin{matrix} upper \\ lower \\ upper \\ lower \end{matrix}\right\}$ *semi-continuous on the* $\left\{\begin{matrix} right \\ left \\ left \\ right \end{matrix}\right\}$

throughout the interval (a, b), *the* $\left\{\begin{matrix} upper \\ upper \\ lower \\ lower \end{matrix}\right\}$ *bounds of the upper*

and lower derivate on the $\left\{\begin{matrix} left \\ right \\ right \\ left \end{matrix}\right\}$, *or any intermediate derivate or*

function, are the same and coincide with the $\left\{\begin{matrix} upper \\ upper \\ lower \\ lower \end{matrix}\right\}$ *bound of the incrementary ratio, the derivates being ordinary, or taken with respect to a monotone increasing function* $g(x)$ *without constant stretches, provided* (i) $g(x)$ *is continuous on the* $\left\{\begin{matrix} right \\ left \\ left \\ right \end{matrix}\right\}$, *or* (ii) $f(x)$ *is not monotone* $\left\{\begin{matrix} decreasing \\ decreasing \\ increasing \\ increasing \end{matrix}\right\}$ *throughout the interval.*

Combining these four theorems in pairs, we have the following six corollaries, forming three pairs of analogous theorems :—

COR. 1.—*If* $f(x)$ *is an upper semi-continuous function (u-function), the upper bound of any left-hand derivate is the upper bound of the incrementary ratio, and the lower bound of any right-hand derivate is the lower bound of the incrementary ratio, the derivates being ordinary, or taken with respect to* $g(x)$, *provided* $g(x)$ *is continuous and* monotone increasing without constant stretches.*

COR. 2.—*If* $f(x)$ *is a lower semi-continuous function (l-function), the lower bound of any left-hand derivate is that of the incrementary ratio, and the upper bound of any right-hand derivate is that of the incrementary ratio, the derivates being ordinary, or taken with respect to* $g(x)$, *provided* $g(x)$ *is continuous.*

COR. 3.—*If* $f(x)$ *is continuous on the right, the upper and lower bounds of any left-hand derivate are the same as those of the incrementary ratio, the derivates being ordinary, or taken with respect to* $g(x)$, *provided* $g(x)$ *is continuous on the right.*

COR. 4.—*If* $f(x)$ *is continuous on the left, the upper and lower bounds of any right-hand derivate are the same as those of the incrementary ratio, the derivates being ordinary, or taken with respect to* $g(x)$, *provided* $g(x)$ *is continuous on the left.*

* For brevity the words "monotone . . . stretches" are sometimes omitted, but must always be tacitly understood of the function $g(x)$.

Cor. 5.—*If $f(x)$ is upper semi-continuous on the left and lower semi-continuous on the right (augmentative),* the lower bound of any derivate is the same as the lower bound of the incrementary ratio, the derivates being ordinary or taken with respect to $g(x)$, provided $g(x)$ is continuous, or* (ii) *$f(x)$ is not monotone decreasing throughout the interval considered.*

Cor. 6.—*If $f(x)$ is lower semi-continuous on the left and upper semi-continuous on the right, the upper bound of any derivate is the same as that of the incrementary ratio, the derivates being ordinary or taken with respect to $g(x)$, provided $g(x)$ is continuous, or* (ii) *$f(x)$ is not monotone increasing throughout the interval considered.*

We might, of course, obtain four more corollaries by combining the four theorems three at a time, but it does not seem worth while to state them here. On the other hand, combining all four theorems, we get an extended form of Dini's theorem as a final corollary:—

Cor. 11 (*Dini's Theorem extended*).—*If $f(x)$ is a continuous function, the upper and lower bounds of any derivate are the same as those of the incrementary ratio, the derivates being ordinary or taken with respect to any continuous monotone increasing function of x.*

If the derivates are taken with respect to a discontinuous monotone increasing function $g(x)$, the theorem still holds for the upper bounds if $f(x)$ is not monotone decreasing, and for the lower bounds if it is not monotone increasing throughout the interval considered.

From the last result we have in particular the following:—

Cor. 12.—*If $f(x)$ is any continuous function, and $g(x)$ any monotone increasing function without constant stretches, the upper bound of the moduli of the derivates of $f(x)$ with respect to $g(x)$ coincides with the upper bound of the modulus of the incrementary ratio, being the same for every such derivate.*

It is easy to see that the extended theorem of Dini states the utmost possible. The following simple example shows that in an interval in which $f(x)$ is monotone increasing, the result is not necessarily true, when $g(x)$ is discontinuous:—

Ex.—Let $f(x) = x$;

and $g(x) = x \quad (0 \leqslant x \leqslant \frac{1}{2})$,

$g(x) = x+1 \quad (\frac{1}{2} < x)$.

* See below (§ 13).

Then on the left $f(x)$ has a differential coefficient with respect to $g(x)$, whose value is unity. On the right, however, there is a differential coefficient everywhere, whose value is unity, except at the point $\frac{1}{2}$, where it is zero.

The lower bound of the left-hand derivates is thus different from that of the right-hand derivates the latter only being equal to the lower bound of the incrementary ratio of $f(x)$ with respect to $g(x)$.

10. Let us consider the nature of the derivates of a function which is not upper semi-continuous on the right. Such a function $f(x)$ must be such that there is at least one point x which can be approached by a sequence of values $x+h$ $(h>0)$, for which

$$\operatorname*{Lt}_{h\to 0} f(x+h) > f(x), \text{ say } = f(x)+e;$$

for, in the contrary case, $f(x)$ is, by definition, upper semi-continuous on the right. Consequently, approaching the point x along this sequence, we get for the incrementary ratio $\{f(x+h)-f(x)\}/h$, the limit $+\infty$.

We see therefore that, *if $f(x)$ is not upper semi-continuous on the right, it has a right-hand derivate $+\infty$ at one point at least.*

The converse is, of course, not true; a function may have a right-hand derivate $+\infty$, and yet be upper semi-continuous on the right.

Similarly we have the alternate statements:—

If $f(x)$ is lower semi-continuous on the right, it has a right-hand derivate $-\infty$ at at least one point.

If $f(x)$ is upper (lower) semi-continuous on the left, it has a left-hand derivate $-\infty$ $(+\infty)$ at at least one point.

Another way of stating these results is evidently the following:—

If the right-hand derivates of $f(x)$ are bounded above, or even if they are finite above (that is never $+\infty$), $f(x)$ is upper semi-continuous on the right.

If the left-hand derivates of $f(x)$ are bounded above, or even finite above, $f(x)$ is lower semi-continuous on the left.

If the right-hand derivates of $f(x)$ are bounded below, or even finite below, $f(x)$ is lower semi-continuous on the right.

If the left-hand derivates of $f(x)$ are bounded below, or even finite below, $f(x)$ is upper semi-continuous on the left.

These results hold for derivates with respect to $g(x)$, provided $g(x)$ is continuous; or, if $g(x)$ is discontinuous, with exception made of the dis-

continuities of $g(x)$, which are, of course, countable. Thus, for instance, we have the following result :—

If the right-hand derivates of $f(x)$ with respect to $g(x)$ are finite above, $f(x)$ is upper semi-continuous on the right, except possibly at the discontinuities of $g(x)$.

11. Combining these results with those of the preceding articles, we get a number of interesting results, for instance, the following :—

Theorem.—*If $f(x)$ is upper semi-continuous on the right throughout the interval (a, b), and any one (and therefore each) of the left-hand derivates is bounded above, then the right-hand derivates are also bounded above, having the same upper bound as the left-hand derivate—viz., the upper bound of the incrementary ratio, while the function $f(x)$ is also lower semi-continuous on the left [being in fact a lower semi-integral].*

In fact, since $f(x)$ is upper semi-continuous on the right, the theorem of § 5 tells us that the incrementary ratio has the same upper bound as the left-hand derivates, and is therefore, by hypothesis, bounded. Hence the right-hand derivates, being limits of the incrementary ratio, are also bounded. It follows from the preceding article that $f(x)$ is lower semi-continuous on the left. Hence, by the second case of the theorem of § 9, the upper bound of the left-hand derivates is again the same as that of the incrementary ratio, and of the right-hand derivates.

The proof that $f(x)$ is, in fact, a lower semi-integral, does not belong to the present discussion; it is simply effected by using the method of De la Vallée Poussin in his proof of Lebesgue's theorem, and is given at length at another place.*

Another of these theorems is the following; it is unnecessary to give the proof at length :—

Theorem.—*If $f(x)$ is continuous on the right, and any one of its left-hand derivates is known to be bounded above and below, $f(x)$ is continuous on the left also [the upper function $f(x)$ being in fact an integral], and all the derivates have the same upper and lower bounds.*

These theorems hold for derivates with respect to $g(x)$, if $g(x)$ is continuous; or, when $g(x)$ is discontinuous, with exception made of the points at which $g(x)$ is discontinuous.

* *Infra*, § 15 *seq.*

12. We see from the theorem of § 9 that, *if $f(x)$ is upper semi-continuous on the left, and has a right-hand derivate which is not negative at any point of the interval (a, b), $f(x)$ is a monotone increasing function.* We may state this result in another way, using § 10—viz., *if the upper right-hand derivate of $f(x)$ is not negative at any point of the interval (a, b), and the lower left-hand derivate is nowhere $-\infty$, then $f(x)$ is a monotone increasing function.*

That the condition that $f(x)$ should be upper semi-continuous on the left [which is, of course, *necessary* in order that $f(x)$ should be monotone increasing], is in no way redundant in this enunciation, will be apparent if we consider the following simple example:—

Ex.—Let $$f(x) = x \quad (0 < x < \tfrac{1}{2}),$$

and $$f(x) = \tfrac{1}{2}(x - \tfrac{1}{2}) \quad (\tfrac{1}{2} \leqslant x \leqslant 1).$$

This function is not upper semi-continuous on the left at the point $\frac{1}{2}$, and at this point only.

It is instructive to prove the theorem of the present article by means of a chain of intervals; this enables us to illustrate the great care which has to be taken in the use of Cantor's numbers; arguments of this kind require the control of the intelligence at every step.

Since the upper right-hand derivate at each point of (a, b) is positive, or zero, we can, corresponding to each point x of that closed interval, find a positive value of h, such that

$$\{f(x+h) - f(x)\}/h \geqslant -e,$$

where e is any assigned positive quantity. Hence, by Lebesgue's Lemma, we can find a chain of these intervals, stretching from a to b, and containing every point of (a, b) as internal point, or as left-hand end-point, except possibly b. Over each of these intervals $(x, x+h)$ the increment of the function is greater than $-eh$. We might be tempted to jump to the conclusion that, adding all these increments together, we could assert that the total increment of the function over (a, b) was greater than $-e(b-a)$, and therefore $\geqslant 0$, since e is any positive quantity.

This argument would, however, be fallacious, since we have nowhere used the semi-continuity of $f(x)$ on the left, and, if this were superfluous, the conclusion would be true of the function given in the above example; which is not the case. The fallacy consists in the supposition that "what is true always is true in the limit." It is true that starting from a, we can add the increments corresponding to the abutting intervals of the chain, so that $x_1, x_2, \ldots$ be the left-hand end-points of these intervals

in order, the increment of $f(x)$ from a to x_n is for all values of n greater than $-e(x_n-a)$; if, however, x be the limiting point of the points x_n, we cannot assert that $f(x)-f(a)$ is not less than $-e(x-a)$, unless we also know that at least one of the limits of $f(x_n)$ is not greater than $f(x)$.

Since the position of the points x_n cannot be made to depend on that of x—for the points x_n are first determined, and x is their limiting point—we cannot be sure that any one of the limits of $f(x_n)$ is not greater than $f(x)$, unless we know that this is true of all the limits, that is, unless f is upper semi-continuous at the point x, and therefore at every point of (a, b), since the point x is in no way differentiated from other points of the half open interval $(a < x \leqslant b)$.

If, however, we know that f is upper semi-continuous on the left, the proof can be completed, by pointing out that, in this case, we can always proceed to the limit with our inequality

$$f(x_n)-f(a) \geqslant -e(x_n-a),$$

and that this is true not only for finite integers, but for all the numbers of Cantor's second class of transfinite ordinals.

This proves the theorem by this method.

13. We now pass to a different class of theorems, characterised by the presence among them of the extended form of Scheeffer's theorem.

Theorem.—*If $f(x)$ is, at every point of a closed interval (a, b),*

lower semi-continuous on the left,

and *upper semi-continuous on the right*

(in particular if it is a continuous function of x), and if the value at the left-hand end-point a is less than that at the right-hand end-point b, then the points at which the lower right-hand derivate of $f(x)$ with respect to a monotone increasing function $g(x)$ (without constant stretches) is positive (not zero), have the potency of the continuum. The same is true of the points at which the lower left-hand derivate is $\geqslant 0$.

By hypothesis $$f(a) < f(b),$$

we can therefore choose a quantity α greater than $f(a)$ and less than $f(b)$, and then a quantity β greater than α and less than $f(b)$.

Let us write $$m = (\beta-\alpha)/\{g(b)-g(a)\},$$

and $$c = \alpha-mg(a) = \beta-mg(b).$$

Then $g(x)$ is a monotone increasing function, and therefore

upper semi-continuous on the left,

and lower semi-continuous on the right,

and m is positive, since by hypothesis its numerator is so, and, $g(x)$ being monotone, increasing (without constant stretches), the same is true of the denominator. Thus $-mg(x)$ is a function of the opposite type to $g(x)$, that is, of the same type as $f(x)$, and therefore $f(x)-mg(x)+c$ is a function of the same type as $f(x)$, and therefore assumes all values between any two of its values.* But at the point a this function is negative, and at b it is positive, therefore between a and b it assumes the value zero, and all the values between $f(a)-mg(a)+c$ and zero. But the set of points at which $f(x)-mg(x)+c$ is $\leqslant 0$ has an extreme right-hand end-point, which must, of course, lie to the left of b, and be one of these zeroes, by reason of the type of the function $f(x)$; let it be the point X. Then from X to b the expression $f(x)-mg(x)+c$ is never negative, and therefore

$$f(X+h)-f(X)-mg(X+h)+mg(X) \geqslant 0.$$

But $g(X+h)-g(X)$ is not negative, since $g(x)$ is monotone increasing, and is not zero since $g(x)$ is not constant in any stretch; therefore we may divide by this quantity, and proceed to the limit $h \to 0$. We thus see that all the right-hand derivates of $f(x)$ at the point X are greater than or equal to m, and are therefore positive not zero.

Now, if we take another value of β, we shall get a different point X; for two curves

$$y = mg(x)+c,$$

and

$$y = Mg(x)+C,$$

which both pass through the point (a, a), have only the one point of intersection (a, a), since these two equations, together with the two got by giving to (x, y) the value (a, a), lead to the two equations

$$(m-M)\,g(x)+(c-C) = 0,$$

$$(m-M)\,g(a)+(c-C) = 0,$$

whence

$$(m-M)\{g(x)-g(a)\} = 0,$$

which, since $g(x)$ is monotone increasing without constant stretches cannot be satisfied by any value of x less than or greater than a.

* *Loc. cit.*, § 6, *supra*.

Thus to each possible value of β, that is, to each value of the interval (α, β), there corresponds one and only one point X; these points have therefore the potency c.

Similarly the set of points at which

$$f(x) - mg(x) + c \geqslant 0$$

has an extreme left-hand end-point, at which all the left-hand derivates are $\geqslant 0$, and the potency of these points is c.

This proves the theorem.

Cor.—*If $f(x)$ is at every point of a closed interval,*

lower semi-continuous on the left,

and *upper semi-continuous on the right,*

and if the lower right-hand (left-hand) derivate of $f(x)$ with respect to a monotone increasing function $g(x)$ without constant stretches is known to be everywhere $\leqslant 0$, except at points forming a set whose potency is not c (e.g., except at a countable set), then $f(x)$ is a monotone non-increasing function of x throughout the interval, so that at the exceptional points also the derivates with respect to $g(x)$ are $\leqslant 0$.

In fact, in this case the lower right-hand (left-hand) derivate of $f(x)$, with respect to $g(x)$ is $\leqslant 0$, except at a set whose potency is not c. Thus the conclusion of the preceding theorem cannot be drawn, whence it follows that its premisses are not satisfied, which shows that $f(b) \leqslant f(a)$. Since a and b are any points, this shows that $f(x)$ is a monotone non-increasing function of x, which proves the corollary.

14. Hence we get the following, which constitutes a considerable extension of the original theorem given by Scheeffer.

Theorem (*Extension of Scheeffer's Theorem*).—*If $f(x)$ and $h(x)$ are two functions, of which the first is throughout an interval,*

lower semi-continuous on the left

and *upper semi-continuous on the right,*

and the latter is *upper semi-continuous on the left,*

and *lower semi-continuous on the right,*

and if one of the right-hand (left-hand) extreme derivates of $f(x)$ with respect to a monotone increasing function $g(x)$ without constant stretches, be everywhere less than the corresponding derivate of $h(x)$, or equal to it and finite, except possibly at points forming a set whose potency is not c (e.g., a countable set), then

$$f(x)-h(x)$$

is a monotone non-increasing function of x throughout the interval.

Put $$k(x) = f(x)-h(x),$$

so that $k(x)$ is a function of the same type as $f(x)$.

Then, denoting by $f^+(x)$ and $f_+(x)$, the upper and lower right-hand derivates of $f(x)$ with respect to $g(x)$, and by h^+ and h_+, k^+ and k_+, those of $h(x)$ and $k(x)$ respectively, we have the inequalities*

$$k_+(x) \leqslant f_+(x)-h_+(x),$$

and also $$k_+(x) \leqslant f^+(x)-h^+(x),$$

with similar inequalities on the left.

Hence we see that, whether it is the upper or lower right-hand (left-hand) derivate alluded to in the statement of the theorem,

$$k_+(x) \leqslant 0,$$

(or $$k_-(x) \leqslant 0),$$

except at the doubtful set. Hence the result follows by the corollary to the preceding theorem.

15. The extension of Scheefer's theorem enables us to follow De la Vallée Poussin† in his proof of Lebesgue's theorem, and in this way to obtain the extensions of that theorem in the form already obtained by myself.‡ The following theorem completes that given in § 11.

* In fact, approaching the point x along any sequence, we have in the limit a certain derivate of $k(x)$ = a certain derivate of $f(x)$ − a certain derivate of $h(x)$. Hence, choosing the sequence in the first instance to give the lower right-hand derivate of $f(x)$, in the second case the upper right-hand derivate of $h(x)$, we get the above inequalities.

† *Cours d'Analyse Infinitesimale*, Vol. 1 (1914), Third Edition, p. 272. De la Vallée Poussin does not use u-functions; I have adapted his method to my own, as it follows naturally from my point of view.

‡ "On Derivates and their Primitive Functions," 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 208 *seq.*

Theorem.—*If $F(x)$ is a function which is*

lower semi-continuous on the right,

and *upper semi-continuous on the left,*

and has a right-hand (left-hand) derivate $f(x)$ with respect to $g(x)$, which is summable with respect to $g(x)$ over the set of points at which $f(x) < 0$, then either

(i) *$f(x) = -\infty$ at a set of points of potency c;*

or (ii) *$F(x)$ is an upper semi-integral with respect to $g(x)$, $F(x)-F(a)$ exceeding the integral of $f(x)$ with respect to $g(x)$ over the set of points at which $f(x) < 0$ by a positive monotone increasing function.*

Let us suppose that $f(x)$ is not $-\infty$ at every point of any set of potency c, and let its integral over the set of points at which $f(x) < 0$ with respect to $g(x)$ be I.

Then, since I is finite by hypothesis, we can find a negative u-function $a(x) \leqslant f(x)$, and having an integral

$$A(x) = \int_a^x a(x)\, dg(x),$$

as near I as we please, the upper bound of these integrals for different choices of the function $a(x)$ being I.

Then, since $a(x)$ is a u-function, the derivates with respect to $g(x)$ of $A(x)$ are all $\leqslant a(x)$,* and therefore are all negative and $\leqslant f(x)$.

Let us write $$h(x) = F(x)-F(a)-A(x),$$

then approaching the point x along such a sequence of points on the right (left) that the limit of the incrementary ratio of $F(x)$ with respect to $g(x)$

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* For $$\frac{A(x+h)-A(x)}{g(x+h)-g(x)} = \frac{\int_x^{x+h} a(x)\, dg(x)}{g(x+h)-g(x)}.$$

Now since $a(x)$ is a u-function, we can find h_0 so that, for all values of t numerically less than h_0 the values of the function $a(x+t)$ are less than $a(x)+e$, where e is as small as we please. Hence, provided $h \leqslant h_0$, the right-hand side of the above inequality is less than $a(x)+e$. Proceeding to the limit with h, we see that all the derivates of $A(x)$ with respect to $g(x)$, that is, all the limits of the left-hand side of the above inequality, are less than $a(x)+e$. Since e is as small as we please, this proves that *all the derivates of $A(x)$ with respect to $g(x)$ are $\leqslant a(x)$.*

is $f(x)$, we get in the limit an equation of the form

$$a\ certain\ derivate\ of\ h(x) = f(x) - a\ certain\ derivate\ of\ A(x).$$

Now the derivates of $A(x)$ are negative, therefore at a point at which $f(x) \geqslant 0$, the right hand side of this equation is $\geqslant 0$. Also the derivates of $A(x)$ are $\leqslant f(x)$, therefore the same is true at the points at which $f(x)$ is negative and finite. Hence, since the left-hand side of the equation does not exceed the upper right-hand derivate of $h(x)$ with respect to $g(x)$, we have

$$upper\ right\ (left)\ hand\ derivate\ of\ h(x) \geqslant 0,$$

except at most at the points of a set which has not the potency c.

But $h(x)$ is a function of the same type as $F(x)$, since $A(x)$, having all its right-hand derivates negative, is monotone decreasing. Therefore, by the Corollary to the theorem of § 13, $h(x)$ is a monotone increasing function, and therefore

$$F(x) - F(a) = A(x) + h(x)$$

is an upper semi-integral with respect to $g(x)$.

Taking a monotone ascending sequence of functions $a(x)$ whose integrals have as limit the integral $I(x)$ of $f(x)$ over the set at which it is negative with respect to $g(x)$, we get, since the limit of a sequence of monotone functions is a monotone function,

$$F(x) - F(a) = I(x) + a\ positive\ monotone\ increasing\ function\ of\ x.$$

This proves the whole theorem.

16. If in the theorem of § 15 we make the further assumption that the derivate $f(x)$ is itself summable with respect to $g(x)$, we may, in the proof, take $I(x)$ to denote $\int_a^x f(x)\,dx$, instead of only the positive part of this integral. In this case the whole argument holds, omitting, of course, that part of it which involves the derivates of $A(x)$ being negative. We get, therefore, as before,

$$a\ certain\ derivate\ of\ h(x) = f(x) - a\ certain\ derivate\ of\ A(x),$$

the right hand side of which inequality is $\geqslant 0$, since the derivates of $A(x)$ do not exceed $a(x)$, and are therefore $\leqslant f(x)$, provided at the point x we have not $f(x) = +\infty$ or $f(x) = -\infty$.

In order to use this result, we are obliged therefore to make the assumption that $f(x)$ is not infinite, whether positively or negatively, at

all the points of any set of potency c. The result obtained is evidently the following:—

Theorem.—*If $F(x)$ is a function which is*

lower semi-continuous on the right,

and *upper semi-continuous on the left,*

and has a right-hand (left-hand) derivate $f(x)$ with respect to $g(x)$ which is summable with respect to $g(x)$, then either

(i) *$f(x)$ is infinite, whether positively or negatively, at a set of points of potency c;*

or (ii) *$F(x)$ is an upper semi-integral with respect to $g(x)$, $F(x)-F(a)$ exceeding the integral of $f(x)$ with respect to $g(x)$ by a positive monotone increasing function of x.*

17. The alternative theorem need not be written out at length. Combining these two we get the following important theorem, the extension in the theory of integration with respect to a monotone increasing function of the celebrated theorem of Lebesgue, as extended by the present author. In consequence of the importance of this theorem, it is thought best to give the proof of it at length, this proof being for the rest by no means lengthy.

Theorem.—*If $F(x)$ is a function which is continuous throughout the interval (a, b), and has a right-hand (left-hand) derivate $f(x)$ with respect to the monotone increasing function $g(x)$ which is summable with respect to $g(x)$, then either*

(i) *$f(x)$ is infinite ($+\infty$ or $-\infty$ indifferently) at a set of points of potency c;*

or (ii) $F(x)-F(a) = \int_a^x f(x)\, dg(x)$.

Let us suppose that $f(x)$ is finite at some point of every set of points of potency c.

Then, since $\int_a^x f(x)\, dg(x)$ exists by hypothesis, we can find a u-function $a(x) \leqslant f(x)$ and an l-function $b(x) \geqslant f(x)$, whose integrals $A(x)$ and $B(x)$ differ from one another, and therefore from that of $f(x)$ by less than e; that is,

$$A(x)-e \leqslant \int_a^x f(x)\, dg(x) \leqslant B(x)+e. \qquad (1)$$

And since $a(x)$ is a u-function, the derivates of $A(x)$ with respect to $g(x)$ are all $\leqslant a(x)$, and therefore $\leqslant f(x)$. Let us write

$$h(x) = F(x) - F(a) - A(x).$$

Then approaching the point x along such a sequence of points on the right (left) that the limit of the incrementary ratio of $F(x)$ with respect to $g(x)$ is $f(x)$, we get in the limit an equation of the form

$$a \text{ certain derivate of } h(x) = f(x) - a \text{ certain derivate of } A(x).$$

Hence, at a point at which $f(x)$ is not infinite, we can certainly say that the upper derivate of $h(x)$ on the right (left) is not negative; by hypothesis, therefore, this is true except at a set of points whose potency is not c. But $h(x)$ is a continuous function, since $F(x)$ is continuous by hypothesis, and $A(x)$ is an integral, and therefore continuous. Hence, by the extended theorem of Scheeffer (§ 14), $h(x)$ is a monotone increasing function of x, and therefore also positive, since it is zero when $x = a$. Thus

$$A(x) \leqslant F(x) - F(a).$$

Similarly
$$F(x) - F(a) \geqslant B(x).$$

Therefore, by (1),

$$F(x) - F(a) - e \leqslant \int_a^x f(x)\,dx \leqslant F(x) - F(a) + e.$$

Since this is true, however small e may be, it follows that

$$F(x) - F(a) = \int_a^x f(x)\,dx,$$

which proves (ii). Thus, when (i) does not hold, (ii) is verified, which proves the theorem.

18. From the preceding theorem we get the following special case of an extension of a theorem of Lebesgue's to be proved subsequently (§ 20):—

THEOREM.—*The indefinite integral of a bounded function $f(x)$ with respect to $g(x)$ has $f(x)$ for differential coefficient with respect to $g(x)$ almost everywhere with respect to $g(x)$.**

* That is, except at a set over which the variation of $g(x)$ is zero. Such a set is not necessarily of content zero, nor *vice versa*. "Almost everywhere with respect to $g(x)$" is therefore not in general synonymous with "almost everywhere," nor is the one concept contained in the other. See W. H. Young, "On Integration with Respect to a Function of Bounded

In fact, if $$A \leqslant f(x) \leqslant B,$$

$$A \leqslant \frac{1}{g(x+h)-g(x)} \int_x^{x+h} f(t)\, dg(t) \leqslant B\,;$$

therefore, proceeding to the limit $h \to 0$, we see that the derivates of

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Variation," 1913, *Proc. London Math. Soc.*, Ser. 2, Vol. 13, p. 133. The example there quoted deals with a monotone function which is constant in stretches, the following example, due to Grace Chisholm Young, deals with one which has no such constant stretches.

Consider the identity

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots = \frac{2}{8} + 2\left(\frac{1}{8}\right)\left(\frac{2}{8}\right) + 2^2\left(\frac{1}{8^2}\right)\left(\frac{2}{8}\right) + 2^3\left(\frac{1}{8^3}\right)\left(\frac{2}{8}\right) + \ldots,$$

and for brevity let us denote the n-th term by $2^{n-1}u_n$, and the remainder by R_n, so that

$$u_n = 2^{-3n+1}, \quad R_n = 1/3.4^{2n}.$$

Let us construct the black intervals of a certain perfect set S in accordance with this equation. We take the interval (0, 1), divide it into eight equal parts and blacken the two middle ones, that is, in the middle of it we place a black interval of length $\frac{1}{4}$. In the middle of each of the white segments left over, we place by subdividing the divisions already obtained into eight parts, a black interval of length u_2; in each of the 2^2 white parts left over we place in the middle an interval of length u_3, and so on. We thus get a set of black intervals, dense everywhere, whose sum is $\frac{1}{3}$. The complementary set S is therefore a perfect set, dense nowhere, and of content $\frac{2}{3}$.

Let the left-hand end-points of the black intervals in the order constructed be

$$a_1,\ a_2,\ \ldots,$$

those black intervals which are of the same length being supposed constructed in order from left to right. Let the corresponding right-hand end-points be

$$b_1,\ b_2,\ \ldots,$$

We put $$g(a_1) = \tfrac{1}{2}R_1 = \frac{1}{96}, \qquad g(b_1) = g(a_1) + u_1 = \frac{1}{96} + \frac{1}{4} = \frac{1}{3} - \frac{1}{96},$$

$$g(a_2) = \tfrac{1}{4}R_2 = \frac{1}{3.4^5}, \qquad g(b_2) = g(a_2) + u_2,$$

$$g(a_3) = g(b_1) + \tfrac{1}{4}R_2, \qquad g(b_3) = g(a_3) + u_2,$$

and so on. In each case $g(b_n) - g(a_n)$ is the length of the interval (a_n, b_n), say u_k, and, if b is the next end-point of a black interval on the left of a_n, and with lower index $(r < n)$, $g(a_n) - g(b_r) = 2^{-k}R_k$.

The values of $g(x)$ so assigned are evidently monotone increasing, and they are such that in any interval of length e containing a point at which the value of $g(x)$ has been assigned, the oscillation of $g(x)$ as at present defined is less than e. Hence we may extend these values to the remaining points of the set S by continuity, and we shall have $g(x)$ defined as a monotone increasing and continuous function over the whole set S.

In each of the closed black intervals of S we now make $g(x)$ linear. The graph $y = g(x)$ consists accordingly in each black interval of a line inclined at an angle of 45° to the axis of x.

Now let us surround each point of S by a small interval with that point as centre and of length u_k. The finite number of non-overlapping intervals so covered over may be divided into two parts:—

(i) the finite number of intervals outside the $2^k - 1$ black intervals of length not less than u_k;

(ii) the parts of length $\frac{1}{2}u_k$ at each end of each of those $2^k - 1$ black intervals.

$\int_a^x f(x)\,dg(x)$ with respect to $g(x)$ are bounded.* Hence, by the preceding theorem, $\int_a^x f(x)\,dg(x)$ is the integral of any one of its derivates with respect to $g(x)$, which proves that any one of those derivates agrees with $f(x)$, except at a set of content zero with respect to $g(x)$. Thus, in particular, the upper and lower left- and right-hand derivates of $\int_a^x f(x)\,dg(x)$ with respect to $g(x)$ agree with $f(x)$, except at the points of four sets of content zero, that is, except at a set of content zero with respect to $g(x)$. In other words, $\int_a^x f(x)\,dg(x)$ has a differential coefficient with respect to $g(x)$, almost everywhere equal to $f(x)$. This proves the theorem.

19. We can now extend another theorem of Lebesgue's :—

THEOREM.—*If $F(x)$ is a monotone ascending (descending) function of x in the interval (a, b), each of its right (left) hand derivates with respect to $g(x)$ is summable with respect to $g(x)$, and $F(x)$ exceeds (is less than) the integral with respect to $g(x)$ of any one of its right (left) hand derivates with respect to $g(x)$ by a positive monotone increasing function.*

Let $F(x)$ be monotone increasing, and $f(x)$ any one of its derivates with respect to $g(x)$ on the right. Then $f(x)$ is necessarily positive or zero.

Let $f_n(x)$ be defined as equal to $f(x)$, except where $f(x)$ is greater than n, where $f_n(x) = n$. Then

$$0 \leqslant f_n(x) \leqslant n.$$

Let $a(x)$ denote a u-function $\geqslant 0$ and $\leqslant f_n(x)$, and let $A(x)$ denote its

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By construction the increments of $g(x)$ over the intervals (i) sum up to R_k.

The graph in the parts (ii) being a straight line inclined at 45° to the axis, it follows that the increment of $g(x)$ in each of the parts (ii) is $\frac{1}{2}u_k$, their sum is therefore less than

$$2^k u_k = 2^{-2k+1}.$$

Thus the increment of $g(x)$ over the chosen set of intervals enclosing S is less than $R_k + 2^{-2k+1}$, which, by choosing k sufficiently large is as small as we please. Thus, by definition, the variation of $g(x)$ over the set S is zero, that is, the content of S with respect to $g(x)$ is zero, whereas, as we have seen, its content proper is $\frac{2}{3}$.

The function $g(x)$ which we have constructed is evidently monotone increasing without constant stretches.

* The upper and lower derivates on either side of a continuous function, being *ul*- and *lu*-function, are summable, if bounded.

integral with respect to $g(x)$. Then the derivates of $A(x)$ with respect to $g(x)$ are all $\geqslant 0$ and $\leqslant a(x)$, and therefore $\leqslant f_n(x)$.

Therefore we may apply the theorem of § 12; for the monotone ascending function $F(x)$ is

lower semi-continuous on the right,

and upper semi-continuous on the left,

and the integral $A(x)$ is a continuous function. Thus

$$F(x)-F(a)-A(x)$$

is a monotone increasing function of x, and therefore positive, since it is zero at $x = a$.

Let us write $$F(x)-F(a) = A(x)+M'(x).$$

Then taking for $a(x)$ in succession different u-functions, so that $A(x)$ approaches as limit its upper bound, we get

$$F(x)-F(a) = \int_a^x f_n(x)\, dg(x)+M(x),$$

where $M(x)$ is a positive monotone increasing function, namely, the limit of $M'(x)$. Since this holds for all values of n, it follows that as $x \to \infty$, $\int_a^x f_n(x)\, dg(x)$ has a finite upper bound $\leqslant F(x)-F(a)$; this is therefore $\int_a^x f(x)\, dg(x)$, by definition, $f(x)$ being in this case summable with respect to $g(x)$. Thus, proceeding to the limit $n \to \infty$,

$$F(x)-F(a) = \int_a^x f(x)\, dg(x) + \textit{a positive monotone increasing function of } x.$$

This proves the theorem in this case. Similarly, or by changing signs, it is seen to be true when $F(x)$ is monotone decreasing.

Cor.—*The right (left) hand derivates with respect to $g(x)$ of a function of bounded variation are summable with respect to $g(x)$.*

20. We can now prove the extension of the theorem of Lebesgue's alluded to in § 18:—

Theorem.—*Any Lebesgue integral with respect to $g(x)$ is the integral with respect to $g(x)$ of any one of its right-hand (left-hand) derivates with respect to $g(x)$.*

Let $$F(x)-F(a)=\int_a^x f(x)\,dg(x).$$

It is clearly sufficient to prove the theorem when $f(x)\geqslant 0$, since in the contrary case we only have to express $f(x)$ as the difference of two such functions, and apply the theorem, supposed proved for a positive integrand.

We assume then that $$0\leqslant f(x),$$

and write $f_n(x)$ for the function which is equal to $f(x)$ except where the latter is greater than n, where $f_n(x)$ has the value zero. Then, writing

$$F_n(x)-F_n(a)=\int_a^x f_n(x)\,dg(x),$$

it follows that $f_n(x)$ being bounded, $F_n(x)$ has $f_n(x)$ for differential coefficient, except at a set of content zero (§ 16). But

$$\frac{F_n(x+h)-F_n(x)}{g(x+h)-g(x)}=\frac{\int_x^{x+h} f_n(x)\,dg(x)}{g(x+h)-g(x)}\leqslant\frac{\int_a^x f(x)\,dg(x)}{g(x+h)-g(x)}\leqslant\frac{F(x+h)-F(x)}{g(x+h)-g(x)}.$$

Hence approaching the point x along such a sequence of points on the right as to give any chosen derivate with respect to $g(x)$, *e.g.*, $F_+(x)$, we have, except at a set of content zero,

$$f_n(x)\leqslant F_+(x).$$

Since this is true for all values of n, and the exceptional sets add up to a set of content zero, we have, almost everywhere,

$$f(x)\leqslant F_+(x). \tag{1}$$

But since $f(x)$ is not negative, $F(x)$ is a monotone increasing function, and therefore exceeds the integral of any one of its derivates by a positive monotone increasing function, which, being in consequence the difference of two integrals, is an integral, say

$$F(x)-F(a)=\int_a^x F_+(x)\,dg(x)+\int_a^x q(x)\,dg(x)=\int_a^x [F_+(x)+q(x)]\,dg(x).$$

Hence $$\int_a^x [f(x)-F_+(x)-q(x)]\,dg(x)=0.$$

This shows that, except at a set of content zero with respect to $g(x)$,

$$f(x)-F_+(x)-g(x)=0\,;$$

but $g(x)$ is $\geqslant 0$, since its integral is monotone increasing; therefore

$$f(x) \geqslant F_+(x). \tag{2}$$

From (1) and (2) the result follows,

$$f(x) = F_+(x),$$

which proves the theorem for the right-hand derivates, and similarly for the left-hand derivates.

Cor. 1.—*The integral of $f(x)$ with respect to $g(x)$ has $f(x)$ for differential coefficient with respect to $g(x)$ almost everywhere with respect to $g(x)$.*

Cor. 2.—*The differential coefficient of $\int_a^x f(x)\,dg(x)$ with respect to x is $f(x)\,g'(x)$, except at a set of content zero and a set of content zero with respect to $g(x)$.*

21. From the preceding we get at once the following theorem, extended from another extension of a celebrated theorem of Lebesgue :—

Theorem.—*A continuous function of bounded variation one of whose right-hand (left-hand) derivates, extreme or intermediate, with respect to $g(x)$ is finite, except at points forming a set which is not of potency c, is the integral with respect to $g(x)$ of any one of its derivates with respect to $g(x)$.*

This follows at once from § 17 and the preceding article. For the derivate specified in the enunciation is summable by § 19.

22. The function of bounded variation considered in the preceding theorem has then a finite differential coefficient almost everywhere with respect to $g(x)$. It will now be shown that this is a property of the most general function of bounded variation, although in the case when it is not an integral, the exceptional set at which any one of the derivates is infinite must, by the preceding theorem, have the potency c.

Theorem.—*If $f(x)$ is a function of bounded variation, continuous or discontinuous,* it has a finite differential coefficient with respect to $g(x)$ except at a set of content zero with respect to $g(x)$.*

* "A Note on Derivates and Differential Coefficients," 1912, *Acta Math.*, 1913, Vol. 37, p. 144 *seq.* The derivates here treated are ordinary, but the argument holds without change for derivates with respect to $g(x)$. (G. E. C. Y.)

It is clearly sufficient to prove the theorem when $F(x)$ is monotone increasing, since in the contrary case $F(x)$ is the difference of two such functions. Let, then, $F(x)$ be monotone increasing. Then, by § 17,

$$F(x)-F(a)=\int_a^x F^+(x)\,dg(x)+M(x),$$

where $F^+(x)$ is the upper derivate on the right of $F(x)$ with respect to $g(x)$ and $M(x)$ is a monotone increasing function. Hence

$$\frac{F(x+h)-F(x)}{g(x+h)-g(x)} \geqslant \int_x^{x+h} \frac{F^+(x)\,dg(x)}{g(x+h)-g(x)}.$$

Letting h assume such a sequence of positive values with zero as limit, that the left-hand side of this inequality approaches the lower right-hand derivate $F_+(x)$, we get except at a set of content zero with respect to $g(x)$,

$$F_+(x) \geqslant F^+(x),$$

which can only be satisfied if $F(x)$ has a right-hand differential coefficient with respect to $g(x)$ at the point x.

Again, since in the first equation, we might have taken any other derivate instead of $F^+(x)$ (§ 19), we get by the same argument, almost everywhere

$$F_+(x) \geqslant F^-(x).$$

But, by a theorem of Grace Chisholm Young's, $F_+(x)$ cannot be greater than $F^-(x)$ except at a countable set of points,* hence we must have

$$F_+(x)=F^-(x),$$

almost everywhere.

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* In De la Vallée Poussin's enunciation of this theorem he hypothecates, with Lebesgue, the *continuity* of the function of bounded variation considered; it was pointed out some years ago by W. H. Young that this hypothesis was unnecessary, and a proof of the general theorem given ("On Functions of Bounded Variation," 1910, *Quarterly Journal*, Vol. 42, p. 79).

The proof given by De la Vallée Poussin, depending as it does on his former theorems, seems to involve the continuity, though this is never explicitly referred to in the proof. On the other hand, the proof itself is not quite complete as printed. De la Vallée Poussin denotes by Λ_F any chosen derivate of the (continuous) function of bounded variation $F(x)$; say, for definiteness, the upper right-hand derivate. He then defines a function $\phi(x)$ as follows

$$\phi(x)=F(x)-\int_a^x \Lambda_F dx. \qquad (1)$$

Now it is only after proving the present theorem that we can assert that, if we take another derivate instead of the upper right-hand derivate for Λ_F, the function $\phi(x)$ given by the new equation (1) will be the same as before. De la Vallée Poussin proceeds as follows:

Similarly, interchanging left and right,

$$F_-(x) = F^+(x),$$

almost everywhere. Hence, finally, all the four derivates are equal, and therefore $F(x)$ has a differential coefficient with respect to $g(x)$, except at a set of points of contact zero. This proves the theorem.

—He writes

$$\text{“}\psi(x) = \phi(x) - \int_a^x \Lambda_\phi \, dx,$$

and points out that ϕ and ψ are monotone increasing functions of x. Hence we get

$$\int_a^x (\Lambda_F + \Lambda_\phi) \, dx = F(x) - \psi(x),$$

whence, remembering that, by a theorem already proved, the left hand side has its integrand for differential coefficient almost everywhere, we see that, except at a set of content zero

$$\Lambda_F + \Lambda_\phi \leqslant \Lambda_F,$$

whence it follows that, almost everwhere, $\Lambda_\phi = 0$.”

Hence, says De la Vallée Poussin, “we conclude that ϕ has zero for differential coefficient almost everywhere, since Λ was any derivate we chose.” It will be seen from the remark made above that this conclusion cannot immediately be drawn without involving a *petitio principii*. The conclusion can, however, for a different reason, be drawn if Λ_ϕ be one of the two upper derivates, since ϕ is a monotone increasing function.—(G. E. C. Y.)

[*Note added July 25th*, 1915.—Prof. De la Vallée Poussin in a recent letter to me has expressed himself as in agreement with these remarks.—(G. E. C. Y.)]

THE EFFECT ON THE TIDES OF VARIATION IN THE DEPTH OF THE SEA

By G. R. Goldsbrough.

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1. In his book on *Tides and Waves*, Airy, after criticising and dismissing both the equilibrium theory and Laplace's dynamical theory, proceeded to substitute his "canal" theory. By stripping off many of the circumstances which modify, in a small way, the tides as they exist, and leaving only the most essential features, he was able to reduce the problem to an extremely simple form. In this way he accounted, qualitatively, for many of the facts of the tides. One point, however, he left unsatisfactory. The harmonic analysis of tidal observations shows that each component tide differs in phase from the corresponding component of tide-producing force. Airy thought he could account for this by assuming the existence of friction. But others doubt whether friction could produce an effect so large as that actually found.* Further, in all the dynamical tidal problems solved so far, not one presents this large phase-difference in the solution. The conjecture has been made that it is due to the irregularities of the continental coast-lines.†

The note here presented is to show that irregularities in the depth may produce just such an effect.‡ Airy's equations are taken with the additional supposition that the depth is no longer constant, but is a function of the position in the canal. A fairly general expression is taken to represent the irregularities of the depth and a general form of solution given. A simple numerical example is completely worked out.

The results show not only the existence of a phase-difference, but also that the tide range varies from point to point—another characteristic of terrestrial tides.

Since the above was written, a paper§ has been published in which it

* Lamb, *Hydrodynamics*, p. 526.

† Lamb, *Hydrodynamics*, p. 527.

‡ This was indicated in a general way by Lamb, *Hydrodynamics*, p. 334.

§ Prof. Lamb and Miss L. Swain, "On a Tidal Problem," *Phil. Mag.*, June 1915.

is shown that phase-differences may arise in canals of uniform depth by stopping the ends. Attention is also drawn to the fact that Airy actually treated a problem of this kind, but apparently did not fully examine it;* so that the statement made above that the phenomenon does not appear in any of the dynamical tidal problems hitherto solved is not quite correct.

2. Consider an extension of Airy's canal theory of the tides to the case where the depth is variable but the breadth uniform. Let the canal be a small circle in co-latitude θ. If ξ be the displacement along the canal of a particle at an undisturbed distance s from a fixed meridian, and the tide height above the normal level there be ζ, then the dynamical equation of the system is

$$\frac{\partial^2 \xi}{\partial t^2} = -g \frac{\partial \zeta}{\partial s} + X, \tag{1}$$

where X is the component of the disturbing force along the canal.

The equation of continuity is

$$\zeta = -\frac{\partial (h\xi)}{\partial s}. \tag{2}$$

In the case of the semidiurnal tides we have for X the expression

$$-H \sin\left(\sigma t + \frac{2s}{R \sin\theta} + \epsilon\right). \tag{3}$$

If the longitude of the point s is ϕ, we have

$$\phi R \sin\theta = s.$$

Hence for (1), (2), and (3), we have the more convenient forms

$$\frac{\partial^2 \xi}{\partial t^2} = -\frac{g}{R \sin\theta} \frac{\partial \zeta}{\partial \phi} + X, \tag{1'}$$

$$\zeta . R \sin\theta = -\frac{\partial (h\xi)}{\partial \phi}, \tag{2'}$$

$$X = -H \sin(\sigma t + 2\phi + \epsilon). \tag{3'}$$

Let $\eta = h\xi$, and eliminate ζ from (1') and (2'). Then

$$\frac{1}{h} \frac{\partial^2 \eta}{\partial t^2} = \frac{g}{R^2 \sin^2\theta} \frac{\partial^2 \eta}{\partial \phi^2} - H \sin(\sigma t + 2\phi + \epsilon). \tag{4}$$

* *Tides and Waves*, Art. 301.

In choosing an expression for the law of depth there is considerable latitude. The following form is convenient in that it leads to equations of well-known form. Let

$$h^{-1} = h_0^{-1}\{1+2\Sigma a_r \cos r\phi\}. \tag{5}$$

Under the sign of summation there may be either a finite or an infinite number of terms. In the latter case the series must be absolutely and uniformly convergent. This assumption as to the law of depth is fairly general. In a practical case where the depth was known at n points the expression could be constructed with h_0 and $(n-1)$ coefficients a, so as to coincide closely with the actual contour of the canal bed.

Now consider an equation similar in form to (4), viz.,

$$\frac{d^2\eta}{d\phi^2} + \left\{\sum_0^\infty b_r \cos r\phi\right\}\eta = H\sin(p\phi+\varpi). \tag{6}$$

This equation has a well-known form and has been very fully discussed.* The complementary function is a convergent series

$$\eta = \sum_{-\infty}^{\infty} B_r \cos\{(k+r)\phi+a\};$$

where k is a constant to be found from a certain infinite determinantal equation; B_0 and a are arbitrary constants.

The particular integral takes the form

$$\eta = \sum_{-\infty}^{\infty} C_r \sin\{(p+r)\phi+\varpi\}.$$

Referring back to (4), substituting the value of h from (5), and putting

$$gh_0/R^2\sin^2\theta = \beta,$$

we have $$\{1+2\Sigma a_r\cos r\phi\}\frac{\partial^2\eta}{\partial t^2} = \beta\frac{\partial^2\eta}{\partial\phi^2} - Hh_0\sin(\sigma t+2\phi+\epsilon). \tag{7}$$

A comparison of (6) and (7) shows that the solution of (7) for the free modes is of the form

$$\eta = \sum_{-\infty}^{\infty} A_m \cos\{\sigma t+(k+m)\phi+\epsilon\}. \tag{8}$$

In this expression, k is arbitrary and σ is to be determined.

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* Forsyth, *Theory of Differential Equations*, Vol. IV, ch. viii; also, p. 431.

On substituting in (7) (omitting the term in H), we find

$$\sigma^2 \Sigma A_m \cos\{\sigma t+(k+m)\phi+\epsilon\}$$

$$+\sigma^2 \Sigma a_r \Sigma A_m [\cos\{\sigma t+(k+m+r)\phi+\epsilon\}+\cos\{\sigma t+(k+m-r)\phi+\epsilon\}]$$

$$-\Sigma\beta(k+m)^2 A_m \cos\{\sigma t+(k+m)\phi+\epsilon\} = 0.$$

From which, on equating coefficients, we find that

$$\{\sigma^2-\beta(k+m)^2\} A_m+\sigma^2 \sum_{r=1}^{\infty} a_r A_{m-r}+\sigma^2 \sum_{r=1}^{\infty} a_r A_{m+r} = 0. \qquad (9)$$

The infinite determinant of these expressions equated to zero would give the various frequencies. However, as the forced tides present more points of importance, we pass on to them.

Equation (7) represents the form for use in the case of the semi-diurnal tides. The solution is

$$\eta = \sum_{-\infty}^{\infty} B_m \sin\{\sigma t+(2+m)\phi+\epsilon\},$$

where the quantities B have to be uniquely determined.

On substitution as before we find

$$B_0\left\{1-\frac{2^2\beta}{\sigma^2}\right\}+\sum_{1}^{\infty} a_r B_r+\sum_{1}^{\infty} a_r B_{-r} = +\frac{h_0 H}{\sigma^2}; \qquad (9)$$

and thereafter

$$B_m\left\{1-\frac{(2+m)^2\beta}{\sigma^2}\right\}+\sum_{r=1}^{\infty} a_r B_{m+r}+\sum_{r=1}^{\infty} a_r B_{m-r} = 0, \qquad (10)$$

for all values of m, positive and negative.

From (9) and (10), the quantities B can be determined in succession. If in the expression for the depth the coefficients a are small compared with unity and diminish rapidly, the series for η is also rapidly convergent. An example of the calculation in a simple case is given later.

The form of expression for the tide height is now

$$-\xi . R \sin\theta = \frac{\partial\eta}{\partial\phi} = \sum_{-\infty}^{\infty} B_m(2+m)\cos\{\sigma t+(2+m)\phi+\epsilon\}. \qquad (11)$$

This can be written

$$-\xi . R \sin\theta = C\cos\{\sigma t+2\phi+\epsilon+\psi\}, \qquad (12)$$

where $$C\cos\psi = 2B_0 + \sum_{1}^{\infty}\{(2+r)B_r + (2-r)B_{-r}\}\cos r\phi, \tag{13}$$

and $$C\sin\psi = \sum_{1}^{\infty}\{(2+r)B_r - (2-r)B_{-r}\}\sin r\phi. \tag{14}$$

The expression (12) is a wave form with an amplitude which is a function of the position, and a phase different from that of the tide-producing force by an amount which is also a function of the position. At a given place there will be a simple harmonic rise and fall of the water surface, but the tide range will differ from point to point. Further, except at special places neither high nor low water will occur when the moon is on the meridian. It is easily seen by reference to (14) that such special places are $\phi = 0$ and π; but that, in general, no others occur.

3. As a simple illustration of these points I have worked out in detail the case where the depth has only *one* variable term, viz.,

$$h^{-1} = h_0^{-1}(1 + 0{\cdot}8\cos 4\phi). \tag{15}$$

Only a_1 occurs here; and the law is such that the greatest depth is nine times the least. This should serve to exaggerate any peculiarities that may appear.

Further, assume that the canal is in latitude 60°. The solution then takes the form

$$\eta = \sum_{-\infty}^{\infty} B_m \sin\{\sigma t + (2+4m)\phi + \epsilon\}. \tag{16}$$

The relations, (9) and (10), between the coefficients become

$$B_0\left\{1 - \frac{2^2\beta}{\sigma^2}\right\} + a_1B_1 + a_1B_{-1} = \frac{h_0H}{\sigma^2} = H', \tag{17}$$

and afterwards, for all values of m from $-\infty$ to $+\infty$,

$$B_m\left\{1 - \frac{(2+4m)^2\beta}{\sigma^2}\right\} + a_1B_{m-1} + a_1B_{m+1} = 0. \tag{18}$$

If we put $$-M_r = \frac{1 - (2+4r)^2\,\beta/\sigma^2}{a_1},$$

$$P_m = \frac{B_{m+1}}{B_m},$$

and $$Q_{-m} = \frac{B_{-m-1}}{B_{-m}}$$

the relation (18) gives, for determination of the ratios P_m and Q_{-m},

$$1/P_{m-1} = M_m - P_m,$$

and

$$1/Q_{-m+1} = M_{-m} - Q_{-m}.$$

The determination of P_{m-1} involves an infinite continued fraction

$$P_{m-1} = \frac{1}{M_m} - \frac{1}{M_{m+1}} - \frac{1}{M_{m+2}} - \dots \text{ ad inf.} \tag{19}$$

Owing to the form of M_m for increasing values of m it is easy to establish that this fraction converges for all values of m. The same remarks apply to the determination of Q_{-m+1}.

The coefficient B_0 is readily determined from (17); with this and the ratios P and Q the other coefficients follow.

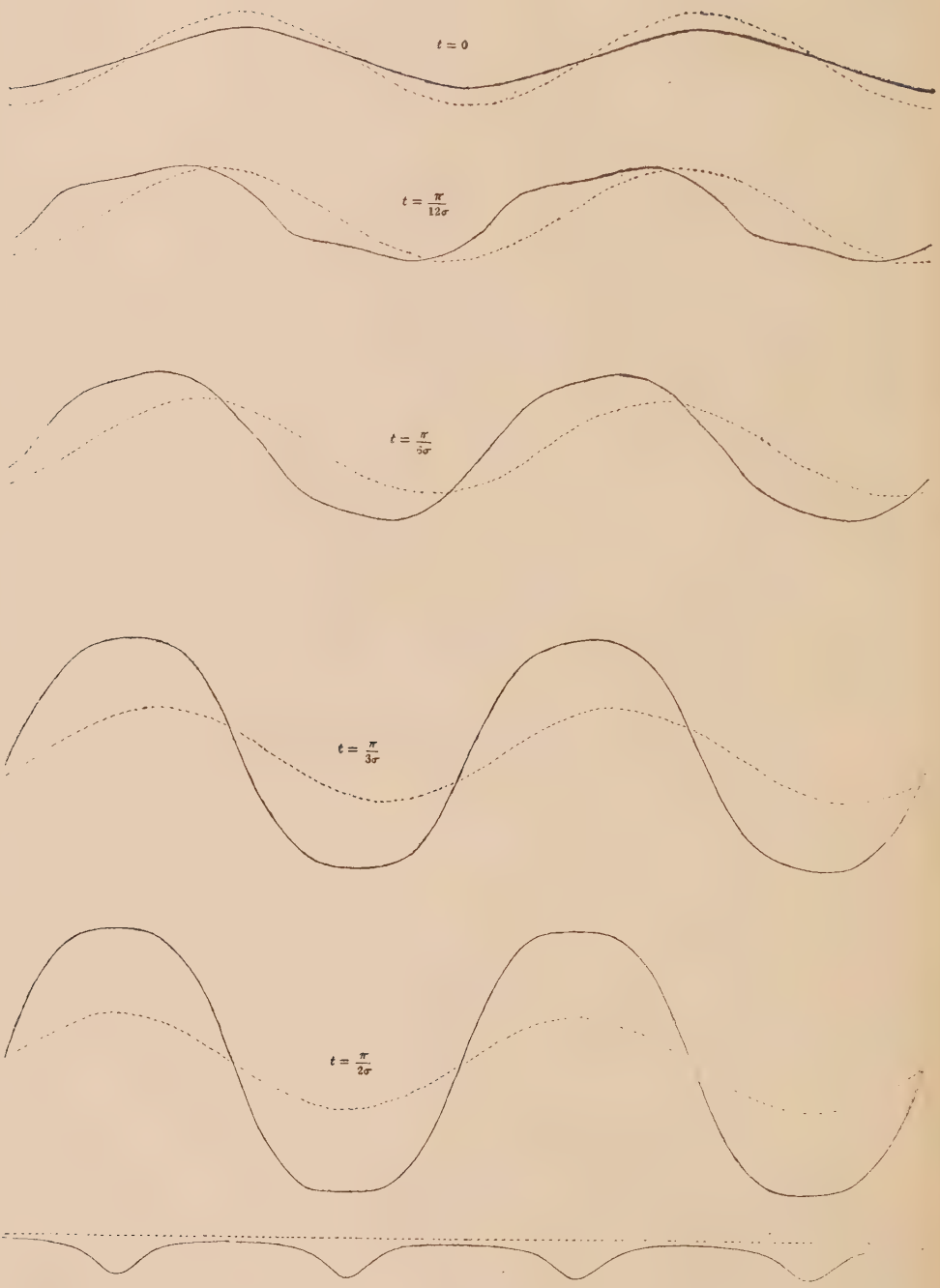
If we take the value of h_0 as 7260 feet, then for the semi-diurnal tide β/σ^2 has the value 0·4. The final result is, with these data

$$\begin{aligned} -\eta/H' = {} & 3{\cdot}166 \sin(\sigma t + 2\phi + \epsilon) + 2{\cdot}154 \sin(\sigma t - 2\phi + \epsilon) \\ & + 0{\cdot}0945 \sin(\sigma t + 6\phi + \epsilon) + 0{\cdot}0643 \sin(\sigma t - 6\phi + \epsilon) \\ & + 0{\cdot}00097 \sin(\sigma t + 10\phi + \epsilon) + 0{\cdot}00066 \sin(\sigma t - 10\phi + \epsilon) \\ & + 0{\cdot}000005 \sin(\sigma t + 14\phi + \epsilon) + 0{\cdot}000003 \sin(\sigma t - 14\phi + \epsilon) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \tag{20}$$

From this,

$$\begin{aligned} \xi . R \sin\theta/H' = {} & 6{\cdot}332 \cos(\sigma t + 2\phi + \epsilon) - 4{\cdot}308 \cos(\sigma t - 2\phi + \epsilon) \\ & + 0{\cdot}5670 \cos(\sigma t + 6\phi + \epsilon) - 0{\cdot}3858 \cos(\sigma t - 6\phi + \epsilon) \\ & + 0{\cdot}0097 \cos(\sigma t + 10\phi + \epsilon) - 0{\cdot}0066 \cos(\sigma t - 10\phi + \epsilon) \\ & + 0{\cdot}00007 \cos(\sigma t + 14\phi + \epsilon) - 0{\cdot}00004 \cos(\sigma t - 14\phi + \epsilon) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

In the figure the wave form has been plotted for values of $t = 0, \frac{\pi}{12\sigma}, \frac{\pi}{6\sigma}, \frac{\pi}{3\sigma}, \frac{\pi}{2\sigma}$. The remarkable variation in the amplitudes is shown, and also the different forms of the wave surface due to the different wave velocities from point to point. The greatest range occurs at the places where ϕ is an odd multiple of $\pi/4$, and the least range where



The upper five figures show the wave profiles at the times indicated. The lowest figure is the profile of the canal bed in corresponding lateral position.

ϕ is an even multiple; that is, the greatest range is just where the depth is greatest and conversely.

The corresponding solution for the case of constant depth h_0 is

$$\xi . R \sin\theta / H' = \tfrac{1}{3}\cos(\sigma t + 2\phi + \epsilon).$$

In the figure this wave is plotted over the others by a broken line. The difference of phase is then readily seen.

[For convenience of drawing, ξ and ϕ, instead of being measured normal to and along the circumference of a circle, have been regarded as purely rectangular coordinates. The scales of ξ and h have been much exaggerated.]

THE SECOND THEOREM OF CONSISTENCY FOR SUMMABLE SERIES

By G. H. HARDY.

[Received June 25th, 1915.—Read November 11th, 1915.]

1. My object in writing this paper is to give a full proof of a theorem enunciated without proof in the tract "The general theory of Dirichlet's series", recently published by Dr. Marcel Riesz and myself*. The theorem is as follows:

If (i) *the series* Σc_n *is summable* (λ, κ), *to sum* C;

(ii) μ *is a logarithmico-exponential function of* λ *such that*

$$\mu = O(\lambda^{\Delta}),$$

where Δ *is a constant; then the series* Σc_n *is summable* (μ, κ) *to sum* C.

2. I begin by recalling Riesz's definition† of summability (λ, κ), *i.e.* summability by means of type λ and order κ. Suppose that (λ_n) is an ascending sequence of positive numbers whose limit is infinity; and let

$$C_\lambda(\tau) = c_1 + c_2 + \ldots + c_n$$

if

$$\lambda_n < \tau \leqslant \lambda_{n+1}.$$

Further let

$$C_\lambda^\kappa(\omega) = C_\lambda(\omega) \tag{2.1}$$

if $\kappa = 0$, and

$$C_\lambda^\kappa(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^\kappa c_n = \kappa \int_0^\omega C_\lambda(\tau)(\omega - \tau)^{\kappa-1}\, d\tau \tag{2.2}$$

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* *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 18, p. 33 (Theorem 19). I refer to this tract as "H. and R."

† H. and R., p. 21.

if $\kappa > 0$. Then $$\omega^{-\kappa} C_\lambda^\kappa(\omega)$$ is called the *typical mean of type* λ *and order* κ formed from the series Σc_n, and the series is said to be *summable* (λ, κ), to sum C, if this "typical mean" tends to the limit C when $\omega \to \infty$.

When $\lambda_n = n$, the means are said to be *arithmetic*. Arithmetic means are equivalent to Cesàro's means, or to the generalisations of Cesàro's means considered by Knopp and Chapman: a series is summable (n, κ) if and only if it is summable (C, κ).*

The *first theorem of consistency*† asserts that, if a series is summable (λ, κ), then it is summable (λ, κ'), to the same sum, for any value of κ' greater than κ. In particular a convergent series is summable by typical means of any positive order, since summability $(\lambda, 0)$ is equivalent to convergence. The general idea expressed by the first theorem of consistency is that, so long as the type remains the same, *the efficacy of a method of summation increases with the order.*

The *second theorem of consistency* lies somewhat deeper. The general idea which it expresses is that, when the order of a method of summation remains the same, *its efficacy increases as the type decreases*, that is to say as the rate of increase of the function λ_n which defines the type decreases. If $$\lambda_n - \lambda_{n-1} > K\lambda_n,$$ that is to say if the rate of increase of λ_n is as great as that of an exponential e^{An}, then the efficacy of the method is *nil*: it will sum convergent series and no others‡. If λ_n runs through the functions of the logarithmico-exponential scale, such as $$e^n,\ n,\ \log n,\ \log\log n,\ \ldots,$$ then we obtain a succession of systems of methods of gradually increasing efficacy.

The theorem suggested by this general idea is that *if a series is summable* (λ, κ) *then it is summable* (μ, κ), μ *being any function of* n *whose rate of increase is less than that of* λ. The actual theorem stated in § 1 is in one way less general and in another more general than this. In the first place, in order to ensure the truth of the theorem, we must suppose that the relation between the rates of increase of μ and λ is characterised by a certain regularity; and the most convenient way of

* Riesz, "Sur une méthode de sommation équivalente à la méthode des moyennes arithmétiques", *Comptes Rendus*, 12 June 1911.

† H. and R., p. 29 (Theorem 16).

‡ H. and R., p. 46 (Theorem 36).

ensuring this is to suppose that μ is a logarithmico-exponential function of λ, a phrase which we will define more precisely in a moment. But, when this limitation is made, we are able to assert rather more than our general principle suggests. The efficacy of the method increases, or at any rate does not decrease, as the rate of increase of the type decreases; a series summable (λ, κ) is certainly summable (μ, κ) if μ increases more slowly than λ. But the converse implication will also be true, and the two methods completely equivalent, if the difference between the rates of increase of λ and μ is not too pronounced, if in fact either function increases with a rapidity comparable to that of a power of the other. If, for example, both λ and μ are powers of n, then any series summable (λ, κ) will be summable (μ, κ), and conversely.

3. Proofs of certain special cases of this theorem have already been published. The most important case is that in which

$$\mu = \log \lambda. \tag{3.1}$$

This case of the theorem was enunciated in 1909 by Riesz*; and his proof was published for the first time in our tract†. Another case is that in which

$$\mu = P(\lambda), \tag{3.2}$$

where P is a polynomial. This case has been treated by Berwald‡, when $\lambda = n$ and κ is an integer. A third case§ is that in which $\kappa = 1$: the theorem then amounts to little more than a restatement in different language of a theorem of Cesàro.

I had conjectured the truth of the general theorem some years ago, when engaged, in collaboration with Mr. Chapman, on a paper dealing with the general theory of summability‖. At that time I had a proof not of the theorem itself, but of its analogue for integrals, and only in the two cases in which (i) κ is an integer or (ii) $0 < \kappa < 1$. As soon as I became familiar with Riesz's methods it became clear to me that my proof applied to series as well; but I was still unable to overcome the algebraical difficulties presented by the proof of the theorem in its most general form.

* "Sur la sommation des séries de Dirichlet", *Comptes Rendus*, 5 July 1909.

† H. and R., p. 30 (Theorem 17).

‡ "Solution nouvelle d'un problème de Fourier", *Arkiv för Matematik*, Vol. 9, 1913, No. 14.

§ Hardy, "On certain oscillating series", *Quarterly Journal*, Vol. 38, 1907, pp. 269–288.

‖ Hardy and Chapman, "A general view of the theory of summable series", *Quarterly Journal*, Vol. 42, 1911, pp. 181–216.

It was only when Riesz, in the course of the preparation of our tract, discovered an important simplification of his method of treatment of the case in which $\mu = \log \lambda$, that I was able to find a completely general proof.

*Definitions and lemmas from the Infinitärcalcul.**

4. A *logarithmico-exponential function*, or, shortly, *an L-function*, is a real one-valued function which can be defined by an explicit formula involving, each only a finite number of times, the ordinary algebraical symbols

$$+, \quad -, \quad \times, \quad \div, \quad \surd,$$

and the symbols $\log(\ldots), \quad e^{\ldots}$

of the logarithmic and exponential functions.

The properties of L-functions which are required for the argument of this paper are as follows.

4.1. *Any L-function $\mu(\lambda)$ is continuous, of constant sign, and monotonic, from a certain value of λ onwards; and the same is true of any of its derivatives.*†

We may suppose, without real loss of generality, that $\mu(\lambda)$, and such of its derivatives as occur in the argument, satisfy these conditions for all values of λ in question.

4.2. *If $\mu \to \infty$, and a number Δ exists such that*

$$\mu = O(\lambda^{\Delta}), \tag{4.21}$$

then

$$\mu^{(r)} = O\left(\frac{\mu}{\lambda^r}\right),$$

$\mu^{(r)}$ denoting the r-th derivative of $\mu(\lambda)$.‡

4.3. *If μ satisfies the conditions of* 4.2, *and ν lies between two posi-*

* See Hardy, "Orders of infinity", *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 12, 1910, or "Properties of logarithmico-exponential functions", *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1912, pp. 54–90. I refer to the first of these publications as "O. I.".

† O. I., p. 18. See also "Properties &c.", p. 40.

‡ O. I., pp. 38 *et seq.*

tive numbers g and G, then positive numbers h and H exist, such that

$$h \leqslant \frac{\mu(\lambda\nu)}{\mu(\lambda)} \leqslant H.* \tag{4.31}$$

For $$\frac{\mu(\lambda\nu)}{\mu(\lambda)} = e^{\log\mu(\lambda\nu)-\log\mu(\lambda)} = e^{\left\{\lambda(\nu-1)\frac{\mu'(\theta)}{\mu(\theta)}\right\}},$$

where θ lies between λ and $\lambda\nu$. Hence

$$\frac{\mu(\lambda\nu)}{\mu(\lambda)} = e^{\lambda O(1/\lambda)} = e^{O(1)}.$$

It is evident that the same result holds for *decreasing* functions which decrease less rapidly than $\lambda^{-\Delta}$ for some value of Δ.

Proof of the theorem when κ is an integer.

5.1. In proving the theorem we may suppose c_n to be *real*: if c_n is complex we can consider the real and imaginary parts of the series separately. We may also suppose, without real loss of generality, that $C = 0$. If C is not zero we begin by proving the theorem for the series

$$(c_1 - C) + c_2 + c_3 + \ldots,$$

and afterwards add to this series the convergent series

$$C + 0 + 0 + \ldots.$$

We are given that

$$\int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\,d\sigma = o(\eta^\kappa),\dagger \tag{5.11}$$

and we wish to prove that

$$\int_{\mu_1}^{\zeta} C_\mu(\tau)(\zeta-\tau)^{\kappa-1}\,d\tau = o(\zeta^\kappa). \tag{5.12}$$

In (5.12) we put $$\tau = \mu(\sigma),$$

and observe that $$C_\mu\{\mu(\sigma)\} = C_\lambda(\sigma).$$

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* More precise results of this character will be found in my paper "Oscillating Dirichlet's integrals", *Quarterly Journal*, Vol. 44, 1913, pp. 1–40 (see p. 23 *et seq.*).

† Since $C_\lambda(\sigma) = 0$ for $0 \leqslant \sigma \leqslant \lambda_1$, it is a matter of indifference whether the lower limit is 0 or λ_1.

We thus obtain

$$\int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\xi-\mu)^{\kappa-1}\mu' d\sigma = o(\xi^\kappa), \tag{5.13}$$

where $$\mu = \mu(\sigma), \qquad \xi = \mu(\eta).$$

From this point onwards our argument depends on the nature of κ. I shall suppose first that κ is an integer.

5.2. If κ is an integer, we have

$$C_\lambda(\sigma) = \frac{1}{\kappa!}\left(\frac{d}{d\sigma}\right)^\kappa C_\lambda^\kappa(\sigma).*$$

The integral (5.13) is therefore a constant multiple of

$$J = \int_{\lambda_1}^{\eta} \frac{d}{d\sigma}(\xi-\mu)^\kappa \left(\frac{d}{d\sigma}\right)^\kappa C_\lambda^\kappa(\sigma)\, d\sigma. \tag{5.21}$$

We transform this integral by κ integrations by parts. Observing that $C_\lambda^\kappa(\sigma)$ and its first $\kappa-1$ derivatives vanish for $\sigma = \lambda_1$, and that $\xi-\mu$ vanishes for $\sigma = \eta$, we obtain

$$\begin{aligned} J = {} & (-1)^{\kappa-1} C_\lambda^\kappa(\eta)\left[\left(\frac{d}{d\sigma}\right)^\kappa (\xi-\mu)^\kappa\right]_{\sigma=\eta} \\ & + (-1)^\kappa \int_{\lambda_1}^{\eta} C_\lambda^\kappa(\sigma)\left(\frac{d}{d\sigma}\right)^{\kappa+1}(\xi-\mu)^\kappa\, d\sigma \\ = {} & J_1 + J_2, \end{aligned} \tag{5.22}$$

say.

5.3. In the first place

$$J_1 = -\kappa!\, C_\lambda^\kappa(\eta)(\xi')^\kappa, \tag{5.31}$$

where ξ' is the value of μ' when $\sigma = \eta$, $\mu = \xi$. Hence

$$J_1 = -\kappa!\,\frac{C_\lambda^\kappa(\eta)}{\eta^\kappa}\left(\frac{\eta\xi'}{\xi}\right)^\kappa \xi^\kappa = o(1)\,O(1)\,\xi^\kappa = o(\xi^\kappa),$$

by 4.2.

* H. and R., p. 28.

On the other hand, it is easily verified that

$$\left(\frac{d}{d\sigma}\right)^{\kappa+1}(\xi-\mu)^{\kappa} = \Sigma A\xi^{\kappa-r}\mu^{s}(\mu')^{s_1}(\mu'')^{s_2}\dots, \tag{5.32}$$

where the A's are constants, and

$$0 < s+s_1+s_2+\dots = r \leqslant \kappa, \tag{5.331}$$

$$s_1+2s_2+3s_3+\dots = \kappa+1. \tag{5.332}$$

Hence our integral reduces to a sum of constant multiples of integrals of the types

$$\xi^{\kappa-r}\int_{\lambda_1}^{\eta} C_{\lambda}^{\kappa}(\sigma)\,\mu^{s}(\mu')^{s_1}(\mu'')^{s_2}\dots\,d\sigma. \tag{5.34}$$

Observing that

$$C_{\lambda}(\sigma) = o(\sigma^{\kappa}),$$

and

$$\mu^{(r)} = O\left(\frac{\mu}{\lambda^{r}}\right),$$

by 4.2, we see that (5.34) is of the form

$$o\left(\xi^{\kappa-r}\int_{\lambda_1}^{\eta}\sigma^{\kappa-s_1-2s_2-3s_3-\dots+1}\,\mu^{s+s_1+s_2+\dots-1}\mu'\,d\sigma\right) \tag{5.35}$$

$$= o\left(\xi^{\kappa-r}\int_{\lambda_1}^{\eta}\mu^{r-1}\mu'\,d\sigma\right) = o(\xi^{\kappa}).$$

This completes the proof of the theorem when κ is an integer.

Proof when $0 < \kappa < 1$.

6.1. We consider next the case in which $0 < \kappa < 1$. I shall suppose first that the increase of μ is greater than that of $\log\lambda$, so that $\lambda\mu'$ tends steadily to infinity with λ.

We observe first that, in virtue of the first theorem of consistency, Σc_n is summable $(\lambda, 1)$, to sum zero, so that

$$C_{\lambda}^{1}(\sigma) = o(\sigma). \tag{6.11}$$

Now let A be any positive constant. Then we can, when η is large enough, determine a unique number η_1 such that

$$\lambda_1 < \eta_1 < \eta, \qquad \xi-\xi_1 = A\eta_1\xi_1', \tag{6.12}$$

where

$$\xi_1 = \mu(\eta_1), \qquad \xi_1' = \mu'(\eta_1). \tag{6.121}$$

For, as σ increases from λ_1 to η, $\xi-\mu$ decreases steadily from $\xi-\mu_1$ (which is large when η is large) to zero, whereas $\sigma\mu'$ increases steadily with σ.

Let us suppose that $0 < A < 1$, and that η_1 has been chosen so as to satisfy (6.12). Then we can determine a positive constant h such that

$$h\eta < \eta_1 < \eta. \tag{6.13}$$

For

$$\xi-\xi_1 = (\eta-\eta_1)\,\xi_2',$$

where ξ_2' is the value of μ' when σ has a certain value η_2 between η_1 and η. Thus

$$A\eta_1\xi_1' = \xi-\xi_1 = (\eta-\eta_1)\,\xi_2' = \frac{\eta-\eta_1}{\eta_2}\,\eta_2\xi_2' > \frac{\eta-\eta_1}{\eta}\,\eta_1\xi_1',$$

since $\sigma\mu'$ increases with λ. Hence

$$A > \frac{\eta-\eta_1}{\eta}, \qquad \eta_1 > (1-A)\,\eta,$$

and we may take

$$h = 1-A.$$

6.2. We now write

$$J = \int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\xi-\mu)^{\kappa-1}\mu'\,d\sigma = \int_{\lambda_1}^{\eta_1} + \int_{\eta_1}^{\eta} = J_1+J_2, \tag{6.21}$$

say. We begin by considering J_1. Integrating by parts, we obtain

$$\begin{aligned} J_1 &= C_\lambda^1(\eta_1)(\xi-\xi_1)^{\kappa-1}\xi_1' - \int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)\frac{d}{d\sigma}\{(\xi-\mu)^{\kappa-1}\mu'\}\,d\sigma \\ &= J_{1,1}+J_{1,2}, \end{aligned} \tag{6.22}$$

say. In the first place

$$J_{1,1} = \frac{C_\lambda^1(\eta_1)}{\eta_1}(A\eta_1\xi_1')^{\kappa-1}\eta_1\xi_1' = o(\eta_1\xi_1')^\kappa = o(\xi_1^\kappa) = o(\xi^\kappa). \tag{6.23}$$

Secondly,

$$\begin{aligned} J_{1,2} &= (\kappa-1)\int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)(\xi-\mu)^{\kappa-2}\mu'^2\,d\sigma - \int_{\lambda_1}^{\eta_1} C_\lambda^1(\sigma)(\xi-\mu)^{\kappa-1}\mu''\,d\sigma \\ &= J_{1,2,1}+J_{1,2,2}, \end{aligned} \tag{6.24}$$

say. Now

$$\frac{\sigma\mu'}{\xi-\mu} < \frac{\sigma\mu'}{\xi-\xi_1} < \frac{\sigma\mu'}{A\eta_1\xi_1'} < \frac{1}{A}, \tag{6.25}$$

since $\lambda_1 < \sigma < \eta_1$ and $\sigma\mu'$ increases with σ. Hence

$$\begin{aligned} J_{1,2,1} &= (\kappa-1)\int_{\lambda_1}^{\eta_1} o(\sigma)(\xi-\mu)^{\kappa-1}\mu'\frac{\sigma\mu'}{\xi-\mu}\,d\sigma \\ &= o\int_{\lambda_1}^{\eta_1}(\xi-\mu)^{\kappa-1}\mu'\,d\sigma \\ &= o(\xi^\kappa). \end{aligned} \tag{6.26}$$

Also

$$\begin{aligned} J_{1,2,2} &= -\int_{\lambda_1}^{\eta_1} o(\sigma)(\xi-\mu)^{\kappa-1} O\left(\frac{\mu'}{\sigma}\right) d\sigma \\ &= o\int_{\lambda_1}^{\eta_1}(\xi-\mu)^{\kappa-1}\mu'\,d\sigma \\ &= o(\xi^\kappa). \end{aligned} \tag{6.27}$$

From (6.22)–(6.27) it follows that

$$J_1 = o(\xi^\kappa). \tag{6.28}$$

6.3. It remains to consider J_2. We have

$$\begin{aligned} J_2 &= \int_{\eta_1}^{\eta} C_\lambda(\sigma)(\xi-\mu)^{\kappa-1}\mu'\,d\sigma \\ &= \xi'\int_{\eta_2}^{\eta} C_\lambda(\sigma)(\xi-\mu)^{\kappa-1}\,d\sigma \end{aligned} \tag{6.311}$$

if μ' increases; and

$$J_2 = \xi_1'\int_{\eta_1}^{\eta_2} C_\lambda(\sigma)(\xi-\mu)^{\kappa-1}\,d\sigma \tag{6.312}$$

if μ' decreases, η_2 denoting in either case a number between η_1 and η. Now

$$\frac{d}{d\sigma}\frac{\xi-\mu}{\eta-\sigma} = \frac{\xi-\mu-(\eta-\sigma)\mu'}{(\eta-\sigma)^2} = \frac{\bar{\xi}'-\mu'}{\eta-\sigma},$$

$\bar{\xi}'$ being the value of μ' for a value $\bar{\eta}$ of its argument between σ and η.

This is positive if μ' increases and negative if μ' decreases. Hence

$$\frac{\xi-\mu}{\eta-\sigma}$$

is monotonic, and varies in the same sense as μ'; so that

$$\left(\frac{\xi-\mu}{\eta-\sigma}\right)^{\kappa-1}$$

is monotonic in the sense opposite to that of the variation of μ'. We have therefore

$$J_2 = \xi' \int_{\eta_2}^{\eta} C_\lambda(\sigma) \left(\frac{\xi-\mu}{\eta-\sigma}\right)^{\kappa-1} (\eta-\sigma)^{\kappa-1}\, d\sigma \tag{6.321}$$

$$= \xi' \left(\frac{\xi-\xi_2}{\eta-\eta_2}\right)^{\kappa-1} \int_{\eta_2}^{\eta_3} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\, d\sigma$$

if μ' increases; and

$$J_2 = \xi_1' \int_{\eta_1}^{\eta_2} C_\lambda(\sigma) \left(\frac{\xi-\mu}{\eta-\sigma}\right)^{\kappa-1} (\eta-\sigma)^{\kappa-1}\, d\sigma \tag{6.322}$$

$$= \xi_1' \left(\frac{\xi-\xi_2}{\eta-\eta_2}\right)^{\kappa-1} \int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\, d\sigma$$

if μ' decreases, η_3 being in either case another number between η_1 and η. And in either case

$$\frac{\xi-\xi_2}{\eta-\eta_2}$$

lies between ξ_1' and ξ'.

The order of μ lies between $\log\lambda$ and λ^Δ, and that of μ' between $1/\lambda$ and $\lambda^{\Delta-1}$, and *a fortiori* between $\lambda^{-\Delta}$ and λ^Δ.* Also

$$h\eta < \eta_1 < \eta.$$

It follows from 4.3 that the ratio ξ_1'/ξ' lies between fixed positive limits. We have therefore in any case

$$J_2 = O(\xi')^\kappa \int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1} d\sigma. \tag{6.33}$$

* Evidently we may suppose $\Delta > 1$.

But*

$$\left| \int_{\eta_3}^{\eta_2} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1} d\sigma \right| < 2 \operatorname*{Max}_{\lambda_1 \leqslant \tau \leqslant \eta} C_\lambda^\kappa(\tau). \tag{6.34}$$

Hence

$$J_2 = O(\xi')^\kappa o(\eta^\kappa) = o(\eta\xi')^\kappa = o(\xi^\kappa). \tag{6.35}$$

From (6.21), (6.28), and (6.35) it follows that

$$J = o(\xi^\kappa). \tag{6.36}$$

6.4. We have thus proved the theorem when $0 < \kappa < 1$ and the order of μ lies between $\log\lambda$ and λ^Δ. Suppose next that the order of μ lies between $\log\log\lambda$ and $(\log\lambda)^\Delta$; and let

$$\nu = (\log\lambda)^a,$$

where $a > 1$. The series is summable (λ, κ), and therefore, by what precedes, summable (ν, κ). But

$$\mu(\lambda) = \mu(e^{\nu^{1/a}})$$

is an L-function of ν whose order lies between $(1/a)\log\nu$ and $\nu^{\Delta/a}$. Hence the series is summable (μ, κ). The theorem is thus proved when $0 < \kappa < 1$ and the order of μ is greater than $\log\log\lambda$. Repeating the argument, we prove it whenever the order of μ is greater than any one of

$$\log\log\log\lambda, \quad \log\log\log\log\lambda, \quad \ldots.$$

Since any L-function which tends to infinity must increase more rapidly than some one of the repeated logarithmic functions†, the theorem is true without restriction on μ. The proof when $0 < \kappa < 1$ is thus complete.

Proof when κ is greater than 1 and not integral.

7.1. The proof of the theorem when κ is greater than 1, but not an integer, presents no fresh difficulty of principle. All that is necessary is to combine in an appropriate manner the arguments used in 5 and 6.

We suppose that

$$k < \kappa < k+1, \tag{7.11}$$

* H. and R., p. 28 (Lemma 7).

† O. I., p. 20; "Properties &c.", pp. 63 *et seq.*

k being an integer, and (for the moment) that μ is of higher order than $\log\lambda$. Integrating the integral (5.13) k times by parts, we obtain a constant multiple of

$$J=\int_{\lambda_1}^{\eta} C_\lambda^k(\sigma)\left(\frac{d}{d\sigma}\right)^{k+1}(\xi-\mu)^\kappa\,d\sigma. \tag{7.12}$$

We write, as in 6.2,

$$J=\int_{\lambda_1}^{\eta_1}+\int_{\eta_1}^{\eta}=J_1+J_2. \tag{7.13}$$

7.2. In order to obtain an upper limit for J_1 we integrate once more by parts. We thus obtain

$$J_1=\frac{C_\lambda^{k+1}(\eta_1)}{k+1}\left(\frac{d}{d\eta_1}\right)^{k+1}(\xi-\xi_1)^\kappa-\frac{1}{k+1}\int_{\lambda_1}^{\eta_1}C_\lambda^{k+1}(\sigma)\left(\frac{d}{d\sigma}\right)^{k+2}(\xi-\mu)^\kappa\,d\sigma \tag{7.21}$$

$$=J_{1,1}+J_{1,2},$$

say. Now

$$\left(\frac{d}{d\sigma}\right)^{k+1}(\xi-\mu)^\kappa=\Sigma A(\xi-\mu)^s(\mu')^{s_1}(\mu'')^{s_2}\ldots, \tag{7.22}$$

where

$$s+s_1+s_2+\ldots=\kappa, \tag{7.231}$$

$$s_1+2s_2+3s_3+\ldots=k+1. \tag{7.232}$$

Hence

$$J_{1,1}=\frac{C^{k+1}(\eta_1)}{k+1}\Sigma A(\xi-\xi_1)^s(\xi_1')^{s_1}(\xi_1'')^{s_2}\ldots. \tag{7.24}$$

But

$$C^{k+1}(\eta_1)=o(\eta_1^{k+1}),$$

since the series is summable (λ,κ), and *a fortiori* summable $(\lambda,k+1)$, to sum 0. Also

$$\xi-\xi_1=O(\eta_1\xi_1'),$$

by (6.12), and

$$\xi_1^{(r)}=O\left(\frac{\xi_1'}{\eta_1^{r-1}}\right),$$

by 4.2. Hence

$$\begin{aligned}J_{1,1}&=\Sigma o\left\{\eta_1^{k+1}(\eta_1\xi_1')^s\,\eta_1^{-s_2-2s_3-\ldots}(\xi_1')^{s_1+s_2+\ldots}\right\}\\&=\Sigma o\left\{\eta_1^{s+s_1+2s_2+\ldots-s_2-2s_3-\ldots}(\xi_1')^{s+s_1+s_2+\ldots}\right\}\\&=\Sigma\,o(\eta_1\xi_1')^{s+s_1+s_2+\ldots}=o(\eta_1\xi_1')^\kappa\\&=o(\xi_1^\kappa)=o(\xi^\kappa).\end{aligned} \tag{7.25}$$

7.3. The integral $J_{1,2}$ is a sum of constant multiples of integrals of the type

$$\int_{\lambda_1}^{\eta_1} C_\lambda^{k+1}(\sigma)(\xi-\mu)^s(\mu')^{s_1}(\mu'')^{s_2}\dots d\sigma, \tag{7.31}$$

where now

$$s+s_1+s_2+\dots=\kappa, \tag{7.321}$$

$$s_1+2s_2+3s_3+\dots=k+2. \tag{7.322}$$

The integral (7.31) is of the form

$$\int_{\lambda_1}^{\eta_1} o(\sigma^{k+1})(\xi-\mu)^{s-\kappa+1}(\mu')^{s_1-1}(\mu'')^{s_2}\dots(\xi-\mu)^{\kappa-1}\mu'\,d\sigma. \tag{7.33}$$

Now

$$\xi-\mu\geqslant\xi-\xi_1=A\eta_1\xi_1'\geqslant A\sigma\mu',$$

for $\lambda_1\leqslant\sigma\leqslant\eta_1$; and

$$s-\kappa+1=1-s_1-s_2-\dots\leqslant 0.$$

Hence

$$(\xi-\mu)^{s-\kappa+1}=O(\sigma\mu')^{s-\kappa+1}; \tag{7.34}$$

and

$$\begin{aligned}\sigma^{k+1}(\xi-\mu)^{s-\kappa+1}(\mu')^{s_1-1}(\mu'')^{s_2}\dots&\\ =O\{\sigma^{k+1}(\sigma\mu')^{s-\kappa+1}\sigma^{-s_2-2s_3-\dots}(\mu')^{s_1+s_2+\dots\ 1}\}&\\ =O(1),&\end{aligned} \tag{7.35}$$

since

$$k+1+s-\kappa+1-s_2-2s_3-\dots=0$$

and

$$s-\kappa+1+s_1+s_2+\dots-1=0,$$

in virtue of (7.321) and (7.322). Thus the integral (7.33) is of the form

$$o\int_{\lambda_1}^{\eta_1}(\xi-\mu)^{\kappa-1}\mu'd\sigma=o(\xi_1^\kappa)=o(\xi^\kappa);$$

so that

$$J_{1,2}=o(\xi^\kappa). \tag{7.36}$$

From (7.25) and (7.36) it follows that

$$J_1=o(\xi^\kappa). \tag{7.37}$$

7.4. It remains to consider

$$J_2 = \int_{\eta_1}^{\eta} C^k(\sigma) \left(\frac{d}{d\sigma}\right)^{k+1} (\xi-\mu)^\kappa \, d\sigma, \tag{7.41}$$

which is a sum of constant multiples of integrals of the type

$$\int_{\eta_1}^{\eta} C^k(\sigma)(\xi-\mu)^s (\mu')^{s_1} (\mu'')^{s_2} \ldots d\sigma, \tag{7.42}$$

where

$$s+s_1+s_2+\ldots = \kappa, \tag{7.431}$$

$$s_1+2s_2+3s_3+\ldots = k+1. \tag{7.432}$$

The integral (7.42) may be written in one or other of the forms

$$(\xi')^{s_1} (\xi'')^{s_2} \ldots \int_{\eta_2}^{\eta} C^k(\sigma)(\xi-\mu)^s \, d\sigma, \tag{7.441}$$

$$(\xi_1')^{s_1} (\xi_1'')^{s_2} \ldots \int_{\eta_1}^{\eta_2} C^k(\sigma)(\xi-\mu)^s \, d\sigma. \tag{7.442}$$

Arguing as in 6.3, we replace each of these integrals by one of the form

$$j = (\xi_3')^{s_1} (\xi_3'')^{s_2} \ldots \left(\frac{\xi-\xi_4}{\eta-\eta_4}\right)^s \int_{\eta_5}^{\eta_6} C^k(\sigma)(\eta-\sigma)^s \, d\sigma, \tag{7.45}$$

where η_3, η_4, ... are numbers between η_1 and η, and ξ_3, ξ_4, ..., ξ_3', ξ_4', ... are the corresponding values of μ and μ'. We write (7.45) in the form

$$j = j_1 j_2, \tag{7.46}$$

where j_1 and j_2 denote the external factor and the integral in (7.45) respectively.

7.5. It follows from arguments similar to those employed in 6.3 that

$$j_1 = O\{(\xi')^{s+s_1}(\xi'')^{s_2} \ldots\} = O\{\eta^{-s_2-2s_3-\ldots}(\xi')^{s+s_1+s_2+\ldots}\}. \tag{7.51}$$

In order to obtain an upper limit for j_2, we observe that

$$[s] = k-s_1-s_2-\ldots = k', \tag{7.52}$$

say, and integrate $k'+1$ times by parts. We thus obtain

$$j_2 = \int_{\eta_5}^{\eta_6} C^k(\sigma)(\eta-\sigma)^s \, d\sigma = F(\eta_6)-F(\eta_5)+j_4 = j_3+j_4, \tag{7.53}$$

say, where

$$(7.54)\quad F(\sigma)=\frac{1}{k+1}C^{k+1}(\sigma)(\eta-\sigma)^{s}+\frac{s}{(k+1)(k+2)}C^{k+2}(\sigma)(\eta-\sigma)^{s-1}+\dots$$
$$+\frac{s(s-1)\dots(s-k'+1)}{(k+1)(k+2)\dots(k+k'+1)}C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k'},$$

$$(7.55)\quad j_4=\frac{s(s-1)\dots(s-k')}{(k+1)(k+2)\dots(k+k'+1)}\int_{\eta}^{\eta_6}C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k'-1}\,d\sigma.$$

Any term of $F(\sigma)$ is of the form

$$o(\eta^{k+r+s-r+1})=o(\eta^{k+s+1});$$

and so

$$(7.56)\quad j_3=o(\eta^{k+s+1}).$$

On the other hand $s-k'$ lies between 0 and 1, and so*

$$(7.57)\quad \left|\int_{\eta_5}^{\eta_6}C^{k+k'+1}(\sigma)(\eta-\sigma)^{s-k-1}\,d\sigma\right|$$
$$\leqslant 2\,\frac{\Gamma(k+k'+2)\,\Gamma(s-k')}{\Gamma(s+k+2)}\operatorname*{Max}_{\lambda_1\leqslant\tau\leqslant\eta}|C^{k+k'+1}(\tau)|$$
$$=o(\eta^{k+s+1}).$$

Thus both j_3 and j_4 are of this form, and so, therefore, is j_2; and therefore

$$(7.58)\quad j=j_1j_2=o\{\eta^{k+s+1-s_2-2s_3-\dots}(\xi')^{s+s_1+s_2+\dots}\}$$
$$=o(\eta\xi')^{s+s_1+s_2+\dots}$$
$$=o(\eta\xi')^{\kappa}=o(\xi^{\kappa}).$$

Hence

$$(7.59)\quad J_2=o(\xi^{\kappa}).$$

7.6. From (7.13), (7.37), and (7.59) it follows that

$$(7.61)\quad J=o(\xi^{\kappa}).$$

The proof of the theorem is thus completed, provided that the order of μ is greater than that of $\log\lambda$. In order to extend the result to cover all possible cases we have only to repeat the argument of 6.4.

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* H. and R., p. 29 (Lemma 8).

Conclusion.

8. It remains only to show that the theorem is the best possible theorem of its kind. In order to prove this it is necessary to show that *if μ is any L-function of λ which tends to infinity more rapidly than any power of λ, then we can determine a number κ and a series Σc_n which is summable (λ, κ) and not summable (μ, κ).*

We may take $\lambda = n$, and we may suppose that μ is of lower order than e^n, since methods of type as high as e^n will sum convergent series only*. Consider the series

$$\Sigma (-1)^n \left(\frac{\mu_n}{\mu_n'}\right)^\kappa, \tag{8.1}$$

which may be written in the form

$$\Sigma (-1)^n n^\kappa \left(\frac{\mu_n}{n\mu_n'}\right)^\kappa.$$

Since μ_n is of order higher than any power of n, we have

$$\frac{\mu_n}{n\mu_n'} = o(1).\dagger$$

Hence

$$\left(\frac{\mu_n}{n\mu_n}\right)^\kappa$$

is an L-function which tends to zero, and so also are all its derivatives. All of these derivatives, moreover, are ultimately of constant sign; and the same is true of all the successive differences of the function. Also the series

$$\Sigma (-1)^n n^\kappa$$

is *finite* (C, κ). It follows from known theorems‡ that the series (8.1) is summable (C, κ), *i.e.* (n, κ).

But the series (8.1) is *not* summable (μ, κ). For if it were, we should have

$$\left(\frac{\mu_n}{\mu_n'}\right)^\kappa = o\left(\frac{\mu_{n+1}}{\mu_{n+1} - \mu_n}\right)^\kappa \S,$$

* H. and R., p. 46 (Theorem 36).

† O.I., p. 38.

‡ The theorem required is a special case of Theorem 1*a* (p. 61) of Bohr's dissertation "Bidrag til de Dirichlet'ske Rækkers Theori" (Copenhagen, 1910).

§ H. and R., p. 36 (Theorem 21).

and this is untrue, since $\mu_{n+1} \sim \mu_n$

and $\mu_{n+1} - \mu_n \sim \mu_n'$,

when μ_n is an L-function of n of order less than e^{n}*.

For example, the series

$$1-1+1-1+\ldots$$

is summable $(1, \kappa)$ for any positive value of κ, but is not summable (e^n, κ) for any value of κ. The series

$$\sqrt{1}-\sqrt{2}+\sqrt{3}-\ldots$$

is summable $(n, 1)$ but not summable $(e^{\sqrt{n}}, 1)$; and so on.

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* O.I., pp. 41 *et seq.*

THE KINETIC THEORY OF THE MOTION OF IONS IN GASES

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1. *Introduction.*

Maxwell's famous "second method" in the kinetic theory of gases* has generally been recognised as affording the most rigorous treatment of the subject at present available. It rests on the assumption that when a given molecule meets a stream of other molecules of definite velocity, collisions at all points of the cross-section of the cylinder of molecular action are equally likely. Little advantage is gained by going behind this assumption; and, in fact, it is in the development of Maxwell's principles that progress has subsequently been made. In its original form Maxwell's theory was worked out on the hypothesis of a force between two molecules varying inversely as the fifth power of the distance, which has mathematical advantages, in that a knowledge of the form of the distribution function is unnecessary. But for other laws of force the equation of transfer involves the distribution function. It is therefore better to use the equation for the function itself, which is equivalent to the aggregate of all the equations of transfer, and which was first given by Boltzmann.† This equation has been treated by expanding the distribution function in powers of the components of velocity relative to the mean; but this is not altogether satisfactory, and has led in one case to a false assumption respecting the form of the function. The rigorous treatment of Boltzmann's equation appears to begin with a paper of Hilbert,‡ in which a certain integral was reduced to a form resembling the left-hand side of an integral equation. In the present paper I have treated the motion of ions in a gas, where the practical absence of collisions among the ions themselves gives rise to great simplification. In one case an integral equation has

* J. C. Maxwell, *Phil. Trans.*, Vol. 157, p. 49, 1866; *Collected Papers*, Vol. 2, p. 26.

† L. Boltzmann, *Wien. Ber.*, II, Vol. 66, p. 324, 1872; Vol. 72, p. 434, 1876; *Gastheorie*, Vol. I, p. 114.

‡ D. Hilbert, *Math. Ann.*, Vol. 72, p. 562, 1912.

been found and solved numerically; but the chief use made of Hilbert's transformation has been to simplify calculations which might otherwise have been inextricable.

2. *Reduction of Boltzmann's Equation.*

Let m be the mass of an ion, n the number of ions per cubic centimetre at the point (x, y, z), and let M, N be the corresponding quantities for the molecules of the gas. Let u, with components u_x, u_y, u_z, be the velocity of an ion before collision, u' the velocity after, and U, U' the velocities of the molecule with which it collides. The number of ions per cubic centimetre with velocities between u_x, u_y, u_z and u_x+du_x, u_y+du_y, u_z+du_z is $f(u_x, u_y, u_z)\,du_x du_y du_z$, where f is the distribution function, and similarly $F(U_x, U_y, U_z)$ is the distribution function for the gas molecules. We shall write $d\omega = du_x du_y du_z$, $d\Omega = dU_x dU_y dU_z$, and a function such as $f(u_x, u_y, u_z)$ will be written $f(\omega)$ or simply f, the notation $f(u)$ meaning that it is a function of the resultant velocity u only. Since n is small in comparison with N, we may neglect the collisions of the ions among themselves and write for F its equilibrium value $N(hM/\pi)^{\frac{3}{2}} \exp(-hMU^2)$. Boltzmann's equation becomes

$$\int (f'F' - fF)\, gbdbd\epsilon d\Omega = Df, \tag{1}$$

where
$$D \equiv \frac{\partial}{\partial t} + u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z} + F_x\frac{\partial}{\partial u_x} + F_y\frac{\partial}{\partial u_y} + F_z\frac{\partial}{\partial u_z}. \tag{2}$$

Here (F_x, F_y, F_z) is the external force on the ions per unit mass, and the rest of the notation is Boltzmann's. If the collisions are those between elastic spheres, then Hilbert's transformation applied to the left-hand side of (1) converts it into

$$\int K(\omega, \omega') f(\omega')\, d\omega' - k(u) f(\omega),$$

where $K(\omega, \omega')$ is a certain function of the six components of velocity. I propose to give a new proof of this result, and extend it to a law of force between ion and molecule which is any function of the distance,* which would not be possible by Hilbert's method. It is convenient to neglect all "collisions" in which the ion fails to come within a certain distance σ of a molecule, and the limits of b are taken as 0 to σ instead of 0 to ∞.

* The corresponding extension for a single gas has been announced by Lunn (*Bull. Amer. Math. Soc.*, Vol. 19, p. 455, 1913), but his paper has apparently never been published.

Write

$$U_x = u_x + g \sin\phi \cos\eta, \quad U_y = u_y + g \sin\phi \sin\eta, \quad U_z = u_z + g \cos\phi,$$

and let ϵ be measured from a plane through the axis of z and the line (ϕ, η). Let 2θ be the angle between the asymptotes of the relative orbit, measured so that $\theta = 0$ corresponds to a complete reversal of path. Then θ is a function of g and b only, and we have

$$u'_x - U'_x = g(\sin 2\theta \cos\epsilon \cos\phi \cos\eta - \sin 2\theta \sin\epsilon \sin\eta + \cos 2\theta \sin\phi \cos\eta),$$

$$u'_y - U'_y = g(\sin 2\theta \cos\epsilon \cos\phi \sin\eta + \sin 2\theta \sin\epsilon \cos\eta + \cos 2\theta \sin\phi \sin\eta),$$

$$u'_z - U'_z = g(-\sin 2\theta \cos\epsilon \sin\phi + \cos 2\theta \cos\phi).$$

Combining these with the three equations of the type

$$mu'_x + MU'_x = mu_x + MU_x,$$

we have

$$u'_x = u_x + R(\sin\theta \cos\epsilon \cos\phi \cos\eta - \sin\theta \sin\epsilon \sin\eta + \cos\theta \sin\phi \cos\eta),$$

$$u'_y = u_y + R(\sin\theta \cos\epsilon \cos\phi \sin\eta + \sin\theta \sin\epsilon \cos\eta + \cos\theta \sin\phi \sin\eta),$$

$$u'_z = u_z + R(-\sin\theta \cos\epsilon \sin\phi + \cos\theta \cos\phi),$$

$$U'_x = u_x + g \sin\phi \cos\eta - Q(\sin\theta \cos\epsilon \cos\phi \cos\eta - \sin\theta \sin\epsilon \sin\eta + \cos\theta \sin\phi \cos\eta),$$

$$U'_y = u_y + g \sin\phi \sin\eta - Q(\sin\theta \cos\epsilon \cos\phi \sin\eta + \sin\theta \sin\epsilon \cos\eta + \cos\theta \sin\phi \sin\eta),$$

$$U'_z = u_z + g \cos\phi - Q(-\sin\theta \cos\epsilon \sin\phi + \cos\theta \cos\phi),$$

where
$$R = \frac{2Mg\cos\theta}{m+M}, \qquad Q = \frac{2mg\cos\theta}{m+M}. \tag{3}$$

From the above we have

$$\partial(u'_x, u'_y, u'_z)/\partial(g, \phi, \eta)$$

$$= \begin{vmatrix} \frac{\partial R}{\partial g}\frac{u'_x - u_x}{R} + R\frac{\partial\theta}{\partial g}(\cos\theta \cos\epsilon \cos\phi \cos\eta - \cos\theta \sin\epsilon \sin\eta - \sin\theta \sin\phi \cos\eta), & (u'_z - u_z)\cos\eta, & -(u'_y - u_y) \\ \frac{\partial R}{\partial g}\frac{u'_y - u_y}{R} + R\frac{\partial\theta}{\partial g}(\cos\theta \cos\epsilon \cos\phi \sin\eta + \cos\theta \sin\epsilon \cos\eta - \sin\theta \sin\phi \sin\eta), & (u'_z - u_z)\sin\eta, & u'_x - u_x \\ \frac{\partial R}{\partial g}\frac{u'_z - u_z}{R} + R\frac{\partial\theta}{\partial g}(-\cos\theta \cos\epsilon \sin\phi - \sin\theta \cos\phi), & -(u'_x - u_x)\cos\eta - (u'_y - u_y)\sin\eta, & 0 \end{vmatrix}$$

Expanding in terms of the first column, we find

$$\partial(u_x', u_y', u_z')/\partial(g, \phi, \eta) = (\Delta_0+\Delta_1)\{(u_x'-u_x)\cos\eta+(u_y'-u_y)\sin\eta\},$$

where $\Delta_0 = R^{-1}\partial R/\partial g\{(u_x'-u_x)^2+(u_y'-u_y)^2+(u_z'-u_z)^2\}$,

$\Delta_1 = R\partial\theta/\partial g\times$

$$\{(u_x'-u_x)(\cos\theta\cos\epsilon\cos\phi\cos\eta-\cos\theta\sin\epsilon\sin\eta-\sin\theta\sin\phi\cos\eta)$$
$$+(u_y'-u_y)(\cos\theta\cos\epsilon\cos\phi\sin\eta+\cos\theta\sin\epsilon\cos\eta-\sin\theta\sin\phi\sin\eta)$$
$$+(u_z'-u_z)(-\cos\theta\cos\epsilon\sin\phi-\sin\theta\cos\phi)\}.$$

On simplifying we find $\Delta_1 = 0$. Except in $\partial R/\partial g$ we may put

$$R = \{\Sigma(u_x'-u_x)^2\}^{\frac{1}{2}},$$

so that $\partial(u_x', u_y', u_z')/\partial(g, \phi, \eta) = \{(u_x'-u_x)\cos\eta+(u_y'-u_y)\sin\eta\}R\partial R/\partial g.$

Since $$\partial(U_x, U_y, U_z)/\partial(g, \phi, \eta) = g^2\sin\phi,$$

we have $$\frac{\partial(U_x, U_y, U_z)}{\partial(u_x', u_y', u_z')} = \frac{g^2\sin\phi}{\{(u_x'-u_x)\cos\eta+(u_y'-u_y)\sin\eta\}R\partial R/\partial g}. \quad (4)$$

We find

$$U'^2 = u^2+g^2+Q^2-2Qg\cos\theta-2m\{\Sigma u_x(u_x'-u_x)\}/M$$
$$+2g(u_x\sin\phi\cos\eta+u_y\sin\phi\sin\eta+u_z\cos\phi),$$

so that

$$F' = N\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}\exp\left[-hM\left\{g^2+u^2-\frac{mR^2}{M}-\frac{2m}{M}\Sigma[u_x(u_x'-u_x)]\right.\right.$$
$$\left.\left.+2g(u_x\sin\phi\cos\eta+u_y\sin\phi\sin\eta+u_z\cos\phi)\right\}\right].$$

Transforming the variables in equation (1) from Ω to ω' by means of (4), and writing

$$P = \Sigma u_x(u_x'-u_x), \quad (5)$$

we find $$\int f'F'gb\,db\,d\epsilon\,d\Omega = \int K(\omega, \omega')f(\omega')\,d\omega', \quad (6)$$

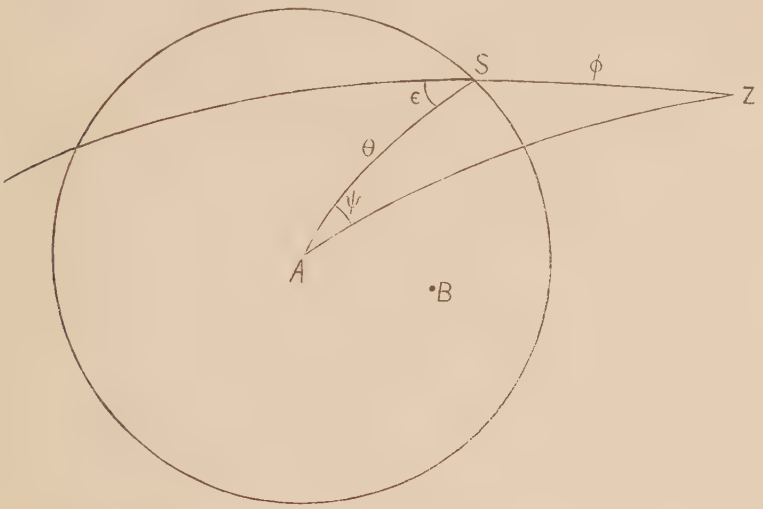
where $K(\omega, \omega')$

$$= N\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}\frac{\exp[-hM(u^2-mR^2/M-2mP/M)]}{R}\int_0^\tau\frac{\exp(-hMg^2)}{\partial R/\partial g}Lg^3b\,db, \quad (7)$$

L standing for the integral

$$\int_0^{2\pi}\frac{\exp[-2hMg(u_x\sin\phi\cos\eta+u_y\sin\phi\sin\eta+u_z\cos\phi)]}{(u_x'-u_x)\cos\eta+(u_y'-u_y)\sin\eta}\sin\phi\,d\epsilon.$$

The integral L can be evaluated by a geometrical artifice. Writing (p, q, r) for $(u'_x-u_x, u'_y-u_y, u'_z-u_z)/R$, A is the point (p, q, r) on a unit



sphere with the origin as centre. Let B be the point $(u_x, u_y, u_z)/u$, S the point (ϕ, η) in polar coordinates. Then as ϵ changes S moves round a small circle with A as pole, and ψ increases by 2π. We have

$$(u'_x-u_x)\cos\eta+(u'_y-u_y)\sin\eta = R(\sin\theta\cos\epsilon\cos\phi+\cos\theta\sin\phi),$$

and
$$-\sin\theta\cos\epsilon\sin\phi+\cos\theta\cos\phi = r.$$

Differentiating the last equation we have, since r is constant,

$$\sin\theta\sin\epsilon\sin\phi\,d\epsilon = (\sin\theta\cos\epsilon\cos\phi+\cos\theta\sin\phi)\,d\phi.$$

Also
$$\sin\psi = \sin\phi\sin\epsilon/\sin Az,$$

and
$$\cos\phi = \cos\theta\cos Az+\sin\theta\sin Az\cos\psi.$$

Differentiating the last equation we find

$$d\phi = \sin\theta\sin\epsilon\,d\psi,$$

whence
$$\frac{\sin\phi\,d\epsilon}{(u'_x-u_x)\cos\eta+(u'_y-u_y)\sin\eta} = \frac{d\psi}{R}.$$

Since
$$u_x\sin\phi\cos\eta+u_y\sin\phi\sin\eta+u_z\cos\phi = u\cos BS,$$

we have
$$L = \frac{1}{R}\int_0^{2\pi}\exp\left[-2hMgu\cos BS\right]d\psi.$$

We may take Az to pass through B without loss of generality. Hence, writing

$$a = 2hMgu \sin\theta \sin AB = 2hMg \sin\theta \{\Sigma(u_y u'_z - u_z u'_y)^2\}^{\frac{1}{2}}/R, \qquad (8)$$

we have

$$L = \frac{\exp(-2hMgu\cos\theta\cos AB)}{R}\int_0^{2\pi}\exp(-a\cos\psi)\,d\psi$$

$$= \frac{2\pi\exp[-h(m+M)P]}{R} I_0(a),$$

where $$I_0(x) = J_0(ix).$$

Hence

$$K(\omega,\omega') = \frac{2\pi}{R^2}\left(\frac{hM}{\pi}\right)^{\frac{3}{2}} N\exp\left[-hM\left(u^2 - \frac{m}{M}R^2 + \frac{M-m}{M}P\right)\right]$$

$$\times\int_0^{\sigma}\frac{\exp(-hMg^2)}{\partial R/\partial g} I_0(a)\, g^3 b\, db. \qquad (9)$$

Here g and $\dfrac{\partial R}{\partial g} = \dfrac{2M}{m+M}\left(\cos\theta - g\sin\theta\dfrac{\partial\theta}{\partial g}\right)$ are to be expressed in terms of b by means of the equations

$$\theta = f(g, b), \qquad R = \{\Sigma(u'_x - u_x)^2\}^{\frac{1}{2}}.$$

The second part of Boltzmann's integral, namely $\int fF\, gb\, db\, d\epsilon\, d\Omega$, is equal to $k(\omega)f(\omega)$, where

$$k(\omega) = \int F\, gb\, db\, d\epsilon\, d\Omega.$$

Taking σ for the upper limit of b as before, we have

$$k(\omega) = N\pi\sigma^2\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}\int\{(U_x-u_x)^2+(U_y-u_y)^2+(U_z-u_z)^2\}^{\frac{1}{2}}\exp(-hMU^2)d\Omega.$$

$k(\omega)$ depends on u only, and is equal to

$$k(u) = \pi^{\frac{1}{2}} N\sigma^2\psi\{u(hM)^{\frac{1}{2}}\}/uhM, \qquad (10)$$

where $\psi(x)$, in Jeans's notation,* is the function introduced by Tait,† and called by him X_2. Hence Boltzmann's equation becomes

$$\int K(\omega,\omega')f(\omega')\,d\omega' - k(u)f(\omega) = Df. \qquad (11)$$

* *Dynamical Theory of Gases*, p. 231.

† P. G. Tait, *Phil. Trans. R. S. E.*, Vol. 33, p. 74, 1886.

3. *Case of Elastic Collisions.*

Here θ is independent of g and equal to $\sin^{-1}(b/\sigma)$ if $b < \sigma$, $\frac{1}{2}\pi$ if $b > \sigma$, σ being the sum of the radii of an ion and molecule. Putting

$$\gamma = 2M/(m+M), \tag{12}$$

and expressing everything in terms of θ, we have

$$R = \gamma g \cos\theta, \qquad \partial R/\partial g = \gamma \cos\theta,$$

and the integral in (9) becomes

$$I = \sigma^2 \int_0^{\frac{1}{2}\pi} \exp\left(-\frac{hMR^2}{\gamma^2 \cos^2\theta}\right) \frac{R^3}{\gamma^4 \cos^4\theta} I_0\left(\frac{2hMRu \sin AB}{\gamma} \tan\theta\right) \cos\theta \sin\theta \, d\theta.$$

Write $\xi = hMR^2/\gamma^2, \quad \eta = 2hMRu \sin AB/\gamma, \quad x = \tan\theta.$

Then $$I = \frac{\sigma^2 R^3}{\gamma^4} \exp(-\xi) \int_0^\infty \exp(-\xi x^2)\, I_0(\eta x)\, x\, dx.$$

This last integral is known,* and thus we have, after a little reduction,

$$K(\omega, \omega') = \left(\frac{hM}{\pi}\right)^{\frac{1}{2}} \frac{\sigma^2 N}{\gamma^2 R} \exp\left[-\frac{hM}{\gamma^2 R^2}\left\{P^2\gamma^2 + 2\gamma(\gamma-1)PR^2 + (\gamma-1)^2 R^4\right\}\right]. \tag{13}$$

I have verified that the same result would be obtained by Hilbert's transformations. As regards the most general case, I had hoped to find some property of the kernel K which would reveal the part played by the inverse fifth power law, but as there does not appear to be one, most of the detailed calculations are restricted to the case of elastic collisions.

4. *Change of Variables.*

The objects of making a further change of variable are to obtain a kernel which is symmetrical in ω and ω', and to get rid of the function $k(u)$. It is also convenient to remove the factor hm from the exponentials.

The function $K(\omega, \omega')$ only differs from a symmetrical function by a function of ω. For since a is symmetrical in ω and ω', the integral in (9)

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* Nielsen, *Cylinderfunktionen*, p. 185.

is also. Hence

$$\frac{K(\omega, \omega')}{K(\omega', \omega)} = \exp\Big[-hM\Big\{\Sigma(u_x^2-u_x'^2)+\frac{M-m}{M}\Sigma u_x(u_x'-u_x) - \frac{M-m}{M}\Sigma u_x'(u_x-u_x')\Big\}\Big] = \exp\left[-hm(u^2-u'^2)\right],$$

so that $\exp(hmu^2)\,K(\omega, \omega')$ is symmetrical in ω and ω'. The reduction therefore proceeds on known lines, and we have only to quote the final results. The variables u_x, u_y, u_z, u_x', u_y', u_z' are changed to p_x, p_y, p_z, p_x', p_y', p_z', where

$$p_x = (hm)^{\frac{1}{2}}u_x, \quad \ldots, \quad p_x' = (hm)^{\frac{1}{2}}u_x', \quad \ldots . \tag{14}$$

We shall write $d\rho = dp_x dp_y dp_z$, $\quad d\rho' = dp_x' dp_y' dp_z'$,

and use the notation ρ, ρ' as we have previously used ω, ω'. Let $r = m/M$ denote the ratio of the mass of the ion to that of the molecule, and let

$$\chi_0(p) = r^{\frac{1}{2}}\exp(-\tfrac{1}{2}p^2)\left\{\psi(r^{-\frac{1}{2}}p)/p\right\}^{\frac{1}{2}}. \tag{15}$$

Writing

$$\chi(\rho) = \chi_0(p)\exp(p^2)f(\omega), \tag{16}$$

$\chi(\rho)$ satisfies the equation

$$\chi(\rho)-\int K(\rho, \rho')\chi(\rho')\,d\rho' = -\frac{\tau}{\chi_0(p)}D\left[\frac{\chi(\rho)}{\exp(p^2)\chi_0(p)}\right], \tag{17}$$

where

$$K(\rho, \rho') = \frac{\exp\left[-\frac{1}{2}hm(u^2+u'^2)\right]}{(hm)^{\frac{3}{2}}\left\{k(u)\,k(u')\right\}^{\frac{1}{2}}}\exp(hmu^2)\,K(\omega, \omega'), \tag{18}$$

and

$$\tau = (hm)^{\frac{1}{2}}/\pi^{\frac{1}{2}}N\sigma^2. \tag{19}$$

The limits of integration are $-\infty$ to ∞ for p_x', p_y', p_z'. In the case of elastic spheres we have, on reduction,

$$K(\rho, \rho') = \frac{(1+r)^2}{4\pi r^{\frac{1}{2}}\xi}\left\{\frac{pp'}{\psi(r^{-\frac{1}{2}}p)\,\psi(r^{-\frac{1}{2}}p')}\right\}^{\frac{1}{2}}\exp\left[-\tfrac{1}{4}\left\{r\xi^2+\frac{(p'^2-p^2)^2}{r\xi^2}\right\}\right], \tag{20}$$

where

$$\xi^2 = (p_x'-p_x)^2+(p_y'-p_y)^2+(p_z'-p_z)^2.$$

The equation (17) is fundamental in what follows.

5. *The Associated Fredholm Equation.*

It is convenient to consider the equation

$$\chi(\rho)-\int K(\rho, \rho')\chi(\rho')\,d\rho' = G(\rho), \tag{21}$$

where $G(\rho)$ is a given function of p_x, p_y, p_z. A numerical solution can only be obtained by reducing the number of variables. Lunn (*loc. cit.*) points out that if K is expanded in a series of Legendre functions of the cosine of the angle between p and p', and G in surface-harmonics, then each coefficient in the expansion of χ is given by a Fredholm equation in the single variable p. In fact, if we write

$$\left.\begin{aligned} \chi(\rho) = \sum_{n=0}^{\infty} \frac{Y_n \Theta_n(p)}{p}, \quad K(\rho, \rho') = \sum_{n=0}^{\infty} \frac{(2n+1)\, k_n(p, p')\, P_n(\eta)}{4\pi p p'} \\ G(\rho) = \sum_{n=0}^{\infty} Z_n G_n(p) \end{aligned}\right\}, \tag{22}$$

where η is the cosine of the angle between p and p', and Y_n, Z_n are surface-harmonics, then $Y_n = Z_n$ and the coefficient $\Theta_n(p)$ is given* by the Fredholm equation

$$\Theta_n(p) - \int_0^\infty k_n(p, p')\, \Theta_n(p')\, dp' = p G_n(p). \tag{23}$$

Here $k_n(p, p')$ is a symmetrical function of p, p' given by

$$k_n(p, p') = 2\pi p p' \int_{-1}^{1} K(\rho, \rho')\, P_n(\eta)\, d\eta.$$

Since $$\xi^2 = p^2 + p'^2 - 2pp'\eta,$$

we may change the variable to ξ and write

$$k_n(p, p') = 2\pi \int_{p \sim p'}^{p+p'} K(\rho, \rho')\, P_n\left(\frac{p^2 + p'^2 - \xi^2}{2pp'}\right) \xi\, d\xi. \tag{24}$$

The particular case $n = 0$ requires special consideration. For the equation

$$\Theta_0(p) - \int_0^\infty k_0(p, p')\, \Theta_0(p')\, dp' = p G_0(p), \tag{25}$$

is still of the form (21), the above theory having, from this point of view, merely shown that the harmonics of different orders are separable. Hence the homogeneous equation corresponding to (25) admits the equilibrium solution

$$\chi(\rho) = A\chi_0(p), \quad \text{or} \quad \Theta_0(p) = p\chi_0(p) = \phi_0(p) \text{ say.}$$

If we insert a parameter λ in equation (25) and solve by Fredholm's

* The fact that $K(\rho, \rho')$ becomes infinite for $\rho = \rho'$ certainly gives rise to no difficulty as long as the number of harmonics is finite.

method, there is the peculiarity that $\int_0^\infty k_0(p, p)\,dp$ does not converge. This is covered by a theorem of Haskins,* that diagonal terms like $k_0(p, p)$ can be put zero in Fredholm's resolvent without altering its value. Writing the resolvent in the form

$$\frac{D(p, p', \lambda)}{D(\lambda)} = \frac{A_0(p, p') - \lambda \frac{A_1(p, p')}{1!} + \ldots + (-)^n \lambda^n \frac{A_n(p, p')}{n!} + \ldots}{1 - \lambda \frac{A_1}{1!} + \ldots + (-)^n \lambda^n \frac{A_n}{n!} + \ldots}, \quad (26)$$

where A_n is a constant, and $A_n(p, p')$ a certain linear function of the iterated kernels, the solution of (26) is

$$\Theta_0(p) = pG_0(p) + \lambda \int_0^\infty \frac{D(p, p', \lambda)}{D(\lambda)} p' G_0(p')\,dp'. \quad (27)$$

In the present case $D(1) = 0$. It is known from the general theory that the condition of solubility of (25) is

$$\int pG_0(p)\,\phi_0(p)\,dp = 0;$$

and further if $\phi_1(p)$ is a particular solution of (25), the general solution is

$$\Theta_0(p) = \phi_1(p) + c\phi_0(p).$$

The particular solution is found as follows. Since $\lambda = 1$ is only a simple root of $D(\lambda)$, we have in the vicinity of $\lambda = 1$,

$$\frac{D(p, p', \lambda)}{D(\lambda)} = \frac{D(p, p', 1) + \xi D'(p, p', 1) + \ldots}{\xi D'(1) + \ldots},$$

where $\xi = \lambda - 1$ and dashes on D denote differentiation with respect to λ. The right-hand side of (27) remains finite if

$$\int D(p, p', 1)\, pG_0(p)\,dp = 0;$$

which is the same condition as before, since $D(p, p', 1)$ is known to be a mere multiple of $\phi_0(p)\,\phi_0(p')$. Hence taking the limit when $\xi \to 0$, we have

$$\phi_1(p) = pG_0(p) + \int_0^\infty \frac{D'(p, p', 1)}{D'(1)} p' G_0(p')\,dp', \quad (28)$$

so that $\phi_1(p)$ can be found when the coefficients A are computed.

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* See Kellogg, *Gött. Nachr.*, Vol. 4, p. 174, 1902. A much more general theorem is given by Poincaré, *Acta Mathematica*, Vol. 33, p. 57, 1910.

6. *Steady Diffusion and Motion in an Electric Field.*

The equilibrium distribution is given by

$$\chi(\rho) = A\chi_0(p),$$

A being independent of ρ. For small departures from it we can put $\chi = A\chi_0$ on the right-hand side of (17), and thus we have

$$\chi(\rho) - \int K(\rho, \rho')\,\chi(\rho')\,d\rho'$$

$$= -\frac{\tau}{\chi_0(p)} D[A \exp(-p^2)]$$

$$= -\frac{\tau}{\exp(p^2)\,\chi_0(p)}\left[\frac{\partial A}{\partial t} + \frac{1}{(hm)^{\frac{1}{2}}}\Sigma p_x \frac{\partial A}{\partial x} - 2(hm)^{\frac{1}{2}} A \Sigma F_x p_x\right]. \quad (29)$$

For steady motion $\partial A/\partial t = 0$ and the surface harmonics are of order 1 only. Writing e for the charge on the ion and E for the electric force, we have, with the notation of the last article,

$$G_1(p) = \frac{\tau}{\exp(p^2)\,\chi_0(p)} \frac{p}{(hm)^{\frac{1}{2}}}, \quad (30)$$

$$Z_1 = \frac{p_x}{p}\left(2AehE_x - \frac{\partial A}{\partial x}\right) + \frac{p_y}{p}\left(2AehE_y - \frac{\partial A}{\partial y}\right) + \frac{p_z}{p}\left(2AehE_z - \frac{\partial A}{\partial z}\right). \quad (31)$$

Hence
$$\chi(\rho) = A\chi_0(p) + Z_1\Theta_1(p)/p, \quad (32)$$

where Θ_1 is given by the Fredholm equation

$$\Theta_1(p) - \int_0^\infty k_1(p, p')\,\Theta_1(p')\,dp' = pG_1(p). \quad (33)$$

In the case of elastic spheres

$$k_1(p, p') = \frac{(1+r)^2}{2r^{\frac{1}{2}}}\left\{\frac{pp'}{\psi(r^{-\frac{1}{2}}p)\,\psi(r^{-\frac{1}{2}}p')}\right\}^{\frac{1}{2}} \int_{p\sim p'}^{p+p'} \exp\left[-\tfrac{1}{4}\left\{r\xi^2 + \frac{(p'^2-p^2)^2}{r\xi^2}\right\}\right] \times \frac{p^2+p'^2-\xi^2}{2pp'}\,d\xi. \quad (34)$$

Supposing for the present that equation (33) has been solved, so that Θ_1 is known, we have, for the first approximation to the distribution function in the disturbed motion,

$$f = A\exp(-p^2) + \frac{\tau}{(hm)^{\frac{1}{2}}}\Psi(p)\,\Sigma p_x\left(2AehE_x - \frac{\partial A}{\partial x}\right), \quad (35)$$

where
$$\Psi(p) = \frac{(hm)^{\frac{1}{2}}}{\tau}\frac{\Theta_1(p)}{p^2\exp(p^2)\,\chi_0(p)}. \quad (36)$$

Let $v(v_x, v_y, v_z)$ denote the mean velocity of the ions near the point (x, y, z), so that

$$v_x = \frac{1}{(hm)^{\frac{1}{2}}} \frac{\int f p_x d\rho}{\int f d\rho}, \tag{37}$$

where the integrals are extended over the range $-\infty$ to ∞ of p_x, p_y, p_z. Hence from (35) we find

$$v_x = \frac{4\tau}{3hm\pi^{\frac{1}{2}}} \left(2ehE_x - \frac{1}{A} \frac{\partial A}{\partial v}\right) \int_0^\infty p^4 \Psi(p)\, dp.$$

But if β is the mobility (velocity under unit electric force) and K the coefficient of diffusion,

$$v_x = \beta E_x - Kn^{-1} \partial n/\partial x = \beta E_x - KA^{-1}\, \partial A/\partial x.$$

Comparing these expressions and remembering equation (19), we have

$$\beta = \frac{8eh}{3(hm)^{\frac{1}{2}} N\pi\sigma^2} \int_0^\infty p^4 \Psi(p)\, dp, \qquad K = \frac{4}{3(hm)^{\frac{1}{2}} N\pi\sigma^2} \int_0^\infty p^4 \Psi(p)\, dp. \tag{38}$$

7. *Numerical Calculation for* $r = 1$.

Since $k_1(p, p')$ is symmetrical in p and p', we need only consider the case in which $p \geqslant p'$. Writing

$$x = \tfrac{1}{2}(p + p'), \qquad y = \tfrac{1}{2}(p - p'),$$

we may write (34) in the form

$$k_1(p, p') = \frac{4}{\{pp'\psi(p)\psi(p')\}^{\frac{1}{2}}} \int_y^x \exp\left[-\left(\theta^2 + \frac{x^2 y^2}{\theta^2}\right)\right](x^2 + y^2 - 2\theta^2)\, d\theta,$$

which is convenient for calculation. The integrations have been performed throughout by Simpson's rule, except where simplification could be advantageously made, while the less important numbers have been freely interpolated. The function $k_1(p, 0)$ is zero for all values of p, while the diagonal terms $k_1(p, p)$ are calculated independently from the formula

$$k_1(p, p) = \frac{4}{p\psi(p)} \left[p \exp(-p^2) + (p^2 - 1) \int_0^p \exp(-x^2)\, dx\right].$$

I had first begun to calculate the iterated kernels $k_{12}(p, p')$, $k_{13}(p, p')$, ... with a view to finding Fredholm's resolvent; but as it appears that the

series of iterations converges the work is much simplified. The solving function is

$$S(p, p') = k_1(p, p') + k_{12}(p, p') + k_{13}(p, p') + \ldots, \tag{39}$$

and the solution of equation (33) is

$$\Theta_1(p) = pG_1(p) + \int_0^\infty S(p, p')\, p' G_1(p')\, dp'. \tag{40}$$

In the tables (pp. 102–108) it has been found possible, with a little extrapolation, to calculate $S(p, p')$ for values of p, p' up to 3, provided that $k_1(p, p')$ is first calculated for the ranges $p \leqslant 4$, $p' \leqslant 3$. Decimal points are omitted, as well as numbers below 0·002. The higher numbers are not likely to be in error by more than 2 per cent., and some of them are much closer; but the amount of labour required to obtain all the significant figures correct is so great that I have not attempted it.

It will be seen that the calculation of the kernel $k_1(p, p')$ is complicated in any case; hence the labour of calculation for any assigned law of force need not be much greater than for the collision of elastic spheres. The table on p. 108 having given $S(p, p')$, we find $\chi_0(p)$, $G_1(p)$, $\Theta_1(p)$, $\Psi(p)$ successively from equations (15), (30), (40), and (36). This can be done with a little extrapolation from $p = 0$ to $p = 2{\cdot}8$, and the results are given in the table on p. 109.

The value of $\int_0^\infty p^4 \Psi(p)\, dp$ is found to be 0·356, and thus we have

$$\beta = \frac{0{\cdot}302eh}{(hm)^{\frac{1}{2}} N\sigma^2}, \qquad K = \frac{0{\cdot}151}{(hm)^{\frac{1}{2}} N\sigma^2}. \tag{41}$$

This also gives the coefficient of diffusion of a simple gas into itself, σ being the diameter of a molecule.

8. *Discussion of the Results.*

Prior to Hilbert's discovery, the difficulties in the way of determining the form of the distribution function in any case other than that of equilibrium were very great, and the diffusion problem has usually been treated by assuming a Maxwellian distribution of velocities relative to the mean velocity of drift. In the case of elastic spheres Stefan* and Langevin† have given the formula

$$K = \frac{3}{16N\sigma^2} \left(\frac{m+M}{hmM\pi}\right)^{\frac{1}{2}}, \tag{42}$$

* J. Stefan, *Wien. Ber.*, II, Vol. 65, p. 323, 1872.

† P. Langevin, *Ann. de Chim. et de Phys.* [8], Vol. 5, p. 245, 1905.

Values of $k_1(p, p')$.

p' \ p	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	1·1	1·2	1·3	1·4
0·1	067	033	022	016	013	010	008	006	004	003	003	002	002	002
0·2	132	085	059	044	036	029	023	018	014	011	009	007	006	005
0·3	198	142	115	083	062	049	042	034	027	022	017	014	011	009
0·4	260	195	152	120	094	076	062	049	040	032	025	020	015	012
0·5	322	251	199	154	124	100	081	066	051	041	033	025	020	016
0·6	374	299	237	193	152	122	098	078	061	048	038	030	023	018
0·7	433	340	276	219	177	141	112	090	070	055	043	033	026	020
0·8	482	382	308	248	197	158	126	101	078	061	047	036	023	021
0·9	526	414	337	267	210	169	133	108	084	064	050	038	029	022
1·0	564	442	362	290	238	176	139	112	088	066	052	039	030	023
1·1	597	475	381	302	234	184	144	115	088	067	050	038	029	022
1·2	624	496	390	308	247	192	150	117	089	068	048	037	028	021
1·3	646	510	396	314	250	194	150	117	089	068	048	037	027	020
1·4	662	523	400	320	252	197	150	117	089	067	047	036	026	020
1·5	673	529	406	321	248	192	146	112	083	064	044	035	025	019
1·6	680	533	411	319	244	187	142	108	079	060	041	033	024	018
1·7	681	534	411	314	239	182	137	104	075	056	037	030	021	016
1·8	680	532	410	308	232	176	130	099	071	052	034	027	018	013
1·9	678	529	404	301	225	170	123	093	066	048	031	024	016	011
2·0	673	520	396	294	218	163	116	088	061	044	029	022	014	009
2·1	664	510	388	283	211	156	110	082	057	041	027	020	012	008
2·2	656	500	378	272	203	148	104	076	053	037	025	018	011	007
2·3	645	488	365	262	195	140	098	070	049	033	023	016	010	006
2·4	634	474	352	252	187	132	092	065	045	030	021	014	009	005
2·5	622	461	339	240	178	124	086	061	042	027	018	012	008	004
2·6	611	447	327	230	169	116	080	057	039	025	016	011	007	003
2·7	598	435	311	220	160	109	074	054	036	023	014	010	006	003
2·8	586	422	294	210	151	102	068	051	033	022	012	009	005	
2·9	573	410	275	200	142	095	063	048	031	020	010	007		
3·0	561	398	260	188	137	089	058	045	029	018	008			

p' \ p	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·6	3·7	3·8	3·9	4·0
0·1	002																									
0·2	004	003	002																							
0·3	007	005	004	003	002																					
0·4	011	009	007	005	004	003	002																			
0·5	016	012	009	007	005	004	003	002																		
0·6	023	018	014	011	008	006	005	003	002																	
0·7	033	026	020	015	012	009	007	005	004	003	002															
0·8	047	036	028	023	021	016	013	009	007	005	004	003	002													
0·9	064	050	038	029	022	017	013	009	007	005	004	003	002	002												
1·0	088	066	052	039	030	023	017	012	009	007	005	004	003	002	002											
1·1	115	088	067	050	038	029	022	016	011	008	006	004	003	002	002											
1·2	150	117	089	068	048	037	028	021	015	011	008	006	004	003	002											
1·3	194	150	117	089	068	048	037	027	020	014	010	007	005	003	002											
1·4	250	197	150	117	089	067	047	036	026	020	014	010	007	005	003	002										
1·5	321	248	192	146	112	083	064	044	035	025	019	013	009	006	004	003	002									
1·6	411	319	244	187	142	108	079	060	041	033	024	018	012	009	006	004	003	002								
1·7	534	411	314	239	182	137	104	075	056	037	030	021	016	010	008	005	004	003	002							
1·8	680	532	410	308	232	176	130	099	071	052	034	027	018	013	008	007	005	003	002							
1·9	678	529	404	301	225	170	123	093	066	048	031	024	016	011	007	006	004	003	002							
2·0	673	520	396	294	218	163	116	088	061	044	029	022	014	009	006	005	003	002								
2·1	664	510	388	283	211	156	110	082	057	041	027	020	012	008	005	004	003	002								
2·2	656	500	378	272	203	148	104	076	053	037	025	018	011	007	005	004	002									
2·3	645	488	365	262	195	140	098	070	049	033	023	016	010	006	004	003	002									
2·4	634	474	352	252	187	132	092	065	045	030	021	014	009	005	003	002										
2·5	622	461	339	240	178	124	086	061	042	027	018	012	008	004	003											
2·6	611	447	327	230	169	116	080	057	039	025	016	011	007	003	002											
2·7	598	435	311	220	160	109	074	054	036	023	014	010	006	003												
2·8	586	422	294	210	151	102	068	051	033	022	012	009	005													
2·9	573	410	275	200	142	095	063	048	031	020	010	007														
3·0	561	398	260	188	137	089	058	045	029	018	008															

First Iteration of $k_1(p, p')$.

$p' = 0{\cdot}1$	$p = 0{\cdot}1$	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8
0·2	003	004	005	006	007	007	006	006	005	005	004	004	003	003	002	002	1·9	2·0
0·3	009	012	014	015	015	015	014	014	013	012	010	009	008	006	005	005	004	003
0·4	019	022	024	025	025	024	023	023	021	020	016	014	012	010	008	007	005	004
0·5	032	038	040	041	040	038	036	033	029	026	022	019	016	013	011	009	007	005
0·6	048	053	056	056	055	052	048	042	037	032	028	024	020	017	014	011	009	007
0·7	066	071	074	072	069	066	059	052	045	039	034	029	024	020	016	013	010	008
0·8	084	089	091	087	082	074	066	058	051	044	038	033	027	022	017	013	010	008
0·9	104	108	109	103	096	087	078	068	059	050	042	035	029	023	018	014	011	008
1·0	122	125	124	118	107	096	085	074	063	053	044	036	029	024	019	014	011	008
1·1	140	142	138	132	118	104	092	079	066	055	045	037	029	023	018	013	011	008
1·2	155	154	148	138	125	110	095	081	067	056	046	038	030	023	017	013	010	007
1·3	165	164	155	145	129	113	097	082	067	056	045	037	029	022	017	013	010	006
1·4	174	172	162	149	131	114	097	081	066	055	044	037	028	020	016	012	008	
1·5	180	177	163	150	132	114	096	079	064	053	042	036	025	018	014	010		
1·6	184	180	162	149	130	112	093	075	061	050	039	032	023	017	013			
1·7	185	179	160	147	125	109	089	072	057	044	035	027	021	016				
1·8	183	173	158	145	123	105	085	068	053	041	032	024	019					
1·9	181	166	156	143	118	099	079	064	050	037	029	022						
2·0	179	160	150	136	111	094	073	059	046	034	026							
2·1	172	154	143	129	104	089	068	054	042	031								
2·2	164	147	136	122	098	084	063	049	037									
2·3	154	140	128	115	091	076	059	044										
2·4	144	133	122	106	082	067	051											
2·5	137	126	112	098	074	058												
2·6	129	118	103	085	066													
2·7	121	110	096	074														
2·8	111	103	084															
2·9	105	096																
3·0	099																	

$p' = 0{\cdot}1$	$p = 2{\cdot}1$	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0
0·2	002	002	002	002	002	002	002	002	002	002
0·3	002	002	002	002	002	002	002	002	002	002
0·4	003	003	003	003	003	003	003	003	003	002
0·5	004	004	004	004	004	004	004	004	003	003
0·6	005	005	005	005	005	005	005	004	004	003
0·7	007	007	007	006	006	006	006	005	005	004
0·8	009	009	008	008	008	007	007	006	006	005
0·9	011	011	010	010	009	009	008	008	007	006
1·0	014	013	013	012	012	011	010	010	009	008
1·1	017	016	016	015	014	013	013	012	011	010
1·2	020	020	019	018	017	016	015	014	013	012
1·3	024	023	022	021	020	019	018	017	016	014
1·4	028	027	026	025	024	023	022	021	019	018
1·5	033	032	031	030	029	028	027	025	024	022
1·6	039	038	037	036	035	034	032	031	029	027
1·7	046	045	044	043	042	040	039	037	035	033
1·8	054	053	052	051	049	048	046	044	042	040
1·9	063	062	061	059	058	056	054	052	050	047
2·0	073	072	071	069	067	065	063	061	058	056
2·1	084	083	081	079	077	075	073	070	068	065
2·2	096	095	093	091	089	086	084	081	078	075
2·3	109	108	106	104	101	099	096	093	090	087
2·4	123	122	120	117	115	112	109	106	103	099
2·5	138	136	134	131	128	125	122	119	115	112
2·6	153	151	149	146	143	140	136	133	129	125
2·7	168	166	164	161	158	154	151	147	143	139
2·8	183	181	179	176	172	169	165	161	157	153
2·9	197	195	193	190	186	183	179	175	171	166
3·0	210	208	206	203	199	196	192	188	183	179

Second Iteration of $k_1(p, p')$.

$p' = 0{\cdot}1$	$p = 0{\cdot}1$	$0{\cdot}2$	$0{\cdot}3$	$0{\cdot}4$	$0{\cdot}5$	$0{\cdot}6$	$0{\cdot}7$	$0{\cdot}8$	$0{\cdot}9$	$1{\cdot}0$	$1{\cdot}1$	$1{\cdot}2$
$0{\cdot}2$	002	002	002	$1{\cdot}3$	$1{\cdot}4$	$1{\cdot}5$	$1{\cdot}6$	$1{\cdot}7$	$1{\cdot}8$	$1{\cdot}9$	$2{\cdot}0$	
$0{\cdot}3$	002	003	003	004	004	004	004	003	003	002	002	$2{\cdot}1$ $2{\cdot}2$
$0{\cdot}4$	003	003	005	006	006	007	007	007	008	007	007	006	005	005	004	003	003	002	$2{\cdot}3$ $2{\cdot}4$
$0{\cdot}5$	006	007	008	010	010	011	012	012	011	011	010	010	009	008	007	006	005	004	003	002	$2{\cdot}5$ $2{\cdot}6$
$0{\cdot}6$	010	012	014	015	016	017	016	016	015	015	014	013	012	010	008	007	005	004	003	002	$2{\cdot}7$ $2{\cdot}8$
$0{\cdot}7$	015	018	021	023	024	025	024	023	022	020	018	016	014	012	010	008	006	005	004	003	002	$2{\cdot}9$
$0{\cdot}8$	022	025	028	030	031	030	029	028	026	023	021	018	015	013	011	009	007	005	004	003	002	002	$3{\cdot}0$
$0{\cdot}9$	030	033	036	037	037	036	034	032	029	026	023	020	017	014	012	010	008	006	004	003	002	002	
$1{\cdot}0$	038	041	044	045	044	042	039	036	032	028	025	021	017	014	012	010	008	006	004	003	003	002	
$1{\cdot}1$	046	049	051	050	049	046	043	039	034	030	026	022	018	015	012	009	007	005	004	003			
$1{\cdot}2$	054	055	056	055	053	049	045	041	036	031	026	022	018	015	011	009	007	005	004				
$1{\cdot}3$	059	060	061	059	055	051	047	042	037	032	027	022	018	014	011	009	007	005					
$1{\cdot}4$	065	064	064	061	057	052	047	041	036	031	026	021	017	014	010	008	006						
$1{\cdot}5$	067	067	066	063	058	053	047	041	035	030	025	020	016	013	010	007							
$1{\cdot}6$	069	069	067	062	057	051	045	039	033	028	023	018	014	011	008								
$1{\cdot}7$	070	069	066	061	055	049	043	037	031	026	021	017	013	009									
$1{\cdot}8$	069	068	064	059	053	047	041	035	029	024	019	014	010										
$1{\cdot}9$	068	066	062	057	051	044	038	032	027	021	016	011											
$2{\cdot}0$	066	063	058	053	047	041	035	030	023	017	012												
$2{\cdot}1$	062	059	055	050	044	038	032	025	019	013													
$2{\cdot}2$	058	054	051	045	040	034	028	022	016														
$2{\cdot}3$	053	050	046	041	036	031	025	020															
$2{\cdot}4$	048	045	041	037	033	029	025																
$2{\cdot}5$	044	041	038	035	033	031																	
$2{\cdot}6$	038	037	034	033	032																		
$2{\cdot}7$	035	033	032	031																			
$2{\cdot}8$	032	031	030																				
$2{\cdot}9$	030	029																					
$3{\cdot}0$	028																						

Third Iteration of $k_1(p, p')$.

$p' = 0{\cdot}4$	$p = 0{\cdot}4$	$0{\cdot}5$	$0{\cdot}6$	$0{\cdot}7$	$0{\cdot}8$	$0{\cdot}9$	$1{\cdot}0$	$1{\cdot}1$	$1{\cdot}2$	$1{\cdot}3$	$1{\cdot}4$	$1{\cdot}5$	$1{\cdot}6$	$1{\cdot}7$	$1{\cdot}8$	$1{\cdot}9$	$2{\cdot}0$	$2{\cdot}1$	$2{\cdot}2$	$2{\cdot}3$	$2{\cdot}4$	$2{\cdot}5$	$2{\cdot}6$	$2{\cdot}7$	$2{\cdot}8$	$2{\cdot}9$	$3{\cdot}0$
$0{\cdot}5$	002	002	002	002	003	003	003	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002
$0{\cdot}6$	003	003	004	004	005	005	005	004	004	004	003	003	003	003	003	003	003	002	002	002	002	002	002	002	002	002	002
$0{\cdot}7$	004	005	005	006	006	006	007	006	006	006	005	005	005	005	004	004	004	003	003	003	003	002	002	002	002	002	002
$0{\cdot}8$	005	006	007	008	008	008	009	009	008	008	008	007	006	006	005	005	004	004	003	003	003	003	002	002	002	002	002
$0{\cdot}9$	007	008	009	010	010	011	011	011	011	010	010	010	009	009	008	008	007	006	006	005	005	004	004	003	003	003	002
$1{\cdot}0$	010	011	012	013	013	014	014	014	014	013	013	013	012	011	011	010	010	009	008	008	007	007	006	006	005	005	004
$1{\cdot}1$	013	014	015	016	016	016	016	015	015	015	014	014	013	013	012	012	011	010	010	009	009	008	008	007	007	006	006
$1{\cdot}2$	016	018	018	018	019	019	018	018	018	017	017	016	016	015	015	014	014	013	013	012	012	011	011	010	010	009	009
$1{\cdot}3$	020	020	021	022	022	022	021	021	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013	012	012
$1{\cdot}4$	023	024	024	024	023	023	022	022	021	021	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013
$1{\cdot}5$	027	027	027	026	026	025	025	024	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016	015	015
$1{\cdot}6$	029	029	028	027	027	026	026	025	025	024	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016
$1{\cdot}7$	030	029	029	028	027	027	026	026	025	025	024	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016
$1{\cdot}8$	029	028	028	027	027	026	026	025	025	024	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016
$1{\cdot}9$	028	027	027	026	026	025	025	024	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016	015	015
$2{\cdot}0$	027	026	026	025	025	024	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014
$2{\cdot}1$	026	025	025	024	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013
$2{\cdot}2$	025	024	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013	012	012
$2{\cdot}3$	024	023	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013	012	012	011	011
$2{\cdot}4$	023	022	022	021	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013	012	012	011	011	010	010
$2{\cdot}5$	022	021	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013	012	012	011	011	010	010	009	009
$2{\cdot}6$	021	020	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013	012	012	011	011	010	010	009	009	008	008
$2{\cdot}7$	020	019	019	018	018	017	017	016	016	015	015	014	014	013	013	012	012	011	011	010	010	009	009	008	008	007	007
$2{\cdot}8$	019	018	018	017	017	016	016	015	015	014	014	013	013	012	012	011	011	010	010	009	009	008	008	007	007	006	006
$2{\cdot}9$	018	017	017	016	016	015	015	014	014	013	013	012	012	011	011	010	010	009	009	008	008	007	007	006	006	005	005
$3{\cdot}0$	017	016	016	015	015	014	014	013	013	012	012	011	011	010	010	009	009	008	008	007	007	006	006	005	005	004	004

Fourth Iteration of $k_1(p, p')$.

$p' = 0{\cdot}6$	$p = 0{\cdot}6$	0·7	0·8	0·9	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8
0·7		002	002	002	002	002	002	002	002	002	002	1·9	2·0	2·1
0·8	002	003	003	003	003	003	003	003	003	003	003	002	002	2·2	2·3
0·9	003	004	004	004	004	004	004	004	004	004	004	003	003	002	002	2·4	2·5
1·0	005	005	006	006	006	007	007	007	007	006	006	005	005	004	004	003	003	002	002	2·6
1·1	007	007	007	007	008	008	008	008	007	007	006	005	005	004	004	003	003	002	002	2·7
1·2	008	008	008	009	009	009	009	009	008	007	006	005	004	003	003	002	002	2·8
1·3	009	009	010	010	010	010	009	009	009	008	007	006	005	004	003	002	002	2·9
1·4	010	010	010	010	010	010	010	009	009	008	007	006	005	004	003	002	002	3·0
1·5	011	011	011	011	010	010	010	010	009	009	008	007	006	005	003	002	002	003
1·6	012	012	011	011	011	010	010	010	010	009	008	007	006	005	004	003	002	002
1·7	012	012	012	011	011	011	010	010	010	009	008	007	006	005	004	003	002	002
1·8	012	012	012	012	011	011	011	010	010	010	009	008	007	006	005	004	003	002
1·9	011	011	011	011	011	010	010	010	009	009	008	007	006	005	004	003	002	002
2·0	011	010	010	010	010	010	009	009	008	007	006	005	004	003	003	002	002	002
2·1	010	010	009	009	009	008	008	007	006	005	004	003	003	002	002	002	002	003
2·2	009	009	009	008	008	007	006	005	004	004	003	003	002	002	002	002	003	003
2·3	008	008	007	006	005	004	004	003	003	002	002	002	002	002	002	003	003	003
2·4	007	006	005	004	004	003	003	002	002	002	002	002	002	003	003	003	003	003
2·5	006	005	004	004	003	003	002	002	002	002	002	002	003	003	003	003	003	003
2·6	005	004	004	003	003	002	002	002	002	002	002	003	003	003	003	003	003	003
2·7	004	004	003	003	002	002	002	002	002	002	003	003	003	003	003	003	003	003
2·8	004	003	003	002	002	002	002	002	002	003	003	003	003	003	003	003	003	003
2·9	003	003	002	002	002	002	002	002	003	003	003	003	003	003	003	003	003	003
3·0	003	002	002	002	002	002	002	003	003	003	003	003	003	003	003	003	003	003

Fifth Iteration of $k_1(p, p')$.

$p' = 0{\cdot}9$	$p = 0{\cdot}9$	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5	2·6	2·7
1·0	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002
1·1	003	003	003	003	003	003	003	003	003	003	003	002	002	002	002	002	002	002	002
1·2	003	003	003	003	003	003	003	003	003	003	003	003	002	002	002	002	002	002	002
1·3	004	004	004	004	004	004	003	003	003	003	003	003	003	003	003	002	002	002	002
1·4	004	004	004	004	004	004	004	003	003	003	003	003	003	003	003	003	002	002	002
1·5	005	005	005	004	004	004	004	004	003	003	003	003	003	003	003	003	003	002	002
1·6	005	005	005	005	004	004	004	004	004	003	003	003	003	003	003	003	003	003	002
1·7	005	005	005	005	004	004	004	004	004	004	003	003	003	003	003	003	003	003	002
1·8	005	005	005	004	004	004	004	004	004	004	004	003	003	003	003	003	003	003	002
1·9	004	004	004	004	004	004	004	004	004	004	004	004	003	003	003	003	003	003	002
2·0	004	004	004	004	004	004	004	004	004	004	004	004	004	003	003	003	003	003	002
2·1	004	004	004	003	003	003	003	003	003	003	003	003	003	003	003	003	003	002	002
2·2	003	003	003	003	003	003	003	003	003	003	003	003	003	003	003	003	002	002	002
2·3	003	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002
2·4	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002
2·5	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002	002

Solving Kernel of $k_1(p, p')$.

p' \ p	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3·0
0·1	067	033	022	016	013	010	008	006	004	003	003	002	002																	
0·2	135	089	064	050	043	036	029	024	022	018	016	011	010	008	007	005	004													
0·3	207	154	131	101	080	068	060	052	045	038	031	027	023	018	015	012	009	007	004											
0·4	282	220	181	151	127	109	094	081	072	062	051	043	035	029	023	017	012	010	008	005	004									
0·5	360	282	249	207	177	153	133	116	096	083	069	058	049	040	033	027	021	016	011	009	005	004								
0·6	435	368	312	269	231	199	170	145	121	103	088	074	062	050	040	033	025	019	015	010	007	004	002							
0·7	519	435	380	324	281	243	207	177	148	125	106	089	074	060	048	037	028	022	017	011	008	006	002							
0·8	597	507	439	379	324	277	236	202	170	142	120	100	081	065	052	041	029	022	017	011	008	006	002							
0·9	673	570	498	427	363	313	266	229	192	160	133	106	086	068	055	043	033	025	017	011	009	006								
1·0	744	629	553	478	414	340	289	246	206	169	140	111	088	072	057	043	033	024	017	011	007									
1·1	809	694	598	512	431	364	308	262	214	176	142	115	090	071	056	040	032	022	016	010										
1·2	864	736	626	535	459	384	323	270	220	179	142	116	092	070	052	040	031	021	015											
1·3	906	771	650	556	471	393	327	272	221	182	143	115	090	067	050	039	027	018												
1·4	942	800	667	570	479	400	327	270	220	179	140	112	086	064	047	036	024													
1·5	965	818	679	576	477	396	323	263	210	171	131	108	077	058	044	030														
1·6	980	828	683	572	469	387	314	251	199	160	120	097	070	053	039															
1·7	982	827	680	561	457	375	300	240	186	144	108	084	063	048																
1·8	977	817	673	551	445	362	285	227	174	134	096	074	054																	
1·9	969	802	661	538	428	342	266	209	161	119	086	065																		
2·0	959	781	641	516	406	325	248	196	144	106	075																			
2·1	936	758	619	493	387	307	230	176	130	095																				
2·2	912	733	595	467	365	287	210	159	116																					
2·3	883	705	564	441	340	263	194	145																						
2·4	853	677	538	413	318	241	179																							
2·5	827	648	506	389	298	224																								
2·6	797	618	479	361	279																									
2·7	770	593	452	337																										
2·8	743	568	420																											
2·9	720	546																												
3·0	699																													

p	$\chi_0(p)$	$\frac{(lm)^{\frac{1}{2}}}{\tau} pG_1(p)$	$\frac{(lm)^{\frac{1}{2}}}{\tau} \Theta_1(p)$	$\Psi(p)$	$\Psi(p)$ for relative Maxwellian distribution.
0·1	1·41	0·00707	0·00890	0·630	0·530
0·2	1·40	0·0274	0·0371	0·635	0·510
0·3	1·37	0·0600	0·0842	0·613	0·485
0·4	1·34	0·102	0·145	0·575	0·452
0·5	1·29	0·151	0·218	0·529	0·413
0·6	1·25	0·201	0·291	0·451	0·370
0·7	1·19	0·252	0·378	0·397	0·325
0·8	1·12	0·301	0·459	0·338	0·280
0·9	1·05	0·343	0·530	0·277	0·246
1·0	0·980	0·375	0·594	0·223	0·195
1·1	0·900	0·401	0·646	0·177	0·157
1·2	0·821	0·415	0·684	0·137	0·120
1·3	0·740	0·417	0·702	0·103	0·0978
1·4	0·660	0·418	0·717	0·0780	0·0745
1·5	0·585	0·405	0·704	0·0565	0·0559
1·6	0·512	0·387	0·685	0·0405	0·0410
1·7	0·443	0·363	0·652	0·0284	0·0295
1·8	0·380	0·334	0·611	0·0194	0·0207
1·9	0·322	0·303	0·563	0·0131	0·0138
2·0	0·270	0·272	0·512	0·00867	0·00970
2·1	0·224	0·240	0·458	0·00564	0·00644
2·2	0·184	0·208	0·403	0·00358	0·00414
2·3	0·150	0·178	0·350	0·00222	0·00267
2·4	0·121	0·147	0·296	0·00134	0·00167
2·5	0·097	0·127	0·254	0·00082	0·00104
2·6	0·076	0·103	0·209	0·00047	0·00061
2·7	0·059	0·084	0·171	0·00027	0·00036
2·8	0·045	0·068	0·140	0·00015	0·00021

which for $m = M$ gives a result very close to (41), namely,

$$K = 0{\cdot}150/(hm)^{\frac{1}{2}} N\sigma^2.$$

But the work of Lorentz (Art. 9) shows very clearly that the relative distribution is not Maxwellian when r is small, so that the generality of these formulæ is somewhat impaired. Correct to the first order in v, a relative Maxwellian distribution would make f of the form (35), where

$$\Psi(p) = \frac{3\sqrt{2}}{8} \exp(-p^2).$$

This function is shown in the last column of the table on p. 109, from which it appears that the method still underestimates the contribution of the slower ions to the drift when $r = 1$.

This error also vitiates that part of a former paper* of mine in which the law of force is that of elastic spheres. Langevin, Enskog,† and Chapman‡ have found expressions for the coefficient of diffusion for any law of force, but the foregoing criticisms apply to these also. The extent to which the results of the two latter writers in the kinetic theory of gases require modification would be beyond the scope of the present paper to determine.

It may be mentioned that the elementary theory, corrected for "persistence of velocities,"§ gives a close approximation to formula (41), namely

$$K = 0{\cdot}143/(hm)^{\frac{1}{2}} N\sigma^2.$$

If the diffusion is not steady, harmonics of zero order appear in (29), and we have to solve the equation

$$\Theta_0(p) - \int_0 k_0(p, p')\,\Theta_0(p')\,dp' = pG_0(p), \qquad (43)$$

where

$$G_0(p) = -\tau(\partial A/\partial t)/\exp(p^2)\,\chi_0(p).$$

It is clear from Art. 5 that a numerical solution of this singular equation is quite feasible, though more laborious than that already carried out for the first order harmonic. I do not, however, propose to carry it out.

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* *Proc. Roy. Soc.*, A, Vol. 88, p. 296, 1913.

† D. Enskog, *Phys. Zeitschr.*, Vol. 13, pp. 56, 533, 1911.

‡ S. Chapman, *Phil. Trans.*, A, Vol. 211, p. 433, 1912.

§ Jeans, *Dynamical Theory of Gases*, p. 273.

9. *Equations for Ions of Small Mass.*

In discussing the case of r small we shall begin by obtaining some known results. In equation (17) we have

$$I \equiv \int K(\rho, \rho')\chi(\rho')d\rho'$$

$$= \frac{(1+r)^2}{4\pi r^{\frac{1}{2}}} \left\{\frac{p}{r\psi(r^{-\frac{1}{2}}p)}\right\}^{\frac{1}{2}} \int \left\{\frac{p'}{r\psi(r^{-\frac{1}{2}}p')}\right\}^{\frac{1}{2}}$$

$$\exp\left[-\tfrac{1}{4}\left\{r\xi^2+\frac{(p'^2-p^2)^2}{r\xi^2}\right\}\right]\frac{\chi(\rho')}{\xi}\,d\rho'. \quad (44)$$

If r is small, the kernel is only appreciable when p' is nearly equal to p, as is physically evident since the change of resultant velocity on collision is small. Let P be the point $\rho\,(p_x, p_y, p_z)$, P' the point ρ', and let OP' meet the sphere with centre O and radius OP at Q. Then we may take

$$d\rho' = p^2 dp' dS$$

without serious error, where dS is the element of solid angle at Q. Writing R for PQ, the exponential becomes approximately

$$\exp\left\{-p^2(p'-p)^2/rR^2\right\}.$$

When $x \to \infty$, $\psi(x) \to \pi^{\frac{1}{2}}x^2$, so that as long as $r^{-\frac{1}{2}}p$ is large we can write $1/\pi^{\frac{1}{2}}p$ for $p/r\psi(r^{-\frac{1}{2}}p)$. Hence

$$I \to \frac{p}{4\pi^{\frac{3}{2}}r^{\frac{1}{2}}} \int \frac{\chi\,dS}{R} \int_0^\infty \exp\left[-\frac{p^2(p'-p)^2}{rR^2}\right]dp',$$

where the value of χ is that at Q. Writing

$$p' = p + r^{\frac{1}{2}}Rx/p,$$

the p'-integral becomes

$$\frac{r^{\frac{1}{2}}R}{p}\int_{-p^2/r^{\frac{1}{2}}R}^{\infty} \exp(-x^2)\,dx = \frac{\pi^{\frac{1}{2}}r^{\frac{1}{2}}R}{p} \text{ approximately.}$$

Hence
$$I \to \frac{1}{4\pi}\int \chi\,dS,$$

the mean value of χ over a sphere of radius p. We may take

$$\chi_0(p) = p^{\frac{1}{2}}\exp\left(-\tfrac{1}{2}p^2\right)$$

without loss of generality, and thus equation (17) becomes

$$\chi - \frac{1}{4\pi}\int \chi dS = -\frac{\tau \exp(\frac{1}{2}p^2)}{\pi^{\frac{1}{2}}p^{\frac{1}{2}}} D\left[\frac{\chi}{p^{\frac{1}{2}}\exp(\frac{1}{2}p^2)}\right]. \tag{45}$$

Since $f = \chi p^{-\frac{1}{2}}\exp(-\frac{1}{2}p^2)$, the equation for f is

$$f - \frac{1}{4\pi}\int f dS = -\frac{\tau}{\pi^{\frac{1}{2}}p} Df. \tag{46}$$

An equivalent equation has been used by Lorentz* in his researches on the theory of electronic conduction in metals. I shall call it Lorentz's equation. Since M does not appear it is obvious that the approximation has gone to the extreme extent of supposing the molecules immovable: the limitations of this will be discussed later. The associated equation

$$f - \frac{1}{4\pi}\int f dS = G$$

is only soluble if the mean value of G over a sphere of radius p vanishes, when the particular solution is simply $f = G$. To solve the problem of steady diffusion and mobility put $f = A\exp(-p^2)$ on the right-hand side of (46) as before; hence

$$f = A\exp(-p^2) + \frac{\tau}{(hm\pi)^{\frac{1}{2}}}\frac{\exp(-p^2)}{p}\Sigma p_x\left(2Aehe_x - \frac{\partial A}{\partial x}\right), \tag{47}$$

$$\beta = \frac{4eh}{3(hm\pi)^{\frac{1}{2}}N\pi\sigma^2}, \qquad K = \frac{2}{3(hm\pi)^{\frac{1}{2}}N\pi\sigma^2}. \tag{48}$$

These formulæ are due to Lorentz. That (47) does not represent a relative Maxwellian distribution has already been mentioned. The Stefan-Langevin formula differs from (48) in this case by the factor

$$9\pi/32 = 0{\cdot}884.$$

10. *Next Approximation for r small.*

In the next approximation we can no longer neglect the factor $\exp(-\frac{1}{4}r\xi^2)$, and we have to distinguish more carefully between ξ, which is PP', and R, which is PQ. If $r^{-\frac{1}{2}}p$ is large we may write, correct to terms in r,

$$\frac{p}{r\psi(r^{-\frac{1}{2}}p)} = \frac{1}{\pi^{\frac{1}{2}}p}\left(1 - \frac{r}{2p^2}\right), \qquad \frac{p'}{r\psi(r^{-\frac{1}{2}}p')} = \frac{1}{\pi^{\frac{1}{2}}p'}\left(1 - \frac{r}{2p^2}\right).$$

* H. A. Lorentz, *Archives Néerlandaises*, Vol. 10, p. 336, 1905; *Theory of Electrons*, p. 268.

Thus $\int K(\rho, \rho')\chi(\rho')\,d\rho'$

$$= \frac{1+2r}{4\pi^{\frac{1}{2}}p^{\frac{1}{2}}r^{\frac{1}{4}}}\left(1-\frac{r}{2p^2}\right)\int \exp\left[-\tfrac{1}{4}\left\{r\xi^2+\frac{(p'^2-p^2)^2}{r\xi^2}\right\}\right]\frac{\chi(\rho')}{p'^{\frac{1}{2}}\xi}\,p'^2\,dp'\,dS.$$

We have to consider integrals of the form

$$J = \int \exp\left[-\tfrac{1}{4}\left\{r\xi^2+\frac{(p'^2-p^2)^2}{r\xi^2}\right\}\right]F(\rho')\,dp'\,dS,$$

where $$F(\rho') = p'^{\frac{3}{2}}\chi(\rho')/\xi.$$

Writing ϵ for $p'-p$, we have accurately

$$\xi^2 = R^2+\epsilon^2+R^2\epsilon/p.$$

Hence $$\frac{(p'^2-p^2)^2}{4r\xi^2} = \frac{\epsilon^2(\epsilon+2p)^2}{4r(R^2+\epsilon^2+R^2\epsilon/p)} = \frac{p^2\epsilon^2}{rR^2}\left(1-\frac{\epsilon^2}{R^2}+\frac{\epsilon^2}{4p^2}\right),$$

expanding to terms of the order ϵ^2. Putting

$$\epsilon = r^{\frac{1}{4}}Rx/p,$$

the exponential becomes

$$\exp\left[-\frac{rR^2}{4}-x^2\left(1-\frac{rx^2}{p^2}+\frac{rR^2x^2}{4p^4}\right)\right] = \left(1-\frac{rR^2}{4}+\frac{rx^4}{p^2}-\frac{rR^2x^4}{4p^4}\right)\exp(-x^2).$$

Hence to the order of approximation required

$$J = \frac{r^{\frac{1}{2}}}{p}\int\left(1-\frac{rR^2}{4}+\frac{rx^4}{p^2}-\frac{rR^2x^4}{4p^4}\right)\left(F+\frac{r^{\frac{1}{4}}Rx}{p}\,\frac{\partial F}{\partial p'}+\frac{rR^2x^2}{2p^2}\,\frac{\partial^2 F}{\partial p'^2}\right)R$$

$$\exp(-x^2)\,dx\,dS,$$

where F and its differential coefficients are taken at Q. Correct to terms in r, the x-integral is

$$\int_{-p^2r^{\frac{1}{4}}R}^{\infty}\left(F-\frac{rR^2}{4}F+\frac{rx^4}{p^2}F-\frac{rR^2x^4}{4p^4}F+\frac{r^{\frac{1}{2}}Rx}{p}\,\frac{\partial F}{\partial p'}+\frac{rR^2x^2}{2p^2}\,\frac{\partial^2 F}{\partial p'^2}\right)\exp(-x^2)\,dx.$$

If $p^2/r^{\frac{1}{4}}R$ is large, the first term is $\pi^{\frac{1}{2}}F$ correct to within any power of r, since when x is large

$$\int_0^x \exp(-x^2)\,dx = \frac{\pi^{\frac{1}{2}}}{2}-\frac{\exp(-x^2)}{2x}$$

approximately. The same argument applies to other integrals of the form

$$\int x^n \exp(-x^2)\,dx,$$

and thus the x-integral becomes

$$\pi^{\frac{1}{2}}\left[F-\frac{rR^2}{4}F+\frac{3r}{4p^2}F-\frac{3rR^2}{16p^4}F+\frac{rR^2}{4p^2}\frac{\partial^2 F}{\partial p'^2}\right].$$

Hence

$$J=\frac{\pi^{\frac{1}{2}}r^{\frac{1}{2}}}{p}\left[\int RF\,dS-\frac{r}{4}\int R^3F\,dS+\frac{3r}{4p^2}\int RF\,dS-\frac{3r}{16p^4}\int R^3F\,dS\right.$$
$$\left.+\frac{r}{4p^2}\int R^3\frac{\partial^2 F}{\partial p'^2}\,dS\right],$$

and $\int K(\rho,\rho')\chi(\rho')\,d\rho'$

$$=\frac{1}{4\pi p^{\frac{3}{2}}}\left[\left(1+2r+\frac{r}{4p^2}\right)\int RF\,dS-\left(\frac{r}{4}+\frac{3r}{16p^4}\right)\int R^3F\,dS+\frac{r}{4p^2}\int R^3\frac{\partial^2 F}{\partial p'^2}\,dS\right].$$

To transform from F to χ we have

$$\int RF\,dS=p^{\frac{3}{2}}\int\chi\,dS,\qquad \int R^3F\,dS=p^{\frac{3}{2}}\int R^2\chi\,dS.$$

Also if θ is the angle between OP and OP',

$$\xi^2=p^2+p'^2-2pp'\cos\theta.$$

Hence $\quad \xi\partial\xi/\partial p'=p'-p\cos\theta \quad$ and $\quad \xi\partial^2\xi/\partial p'^2+(\partial\xi/\partial p')^2=1.$

At the point Q,

$$\partial\xi/\partial p'=R/2p \quad\text{and}\quad \partial^2\xi/\partial p'^2=(4p^2-R^2)/4p^2R.$$

Further, by carrying out the differentiations we find at the same point

$$\frac{\partial^2 F}{\partial p'^2}=\frac{p^{\frac{3}{2}}}{R}\frac{\partial^2\chi}{\partial p'^2}+\frac{2p^{\frac{1}{2}}}{R}\frac{\partial\chi}{\partial p'}-\frac{p^{\frac{3}{2}}}{R^3}\chi,$$

so that $\int K(\rho,\rho')\chi(\rho')\,d\rho'$

$$=\frac{1}{4\pi}\left[(1+2r)\int\chi\,dS-\left(\frac{r}{4}+\frac{3r}{16p^4}\right)\int R^2\chi\,dS+\frac{r}{2p^3}\int R^2\frac{\partial\chi}{\partial p}\,dS\right.$$
$$\left.+\frac{r}{4p^2}\int R^2\frac{\partial^2\chi}{dp^2}\,dS\right],$$

$\partial/\partial p$ now denoting differentiation along the normal at Q. From this it is easy to write down the generalisation of (45) and of Lorentz's equation (46). It is in the r term that Maxwell's law for the equilibrium distribution begins to appear. We find that the mobility and coefficient of

diffusion, correct to the order r, are given by

$$\beta = \frac{4eh(1+\frac{2}{3}r)}{3(hm\pi)^{\frac{1}{2}}N\pi\sigma^2}, \qquad K = \frac{2(1+\frac{2}{3}r)}{3(hm\pi)^{\frac{1}{2}}N\pi\sigma^2}, \tag{49}$$

showing that "persistence of velocity" is just beginning to make itself felt in increasing the velocity of drift. But our main object in including the r terms is not to make this slight extension of Lorentz's results. It is clear that if the molecules are absolutely fixed, no energy is transferred from an ion to a molecule on collision, so that the kinetic energy of an ion increases by an amount eEx after traversing a distance x in the direction of an electric field E. There can be no final steady velocity of a stream of ions in an electric field, and Lorentz's formula (48) applies at most to the initial stages of the motion, while the average kinetic energy of an ion is still approximately that of a molecule of the gas. If the mass of the molecule is finite, however, it is possible to have a final state in which the average kinetic energy of an ion exceeds that of a molecule, which was first put in evidence by Townsend,* and which I have treated theoretically in the paper already cited. If r is small there may be an appreciable rise of kinetic energy even in weak electric fields, and the distribution function may differ considerably from Maxwell's form. In the following article an attempt is made to trace the transition. We shall see that the rise of kinetic energy in weak fields is regulated by the quantity E^2/r.

11. *Motion of Small Ions in an Electric Field.*

In discussing this it is convenient not to make use of the generalised equation in either χ or f, but to write

$$f = P\exp(-p^2).$$

Then

$$\chi = p^{\frac{1}{2}}\exp(-\tfrac{1}{2}p^2)\,P,$$

so that from the last article P is given, correct to the first order in r, by the equation

$$\begin{aligned} &P - \frac{1}{4\pi}\int P\,dS \\ &\quad - \frac{r}{4\pi}\left[2\int P\,dS - \frac{1}{p^2}\int R^2 P\,dS + \left(\frac{3}{4p^3} - \frac{1}{2p}\right)\int R^2\frac{\partial P}{\partial p}\,dS + \frac{1}{4p^2}\int R^2\frac{\partial^2 P}{\partial p^2}\,dS\right] \\ &= -\frac{\tau}{\pi^{\frac{1}{2}}p}\left[\frac{\partial P}{\partial t} + \frac{1}{(hm)^{\frac{1}{2}}}\Sigma p_x\frac{\partial P}{\partial x} + (hm)^{\frac{1}{2}}\Sigma F_x\frac{\partial P}{\partial p_x} - 2(hm)^{\frac{1}{2}}P\Sigma F_x p_x\right]. \end{aligned} \tag{50}$$

* J. S. Townsend, *Proc. Roy. Soc.*, A, Vol. 81, p. 464, 1908.

For steady motion in a uniform electric field $(E, 0, 0)$, the right-hand side becomes $-a(\partial P/\partial p_x - 2Pp_x)/p$, where

$$a = (hm)^{\frac{1}{2}} eE\tau/\pi^{\frac{1}{2}}m = eEh/N\pi\sigma^2. \tag{51}$$

Expand P in the form

$$P_0 + P_1(p_x/p) + P_2(p_x/p)^2 + \ldots + P_n(p_x/p)^n + \ldots,$$

where the coefficients are functions of p only. The result of substituting in equation (50) is found from the following integration theorems, which are easily proved :—

(1) $\frac{1}{4\pi}\int p_x^n dS = 0$ if n is odd, $= p^n/(n+1)$ if n is even,

(2) $\frac{1}{4\pi}\int R^2 p_n dS = -2p^{n+1}p_x/(n+2)$ if n is odd, $= 2p^{n+2}/(n+1)$ if n is even,

p_x on the right being taken at the point P.

Equating to zero the coefficients of the various powers of p_x, we have

$$2\Omega\left(P_0 + \frac{P_2}{3} + \ldots + \frac{P_{2n+1}}{2n+1} + \ldots\right) = \frac{aP_1}{rp^2} - \frac{1}{r}\left(\frac{P_2}{3} + \frac{P_4}{5} + \ldots + \frac{P_{2n}}{2n+1} + \ldots\right),$$

$$P_1 + 2r(\Omega - 1)\left(\frac{P_1}{3} + \frac{P_3}{5} + \ldots + \frac{P_{2n+1}}{2n+3} + \ldots\right) = a\left(2P_0 - \frac{1}{p}\frac{dP_0}{dp} - \frac{2P_2}{p^2}\right),$$

and for higher values

$$P_n = a\left(2P_{n-1} + \frac{(n-1)P_{n-1}}{p^2} - \frac{1}{p}\frac{dP_{n-1}}{dp} - \frac{(n+1)P_{n+1}}{p^2}\right),$$

Ω standing for the operator $\frac{1}{4}\frac{d^2}{dp^2} + \left(\frac{3}{4p} - \frac{p}{2}\right)\frac{d}{dp}$.

A practicable solution of these recurrence equations can be obtained when a is small. We write

$$b = 2a^2/3r = 2e^2h^2E^2/3N^2\pi^2\sigma^4 r, \tag{52}$$

which we suppose to be finite. With P_0 as the standard of reference, P_1 is at most of order a, P_2 of order a^2, and so on. Hence retaining only terms

of order **1**, a, b, ab (that is 1, E, E^2/r, E^3/r), we have

$$2\Omega P_0 = \frac{aP_1}{rp^2} - \frac{P_2}{3r}, \tag{53}$$

where $$P_1 = a\left(2P_0 - \frac{1}{p}\frac{dP_0}{dp}\right), \quad P_2 = a\left(2P_1 + \frac{P_1}{p^2} - \frac{1}{p}\frac{dP_1}{dp}\right). \tag{54}$$

Eliminating P_1 and P_2, we have

$$\left(1+\frac{b}{p^2}\right)\frac{d^2P_0}{dp^2} + \left(\frac{3}{p} - 2p + \frac{b}{p^3} - \frac{4b}{p}\right)\frac{dP_0}{dp} + 4b\left(1-\frac{1}{p^2}\right)P_0 = 0.$$

The left-hand side becomes a perfect differential on multiplying by $p^3\exp(-p^2)$, giving

$$(p^3+bp)\exp(-p^2)\frac{dP_0}{dp} - 2bp^2\exp(-p^2)P_0 = C.$$

The constant C must vanish, otherwise P_0 will become infinite at infinity in too high an order. Hence

$$(p^2+b)dP_0/dp = 2bpP_0, \quad \text{or} \quad P_0 = A(p^2+b)^b.$$

From (54), $$P_1 = 2aA(p^2+b)^{b-1}p^2,$$

so that correct to the first power of a we have

$$f = A\exp(-p^2)\left[(p^2+b)^b + 2a(p^2+b)^{b-1}pp_x\right]. \tag{55}$$

The second term of (55), though necessary to explain the flow of ions, is small in comparison with the first, so that the ions still move nearly equally in all directions. We can visualise the effect of the electric field as follows. Since the transfer of energy on collision is small, an ion can go on accumulating energy for some time, the collisions meanwhile ensuring that the velocities shall be nearly uniformly distributed in all directions. Finally a steady state is reached in which the velocities differ by a finite amount from their equilibrium values, and the distribution is further distorted so that it no longer conforms to Maxwell's law. The larger b is, the greater the tendency for all the ions to have the same (high) velocity.

Terms in p_x^2 begin to appear when the approximation is taken to the second order in a, and so on. Actual experiments on the motion of small ions with abnormal kinetic energy* are covered rather by the conditions a finite, b large, than by a small, b finite, but I have so far found the difficulties of treating the former case insuperable.

* J. S. Townsend and H. T. Tizard, *Proc. Roy. Soc.*, A, Vol. 88, p. 336, 1913.

When b is small, equation (55) reduces to (47). This gives the limitations of Lorentz's formula, namely that E^2/r should be small in comparison with $3N^2\pi^2\sigma^4/2c^2h^2$. The present investigation has no meaning if r is taken absolutely zero.

From (37), the mean velocity of drift in the electric field is given by

$$v = \frac{2a}{3(hm)^{\frac{1}{2}}} \frac{\int_0^\infty (p^2+b)^{b-1} p^5 \exp(-p^2)\,dp}{\int_0^\infty (p^2+b)^b p^2 \exp(-p^2)\,dp}. \tag{56}$$

If k is the ratio of the mean kinetic energy of agitation of an ion to that of a molecule, then

$$k = \tfrac{2}{3}hm\,\overline{\{(u_x-v)^2+u_y^2+u_z^2\}} = \tfrac{2}{3}(\bar{p}^2-hmv^2).$$

The last term being of order a^2, we have

$$k = \tfrac{2}{3}\frac{\int fp^2\,d\rho}{\int f\,d\rho} = \tfrac{2}{3}\frac{\int_0^\infty (p^2+b)^b p^4 \exp(-p^2)\,dp}{\int_0^\infty (p^2+b)^b p^2 \exp(-p^2)\,dp}. \tag{57}$$

The coefficient $2a/3(hm)^{\frac{1}{2}}$ in (56) is $\frac{2}{3}\Omega b^{\frac{1}{2}}$, where Ω is the root-mean square value of the velocity of agitation of a gas molecule. Hence v/Ω and k can be found numerically, as is done for the smaller values of b in the following table.

b	v/Ω	k	$\frac{k-1}{v^2/\Omega^2}$
0·2	·312	1·115	1·18
0·4	·419	1·197	1·12
0·6	·498	1·276	1·11
0·8	·554	1·341	1·10
1·0	·604	1·402	1·10

The exact limiting value of the ratio of $k-1$ to v^2/Ω^2, when $b \to 0$, is found to be $3\pi/8$.

12. *Non-Steady Diffusion of Small Ions.*

The general problem of diffusion of small ions is curiously intractable, but the following special treatment is exact as far as it goes, and has

something of the simplicity of the elementary methods. Suppose that the molecules are immovable, and that all the ions have initially the same resultant velocity u. Then if no external forces are acting they will continue to have the velocity u throughout all time. For diffusion in one dimension expand f in the form

$$f = A_0 + A_1 u_x + A_2 u_x^2 + \ldots,$$

where the coefficients A are functions of u which are only appreciable in the neighbourhood of the particular value chosen; that is, they are functions of x and t only. Then substituting in Lorentz's equation

$$f - \frac{1}{4\pi}\int f\,dS = -\frac{1}{N\pi\sigma^2 u}\left[\frac{\partial f}{\partial t} + \Sigma u_x \frac{\partial f}{\partial x}\right], \tag{58}$$

and equating coefficients we have

$$A_2\frac{u^2}{3} + A_4\frac{u^4}{5} + \ldots = \frac{1}{N\pi\sigma^2 u}\frac{\partial A_0}{\partial t}, \tag{59}$$

$$A_1 = -\frac{1}{N\pi\sigma^2 u}\left(\frac{\partial A_1}{\partial t} + \frac{\partial A_0}{\partial x}\right), \qquad A_2 = -\frac{1}{N\pi\sigma^2 u}\left(\frac{\partial A_2}{\partial t} + \frac{\partial A_1}{\partial x}\right), \text{ \&c.} \tag{60}$$

For slow diffusion the coefficients diminish rapidly, and the chief part of A_n is $(-N\pi\sigma^2 u)^{-n}\,\partial^n A_0/\partial x^n$. Hence the most significant terms in (59) give

$$\frac{\partial A_0}{\partial t} = \frac{u}{3N\pi\sigma^2}\frac{\partial^2 A_0}{\partial x^2}.$$

Knowing the orders of magnitude of the terms, the extension to three dimensions is simple. Since A_0 is approximately proportional to the concentration n, we have the usual differential equation

$$\frac{\partial n}{\partial t} = K\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2}\right),$$

with the constant $K = u/3N\pi\sigma^2$.

The result is, of course, not new, but the method of proof has some advantages.

13. *Theory of the Large Ion in Gases.*

When r is large $\psi(r^{-\frac{1}{2}}p)/p = 2/r^{\frac{1}{2}}$ approximately,

and $\chi_0(p) \to 2^{\frac{1}{2}} r^{\frac{1}{4}} \exp(-\frac{1}{2}p^2)$.

Hence the right-hand side of (17) becomes

$$-\frac{\tau}{2r^{\frac{1}{2}}}\left[\frac{\partial\chi}{\partial t}+\frac{1}{(hm)^{\frac{1}{2}}}\Sigma p_x\frac{\partial\chi}{\partial x}+(hm)^{\frac{1}{2}}\Sigma F_x\frac{\partial\chi}{\partial p_x}-\chi(hm)^{\frac{1}{2}}\Sigma F_x p_x\right].$$

We have to calculate the integral $I\equiv\int K(\rho,\rho')\chi(\rho')d\rho'$ correct to terms in $1/r$. From (20) we see that $K(\rho,\rho')$ is only appreciable when ξ is small, *i.e.*, when ρ is nearly equal to ρ', which is the mathematical representation of the fact that the components of velocity of an ion are only slightly changed by collision. Proceeding to a closer approximation for ψ, we find

$$\left(\frac{p}{\psi(r^{-\frac{1}{2}}p)}\right)^{\frac{1}{2}}=\frac{r^{\frac{1}{4}}}{2^{\frac{1}{2}}}\left(1-\frac{p^2}{6r}\right),$$

and the kernel, correct to constant terms, becomes

$$K(\rho,\rho')=\frac{(1+r)^2}{8\pi r\xi}\left(1-\frac{p^2}{6r}\right)\left(1-\frac{p'^2}{6r}\right)\left(1-\frac{(p'^2-p^2)^2}{4r\xi^2}\right)\exp(-\tfrac{1}{4}r\xi^2)$$

$$=\frac{\exp(-\frac{1}{4}r\xi^2)}{8\pi\xi}\left(r+2-\frac{p^2}{6}-\frac{p'^2}{6}-\frac{(p'^2-p^2)^2}{4\xi^2}\right).$$

If P and P' are the representative points of ρ and ρ', let PP' make an angle α with OP, and the plane OPP' an angle ψ with zOP. Then P' is the point $(p_x+\xi A,\ p_y+\xi B,\ p_z+\xi C)$, where

$$A=\cos\theta\cos\phi\sin\alpha\cos\psi-\sin\phi\sin\alpha\sin\psi+\sin\theta\cos\phi\cos\alpha,$$

$$B=\cos\theta\sin\phi\sin\alpha\cos\psi+\cos\phi\sin\alpha\sin\psi+\sin\theta\sin\phi\cos\alpha,$$

$$C=-\sin\theta\sin\alpha\cos\psi\quad+\cos\theta\cos\alpha.$$

To get the required order of approximation we must take two sets of terms in the Taylor expansion of $\chi(\rho')$. Since

$$d\rho'=\xi^2\sin\alpha\,d\xi\,d\alpha\,d\psi$$

and $2-\frac{p^2}{6}-\frac{p'^2}{6}-\frac{(p'^2-p^2)^2}{4\xi^2}=2-\frac{p^2}{3}-p^2\cos^2\alpha-\frac{5\xi^2}{12}-\frac{4}{3}p\xi\cos\alpha$

we have, correct to the order $1/r$,

$$I=\frac{1}{8\pi}\int\exp(-\tfrac{1}{4}r\xi^2)\,\xi\sin\alpha\,d\xi\,d\alpha\,d\psi\left[r\chi(\rho)+r\xi\Sigma A\frac{\partial\chi}{\partial p_x}+\frac{r\xi^2}{2}\Sigma A^2\frac{\partial^2\chi}{\partial p_x^2}\right.$$

$$\left.+r\xi^2\Sigma BC\frac{\partial^2\chi}{\partial p_y\partial p_z}+\left(2-\frac{p^2}{3}-p^2\cos^2\alpha\right)\chi(\rho)\right].$$

Evaluating the integrals we find, after a little reduction,

$$\int \exp(-\tfrac{1}{4}r\xi^2)\,\xi \sin a\, d\xi\, da\, d\psi = \frac{8\pi}{r},$$

$$\int \exp(-\tfrac{1}{4}r\xi^2)\,\xi^2 A \sin a\, d\xi\, da\, d\psi = \ldots = \ldots = 0,$$

$$\int \exp(-\tfrac{1}{4}r\xi^2)\,\xi^3 A^2 \sin a\, d\xi\, da\, d\psi = \ldots = \ldots = \frac{32\pi}{3r^2},$$

$$\int \exp(-\tfrac{1}{4}r\xi^2)\,\xi^3 BC \sin a\, d\xi\, da\, d\psi = \ldots = \ldots = 0,$$

$$\int \exp(-\tfrac{1}{4}r\xi^2)\,\xi \sin a \cos^2 a\, d\xi\, da\, d\psi = \frac{8\pi}{3r}.$$

Hence

$$I = \chi(\rho) + \frac{2}{3r}\left(\frac{\partial^2\chi}{\partial p_x^2} + \frac{\partial^2\chi}{\partial p_y^2} + \frac{\partial^2\chi}{\partial p_z^2}\right) + \frac{2}{r}\chi(\rho) - \frac{2p^2}{3r}\chi(\rho),$$

and χ satisfies the *partial differential equation*

$$\frac{\partial^2\chi}{\partial p_x^2} + \frac{\partial^2\chi}{\partial p_y^2} + \frac{\partial^2\chi}{\partial p_z^2} - (p^2-3)\chi$$

$$= \frac{3r^{\frac{1}{2}}\tau}{4}\left[\frac{\partial\chi}{\partial t} + \frac{1}{(hm)^{\frac{1}{2}}}\Sigma p_x \frac{\partial\chi}{\partial x} + (hm)^{\frac{1}{2}}\Sigma F_x \frac{\partial\chi}{\partial p_x} - \chi(hm)^{\frac{1}{2}}\Sigma F_x p_x\right].$$

The equation for f is

$$\frac{1}{hm}\left(\frac{\partial^2 f}{\partial u_x^2} + \frac{\partial^2 f}{\partial u_y^2} + \frac{\partial^2 f}{\partial u_z^2}\right) + 2\left(u_x\frac{\partial f}{\partial u_x} + u_y\frac{\partial f}{\partial u_y} + u_z\frac{\partial f}{\partial u_z}\right) + 6f = A\,Df, \quad (61)$$

where we have replaced the original variables u_x, u_y, u_z, and

$$A = \frac{3r^{\frac{1}{2}}\tau}{4} = \frac{3m}{4N\sigma^2}\left(\frac{h}{M\pi}\right)^{\frac{1}{2}}. \quad (62)$$

As a check on the accuracy of the work we notice that $f = \exp(-hmu^2)$ makes the left-hand side of (61) vanish, so that we recover Maxwell's law for the equilibrium distribution. Multiply equation (61) by any function Q of the components of velocity and integrate over the range $-\infty$ to ∞. The integrals are transformed by the Gauss-Green theorems for infinite

space, and thus

$$\frac{1}{hm}\int\left(\frac{\partial^2 Q}{\partial u_x^2}+\frac{\partial^2 Q}{\partial u_y^2}+\frac{\partial^2 Q}{\partial u_z^2}\right)f\,d\omega-2\int\Sigma\left(u_x\frac{\partial Q}{\partial u_x}\right)f\,d\omega$$
$$=A\left[\frac{\partial}{\partial t}\int Qf\,d\omega+\Sigma\frac{\partial}{\partial x}\int u_xQf\,d\omega-\int\Sigma\frac{\partial}{\partial u_x}(F_xQ)f\,d\omega\right].$$

If E, H are the electric and magnetic forces respectively, we have

$$F_x=\frac{e}{m}(E_x+u_yH_z-u_zH_y),\qquad F_y=\frac{e}{m}(E_y+u_zH_x-u_xH_z),$$
$$F_z=\frac{e}{m}(E_z+u_xH_y-u_yH_x),$$

where e and E are measured in electromagnetic units. Since

$$\Sigma\partial(F_xQ)/\partial u_x=\Sigma F_x\partial Q/\partial u_x+Q\,\Sigma\,\partial F_x/\partial u_x,$$

and the last term vanishes, the equation for Q may be written

$$\frac{n}{hm}\left(\overline{\frac{\partial^2 Q}{\partial u_x^2}+\frac{\partial^2 Q}{\partial u_y^2}+\frac{\partial^2 Q}{\partial u_z^2}}\right)-2n\Sigma\left(\overline{u_x\frac{\partial Q}{\partial u_x}}\right)$$
$$=A\left[\frac{\partial}{\partial t}(n\bar{Q})+\Sigma\frac{\partial}{\partial x}(n\overline{u_xQ})-n\Sigma\left(\overline{F_x\frac{\partial Q}{\partial u_x}}\right)\right].\qquad(63)$$

This case is therefore peculiarly simple, in that the equation of transfer for any function Q can be found explicitly in terms of mean values. But of course the interpretation of the results is affected to some extent by the restriction on m.

For *steady* motion in a uniform electromagnetic field, we have, putting $Q=u_x$, u_y, u_z in (63),

$$\left.\begin{aligned}v_x=\frac{eA}{2m}(E_x+v_yH_z-v_zH_y),\qquad v_y=\frac{eA}{2m}(E_y+v_zH_x-v_xH_z)\\ v_z=\frac{eA}{2m}(E_z+v_xH_y-v_yH_x)\end{aligned}\right\}.\qquad(64)$$

It follows that the stream will move like a particle under a resisting force proportional to the velocity. The mobility is

$$\beta=\frac{eA}{2m}=\frac{3e}{8N\sigma^2}\left(\frac{h}{M\pi}\right)^{\frac{1}{2}},\qquad(65)$$

which is independent of m. This formula agrees with that of Langevin.

The velocity distribution in weak fields is found by putting

$$f = A \exp(-hmu^2) + u_x \phi(u)$$

in (61), and is found to be approximately Maxwellian. Similarly for slow diffusion we have

$$K = \frac{3}{16 N \sigma^2} \left(\frac{1}{hM\pi}\right)^{\frac{1}{2}}. \tag{66}$$

14. *Magnetic Deflexion of a Stream of Ions.*

(1) *Inverse Fifth-power Law.** — The theory for steady motion in uniform fields is perfectly rigorous, with no restriction except that necessary to exclude multiple collisions. The equation of transfer becomes

$$\frac{ne}{m}\left[\Sigma E_x \left(\frac{\delta Q}{\delta u_x}\right) + \Sigma H_z \left(u_y \frac{\partial Q}{\partial u_x} - u_x \frac{\partial Q}{\partial u_y}\right)\right] + \Delta Q = 0. \tag{67}$$

Putting $Q = u_x, u_y, u_z$, we have three equations of the type

$$v_x/T = e(E_x + v_y H_z - v_z H_y)/m,$$

where T is defined in my former paper. For the crossed fields $(E, 0, 0)$, $(0, 0, H)$, we find

$$v_x = \frac{eE}{m} \frac{T}{1+\omega^2 T^2}, \qquad v_y = -\frac{eE}{m} \frac{\omega T^2}{1+\omega^2 T^2}, \qquad v_z = 0, \tag{68}$$

and the magnetic deflexion ϕ of the stream is given by $\tan \phi = \omega T$. These formulæ have been obtained by Townsend† by the method of mean free paths.

The inverse fifth power law is no more suited than any other law for determining the distribution function, but it furnishes equations by which the mean value of any function of the velocities may be found in time. Thus it furnishes an elegant and rigorous proof of the fact that the magnetic field does not merely turn the stream round through an angle, but produces internal distortions as well. The mean value of $u_x u_y$, referred to axes turned through an angle ϕ, is not zero, but comparable with the

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* See Jeans, *Dynamical Theory of Gases*, Ch. XV.

† J. S. Townsend, *Proc. Roy. Soc.*, A, Vol. 86, p. 571, 1912.

value of $\bar{u}_x\bar{u}_y$ referred to the original axes. In fact, we find that

$$\frac{\overline{u_x u_y}}{\overline{v_x v_y}} = \frac{1-\Theta}{2\,(\Theta^2\cos^2\phi+\sin^2\phi)},$$

where Θ stands for $(2A_1m+3A_2M)/(2A_1m+2A_2M)$. Similarly we find that the abnormal kinetic energy formula $k-1 = v^2/\Omega^2$ still holds when any uniform magnetic field is superposed on the electric field.

(2) *Elastic Spheres, r small.*—The theory is due effectively to Gans,* who has treated the analogous problem of the Hall effect in the electronic theory of metallic conduction. It can be put in a very simple form by assuming that

$$f = A\exp(-p^2)+Xp_x+Yp_y+Zp_z,$$

where X, Y, Z are functions of p only, and using (46). If the axis of z is parallel to the magnetic field, the mean velocity of drift is given by

$$v_x = \beta(I_1E_x+cI_2E_y), \quad v_y = \beta(-cI_2E_x+I_1E_y), \quad v_z = \beta E_z, \tag{69}$$

where
$$I_1 = 2\int_0^\infty \frac{p^5\exp(-p^2)}{p^2+c^2}\,dp, \quad I_2 = 2\int_0^\infty \frac{p^4\exp(-p^2)}{p^2+c^2}\,dp, \tag{70}$$

and
$$c = 3\pi^{\frac{1}{2}}\omega T/4, \tag{71}$$

T denoting as before an interval of time such that $\beta = eT/m$. The magnetic deflexion in crossed fields $(E, 0, 0)$, $(0, 0, H)$ is

$$\phi = \tan^{-1}(cI_2/I_1).$$

The difference between the laws of deflexion for different laws of force may be explained more physically as follows. The magnetic field acts relatively more on the faster ions than on the slower ones, but this effect is modified by inequalities in the persistence of velocity after collision. The inverse fifth power law allows the velocity of each ion to persist to just the extent necessary to make the stream follow the resultant mean electromagnetic force.

15. *Diffusion in a Magnetic Field.*

The theory of steady slow diffusion in a uniform magnetic field is similar in general to that of magnetic deflexion. For the inverse fifth-

* R. Gans, *Ann. d. Phys.*, Ser. 4, Vol. 20, p. 293, 1906.

power law we have

$$nv_x = -K\frac{\dfrac{\partial n}{\partial x}+\omega T\dfrac{\partial n}{\partial y}}{1+\omega^2T^2}, \quad nv_y = -K\frac{-\omega T\dfrac{\partial n}{\partial x}+\dfrac{\partial n}{\partial y}}{1+\omega^2T^2}, \quad nv_z = -K\frac{\partial n}{\partial z}, \quad (72)$$

where K is the ordinary coefficient of diffusion. For elastic ions of small mass Gans' method gives, instead of (69),

$$nv_x = -K\left(I_1\frac{\partial n}{\partial x}+cI_2\frac{\partial n}{\partial y}\right), \quad nv_y = -K\left(-cI_2\frac{\partial n}{\partial x}+I_1\frac{\partial n}{\partial y}\right), \quad nv_z = -K\frac{\partial n}{\partial z}. \quad (73)$$

In this, of course, the distribution of velocities in the normal state is taken as Maxwellian. Non-steady diffusion can be treated, as in Art. 12, by taking all the velocities equal, and thus we find the partial differential equation

$$\frac{\partial n}{\partial t} = \frac{K}{1+\omega^2T^2}\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)+K\frac{\partial^2 n}{\partial z^2},$$

the mean fluxes being given by the same equations (72) as hold for the inverse fifth-power law.

In all these cases a concentration-gradient parallel to the axis of x gives rise to a flux parallel to the axis of y, and *vice versa*. In fact, whatever the law of force, the angle between the flux and the concentration gradient is equal to the angle between the mean velocity and the electric force in the magnetic deflexion of a slow uniform stream. Different conclusions are reached by Townsend,* who obtains the equations

$$nv_x = -\frac{K}{1+\omega^2T^2}\frac{\partial n}{\partial x}, \quad nv_y = -\frac{K}{1+\omega^2T^2}\frac{\partial n}{\partial y}, \quad nv_z = -K\frac{\partial n}{\partial z} \quad (74)$$

by the method of mean free paths. It seems to me, however, that Townsend's geometrical argument should be interpreted in a different way from what he has done. The argument certainly proves that if we have a distribution of ions such that $\partial n/\partial x = 0$, and such that n tends to zero when y becomes either positively or negatively infinite, the total flux across the plane $x = 0$ is zero. But nothing is proved as to the distribution of flux over the surface-elements of the plane. If P is a point on the axis of y above AB in Townsend's figure (p. 575), ions passing P from A have a greater normal component of velocity than those from B, so that a thin layer of ions along the line AB gives rise to a flux at P parallel to AB, while below AB the flux is in the opposite direction.

* J. S. Townsend, *Proc. Roy. Soc.*, A, Vol. 86, p. 571, 1912.

16. *Extension to Collisions of any Degree of Elasticity.*

The whole of the work on elastic collisions can be extended to collisions with any constant coefficient of restitution e by the following theorem :—

If the external force per unit mass is the same, the distribution of velocities is the same for ions of mass m and coefficient of restitution e as for perfectly elastic ions of mass m_0, where

$$m_0 = \frac{2m+M(1-e)}{1+e}.$$

Consider first a single collision. For given initial velocities perpendicular and parallel to the line of impact it is easy to prove by the ordinary methods that the final velocities are the same in the two cases. Hence the effects of a collision are the same, and the effect of external force between collisions is also the same, so that the velocities continue to correspond indefinitely.

Writing f for $\frac{1}{2}(1+e)$, we may keep the electric and magnetic forces the same by altering the charge in the same ratio as the mass. Then

$$e/e_0 = m/m_0 = \kappa,$$

where

$$\kappa = \frac{mf}{m+M(1-f)}, \tag{75}$$

and all the mean velocities and velocities of agitation are the same. It follows from the definition that the relations for the derived quantities β, K, k are $\beta = \beta_0$, $K = K_0$, $k = \kappa k_0$.

Since β, K and e/m are unaltered, many equations, *e.g.*, the formulæ for magnetic deflexion, are unaffected and hold for collisions of any degree of elasticity. Of positive results we may note the following.

(1) In the equilibrium state the velocities of imperfectly elastic ions are distributed according to Maxwell's law, but with an energy factor κ less than unity.

(2) For collisions of conservative type it would appear that the equation $k = 1+v^2/\Omega^2$ represents the facts fairly accurately. Assuming it in general we have

$$k = \kappa(1+v^2/\Omega^2) \tag{76}$$

for collisions involving loss of energy. This may be written

$$k-f+\frac{kM}{m}(1-f) = \frac{v^2}{\Omega^2}f, \tag{77}$$

so that it disagrees with a formula* given by me previously, in which the factor $(4f-3)/(2-f)$ occurs on the right instead of f. Since this latter formula does not conform to the general theorem here given, it is clear that some mistake had been made in the calculation; but it is hardly worth while repeating it as the whole method there used for elastic collisions was inaccurate. I now wish to withdraw this formula and substitute (76) as a semi-theoretical formula giving a general account of the phenomena. The numerical conclusions of the previous paper are unaltered.

* *Proc. Roy. Soc.*, A, Vol. 88, p. 300, 1913.

A THEOREM ON SERIES OF ORTHOGONAL FUNCTIONS

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1. *Introduction.*

It is well known that, extending Fejér's theorem* about the summability of Fourier's series, Lebesgue proved† the following theorem :—

Fourier's series

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=A_0+\sum_{n=1}^{\infty}A_n,$$

corresponding to any summable function $f(x)$, *is summable* (C1) *almost everywhere*‡ : *i.e.*, if

$$s_n=A_0+A_1+\ldots+A_n,$$

then

$$\lim_{n\to\infty}\frac{s_0+s_1+\ldots+s_n}{n+1}=f(x)\quad(=S,\text{ say})$$

almost everywhere.

From this theorem it evidently follows that

$$\lim_{n\to\infty}\frac{(s_0-S)+(s_1-S)+\ldots+(s_n-S)}{n+1}=0$$

almost everywhere.

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* *Mathematische Annalen*, Bd. 58, 1904, p. 51.

† *Ibid.*, Bd. 61, 1905, p. 251.

‡ As is well known : "a series is summable (C1)" means "Cesàro's mean of the first order of a series is convergent", according to G. H. Hardy ; ". . . . almost everywhere" means ". . . . in the whole interval, except perhaps at the points of a set of zero measure."

Messrs. G. H. Hardy and J. E. Littlewood proved* that

$$\lim_{n\to\infty} \frac{(s_0-S)^2+(s_1-S)^2+\ldots+(s_n-S)^2}{n+1} = 0$$

and

$$\lim_{n\to\infty} \frac{|s_0-S|+|s_1-S|+\ldots+|s_n-S|}{n+1} = 0$$

almost everywhere, provided that $\{f(x)\}^2$ *is summable.*

It will be of some interest to prove an analogous theorem for a general series of orthogonal functions.

Let $$\{\phi_n(x)\} = \phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_n(x),\ \ldots$$

be a sequence of normal orthogonal functions for the interval (0, 1). It is well known that, if $\{c_n\}$ denotes a set of numbers, such that the series

$$\sum_{n=1}^{\infty} c_n^2 = c_1^2+c_2^2+\ldots+c_n^2+\ldots$$

converges, then the series

$$\sum_{n=1}^{\infty} c_n\phi_n(x) = c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)+\ldots \tag{1}$$

converges on the average to a summable function $f(x)$, such that

$$c_n = \int_0^1 f(x)\,\phi_n(x)\,dx$$

for every value of n, that $\{f(x)\}^2$ is summable, and that

$$\int_0^1 \{f(x)\}^2\,dx = \sum_{n=1}^{\infty} c_n^2. \tag{2}$$

This function $f(x)$ is uniquely determinate if sets of points of measure zero be neglected.

Prof. E. W. Hobson proved† the theorem:—

If the series

$$\sum_{n=1}^{\infty} c_n^2 \log n = c_1^2 \log 1+c_2^2 \log 2+\ldots+c_n^2 \log n+\ldots \tag{3}$$

is convergent, then the series (1) *is summable* (*C*1) *to the sum* $f(x)$ *almost*

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* *Comptes Rendus*, Tome 156 (1913), p. 1307.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 14, 1915, p. 428. This theorem was first given in a paper of H. Weyl (*Math. Annalen*, Bd. 67, 1909, p. 225), without a complete proof.

everywhere: i.e., if

$$s_n = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x),^*$$

then
$$\lim_{n\to\infty} \frac{s_1+s_2+\ldots+s_n}{n} = f(x)$$

almost everywhere.

By using the method, which Prof. Hobson used in the proof of this theorem, I have succeeded in proving the following theorem:—

THEOREM.—*If the series* (3) *is convergent, then*

$$\lim_{n\to\infty} \frac{(s_1-f)^2+(s_2-f)^2+\ldots+(s_n-f)^2}{n} = 0^\dagger$$

and
$$\lim_{n\to\infty} \frac{|s_1-f|+|s_2-f|+\ldots+|s_n-f|}{n} = 0$$

almost everywhere.

In the proof of this theorem, I have not used the theorem of Weyl and Hobson mentioned above. Consequently it will not be without interest to remark here that the latter theorem can be deduced as a corollary of our theorem.

2. *Lemma.*

Lemma.—
$$\int_0^1 (s_\nu - f)^2\,dx = \sum_{i=\nu+1}^{\infty} c_i^2 \quad (\nu = 1, 2, 3, \ldots).$$

Proof.—

$$\int_0^1 (s_\nu - f)^2\,dx = \int_0^1 f^2\,dx - 2\int_0^1 f s_\nu\,dx + \int_0^1 s_\nu^2\,dx. \tag{4}$$

By hypothesis
$$\int_0^1 \{\phi_i(x)\}^2\,dx = 1 \quad (i = 1, 2, 3, \ldots),$$

$$\int_0^1 \phi_i(x)\,\phi_j(x)\,dx = 0 \quad (i \neq j;\ i, j = 1, 2, 3, \ldots),$$

$$\int_0^1 f(x)\,\phi_i(x)\,dx = c_i \quad (i = 1, 2, 3, \ldots).$$

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* Hereafter, throughout this paper, s_n will be used in the meaning defined by this equation.

† For the sake of simplicity, f is used to denote $f(x)$.

Therefore, after a simple calculation, (4) becomes

$$\int_0^1 (s_\nu - f)^2\, dx = \int_0^1 f^2\, dx - (c_1^2 + c_2^2 + \ldots + c_\nu^2).$$

But, by (2), $$\int_0^1 f^2\, dx = \sum_{t=1}^{\infty} c_t^2.$$

Therefore we obtain $$\int_0^1 (s_\nu - f)^2\, dx = \sum_{t=\nu+1}^{\infty} c_t^2.$$

3. *Proof of the Theorem.*

We shall denote the expression

$$\frac{(s_1-f)^2+(s_2-f)^2+\ldots+(s_n-f)^2}{n}$$

by $T(x, n)$ and sometimes simply by T_n.

(i) By applying the lemma of § 2, we obtain

$$\int_0^1 T(x, n)\, dx = \frac{c_2^2+2c_3^2+\ldots+(n-1)\,c_n^2}{n} + \sum_{t=n+1}^{\infty} c_t^2$$

$$< \frac{c_2^2+2c_3^2+\ldots+(r-2)\,c_{r-1}^2}{n} + \sum_{t=r}^{\infty} c_t^2,$$

where $$r < n.$$

Herein put $$r = 2^\rho, \quad n = 2^p,$$

ρ and p being positive integers such that

$$\rho < p\,;$$

and write $$A(\rho) = c_2^2+2c_3^2+\ldots+(2^\rho-2)\,c^2(2^\rho-1).*$$

Then we have

(5) $$\int_0^1 T(x, n)\, dx < \frac{A(\rho)}{2^p} + \sum_{t=2^\rho}^{\infty} c_t^2.$$

Since the series (3) is convergent, there exists a positive integer ρ_1,

* For the sake of convenience of printing, we shall often use the following notations :—

$$c(n) = c_n, \qquad c^2(n) = c_n^2.$$

corresponding to any prescribed positive number ϵ, such that

$$\sum_{t=2^{\rho_1}}^{\infty} c_t^2 \log t < \epsilon.$$

The logarithms may be supposed to be logarithms to base 2. From this inequality, it can easily be deduced that

$$\sum_{q=0}^{\infty} \sum_{t=2^{\rho_1+q}}^{\infty} c_t^2 = \sum_{2^{\rho_1}}^{\infty} c_t^2 + \sum_{2^{\rho_1+1}}^{\infty} c_t^2 + \sum_{2^{\rho_1+2}}^{\infty} c_t^2 + \ldots < \epsilon. \tag{6}$$

Choose a positive integer p_1 sufficiently large to ensure that

$$\frac{A(\rho_1)}{2^{p_1}} < \epsilon \quad (p_1 > \rho_1). \tag{7}$$

Now put $$\rho = \rho_1 + q, \quad p = p_1 + 2q \quad (q = 0, 1, 2, \ldots),$$

in the inequality (5).

First of all, we obtain

$$\frac{A(\rho_1+q)}{2^{p_1+2q}} = \frac{A(\rho_1) + (2^{\rho_1}-1)\, c^2(2^{\rho_1}) + \ldots + (2^{\rho_1+q}-2)\, c^2(2^{\rho_1+q}-1)}{2^{p_1+2q}}$$

$$< \frac{1}{2^{2q}} \frac{A(\rho_1)}{2^{p_1}} + \frac{1}{2^{p_1-\rho_1+q}} \sum_{t=2^{\rho_1}}^{\infty} c_t^2$$

$$< \frac{\epsilon}{2^{2q}} + \frac{\epsilon}{2^{p_1-\rho_1+q}} \quad [\text{by (6) and (7)}].$$

Therefore we have

$$\int_0^1 \sum_{q=0}^{\infty} T(x, 2^{p_1+2q})\, dx = \sum_{q=0}^{\infty} \frac{A(\rho_1+q)}{2^{p_1+2q}} + \sum_{q=0}^{\infty} \sum_{t=2^{\rho_1+q}}^{\infty} c_t^2$$

$$< \epsilon \sum_{q=0}^{\infty} \left(\frac{1}{2^{2q}} + \frac{1}{2^{p_1-\rho_1+q}} \right) + \epsilon$$

$$< \epsilon \left(\frac{4}{3} + \frac{1}{2^{p_1-\rho_1-1}} + 1 \right).$$

But $$\frac{1}{2^{p_1-\rho_1-1}} \leqslant 1,$$

since $$p_1 - \rho_1 \geqslant 1.$$

Therefore the above inequality becomes

$$\int_0^1 \sum_{q=0}^{\infty} T(x, 2^{p_1+2q})\, dx < 4\epsilon.$$

In this inequality we may change p_1 into p_1+1, and after this change we have

$$\int_0^1 \sum_{q=0}^{\infty} T(x, 2^{p_1+2q+1})dx < 4\epsilon.$$

By adding together these two inequalities, we obtain

$$\int_0^1 \sum_{q=0}^{\infty} T(x, 2^{p_1+q})dx < 8\epsilon. \tag{8}$$

If we write $8\epsilon = \delta^2$, it follows* that, in a set $H(\delta)$ whose measure is greater than $1-\delta$, we have

$$T(x, 2^{p_1+q}) < \delta \quad (q = 0, 1, 2, \ldots).$$

Or, in the set $H(\delta)$,

$$T(x, 2^m) < \delta$$

for every integer $m \geqslant p_1$.

Let η be any prescribed positive number, and let $\delta_1, \delta_2, \ldots, \delta_n, \ldots$ be a sequence of positive numbers, such that

$$\lim_{n\to\infty} (\delta_1+\delta_2+\ldots+\delta_n) = \eta.\dagger$$

Then, from the above result, it immediately follows that, corresponding to $\delta_1, \delta_2, \ldots, \delta_n, \ldots$, there exist positive integers $p_1, p_2, \ldots, p_n, \ldots$, and sets of points $H(\delta_1), H(\delta_2), \ldots, H(\delta_n), \ldots$ whose measures are greater than $1-\delta_1, 1-\delta_2, \ldots, 1-\delta_n, \ldots$ respectively, such that

$$T(x, 2^m) < \delta_n$$

in $H(\delta_n)$, for $m \geqslant p_n$, and for $n = 1, 2, 3, \ldots$.

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* Let E be the set of all the points at which

$$\sum_{q=0}^{\infty} T(x, 2^{p_1+q}) \geqslant \delta.$$

Then, by (8), we have

$$\int_{(E)} \sum_{q=0}^{\infty} T(x, 2^{p_1+q})\, dx < \delta^2.$$

Therefore

$$\delta \int_{(E)} dx < \delta^2, \quad \text{or} \quad \int_{(E)} dx < \delta,$$

which shows that the measure of the set E is less than δ. Therefore the measure of the complementary set $C(E) \equiv H(\delta)$ is greater than $1-\delta$, and in this set $H(\delta)$, we have

$$\sum_{q=0}^{\infty} T(x, 2^{p_1+q}) < \delta \quad \text{and} \quad T(x, 2^{p_1+q}) \geqslant 0,$$

and therefore

$$T(x, 2^{p_1+q}) < \delta \quad (q = 0, 1, 2, \ldots).$$

† For instance, if we choose

$$\delta_n = \frac{\eta}{2^n},$$

then

$$\lim_{n\to\infty} (\delta_1 + \delta_2 + \ldots + \delta_n) = \eta.$$

All the sets $H(\delta_n)$ have in common a set K of measure

$$>1-\sum_{n=1}^{\infty}\delta_n, \quad \text{or} \quad >1-\eta.$$

In this set K we have $\lim_{m\to\infty} T(x, 2^m) = 0,$

since $\lim_{n\to\infty} \delta_n = 0.$

And, since η is any prescribed positive number, it follows that

(9) $$\lim_{m\to\infty} T(x, 2^m) = 0$$

almost everywhere.

Thus the first part of our theorem is proved in the case $n = 2^m$.

(ii) Let μ and ν be any two positive integers, such that

$$\nu < \mu \leqslant 2\nu.$$

Then $$\mu T_\mu = (s_1-f)^2+(s_2-f)^2+\ldots+(s_\mu-f)^2,$$

$$\nu T_\nu = (s_1-f)^2+(s_2-f)^2+\ldots+(s_\nu-f)^2,$$

and $$\mu T_\mu-\nu T_\nu = (s_{\nu+1}-f)^2+(s_{\nu+2}-f)^2+\ldots+(s_\mu-f)^2.$$

But $$\mu T_\mu-\nu T_\nu = \mu(T_\mu-T_\nu)+(\mu-\nu)T_\nu;$$

therefore

$$T_\mu-T_\nu+\frac{\mu-\nu}{\mu}T_\nu = \frac{(s_{\nu+1}-f)^2+(s_{\nu+2}-f)^2+\ldots+(s_\mu-f)^2}{\mu},$$

whence we can easily deduce

$$|T_\mu-T_\nu| < |T_\nu| + \frac{(s_{\nu+1}-f)^2+(s_{\nu+2}-f)^2+\ldots+(s_{2\nu}-f)^2}{\nu},$$

or

(10) $$|T_\mu-T_\nu| < |T_\nu| + U(x, \nu) \quad (\nu < \mu \leqslant 2\nu),$$

where $$U(x, \nu) = \frac{(s_{\nu+1}-f)^2+(s_{\nu+2}-f)^2+\ldots+(s_{2\nu}-f)^2}{\nu}.$$

By applying the lemma of § 2, we obtain

$$\int_0^1 U(x, \nu)dx = \frac{1}{\nu}\left\{\sum_{t=\nu+2}^{\infty} c_t^2+\sum_{t=\nu+3}^{\infty} c_t^2+\ldots+\sum_{t=2\nu+1}^{\infty} c_t^2\right\} < \sum_{t=\nu}^{\infty} c_t^2.$$

Herein put $$\nu = 2^{\rho_1+q} \quad (q = 0, 1, 2, \ldots).$$

Then $$\int_0^1 \sum_{q=0}^{\infty} U(x, 2^{\rho_1+q})\,dx < \sum_{q=0}^{\infty} \sum_{t=2^{\rho_1+q}}^{\infty} c_t^2 < \epsilon \quad [\text{by (6)}].$$

If we write $\epsilon = \eta^2$, we see that, in a set $L(\eta)$ whose measure is greater than $1-\eta$, we have

$$U(x, 2^{\rho_1+q}) < \eta \quad (q = 0, 1, 2, \ldots).$$

Introducing this relation in (10), we find that, in the set $L(\eta)$, we have

$$|T(x, \mu)-T(x, 2^m)| < |T(x, 2^m)| + \eta \tag{11}$$

for every m and μ satisfying the conditions

$$m \geqslant \rho_1, \quad 2^m < \mu \leqslant 2^{m+1}.$$

From the result, which we have already obtained in (i), it follows that, in a set $K(\eta)$ of measure $> 1-\eta$, we have

$$|T(x, 2^m)| < \eta \tag{12}$$

for every $m \geqslant p_1$, p_1 being chosen sufficiently large.

Therefore, if $G(\eta)$ denote the set of all the points which are in common to the sets $L(\eta)$ and $K(\eta)$, then the measure of $G(\eta)$ is greater than $1-2\eta$, and in this set $G(\eta)$, we have

$$|T(x, \mu)-T(x, 2^m)| < 2\eta \tag{13}$$

for every m and μ satisfying the conditions

$$m \geqslant m_0, \quad 2^m < \mu \leqslant 2^{m+1}, \tag{14}$$

where m_0 is a sufficiently large positive integer, whose value $[\geqslant \max(\rho_1, p_1)]$ can be determined so as to correspond to every given value of η.

Let m' and μ' be any set of values of m and μ, satisfying the conditions (14). Then, in $G(\eta)$, we have

$$|T(x, \mu')-T(x, 2^{m'})| < 2\eta, \quad |T(x, 2^{m'})| < \eta. \tag{15}$$

Now
$$\begin{aligned} |T(x, \mu)-T(x, \mu')| \leqslant\ & |T(x, \mu)-T(x, 2^m)| \\ & + |T(x, \mu')-T(x, 2^{m'})| \\ & + |T(x, 2^m)| + |T(x, 2^{m'})|. \end{aligned}$$

Therefore, using (12), (13), and (15), we find that, in the set $G(\eta)$,

$$|T(x, \mu)-T(x, \mu')| < 6\eta$$

for all integral values of μ and μ' which are greater than 2^{m_0} $(= l_0$ say).

Let ζ be any prescribed positive number, and let $\eta_1, \eta_2, \ldots, \eta_n, \ldots$ be

a sequence of positive numbers, such that

$$\lim_{n\to\infty}(\eta_1+\eta_2+\ldots+\eta_n)=\tfrac{1}{2}\xi.$$

Then, from the above result, it follows that, corresponding to $\eta_1, \eta_2, \ldots, \eta_n, \ldots$, there exist positive integers $l_1, l_2, \ldots, l_n, \ldots$, and sets of points $G(\eta_1)$, $G(\eta_2)$, $\ldots, G(\eta_n), \ldots$, whose measures are greater than $1-2\eta_1, 1-2\eta_2, \ldots, 1-2\eta_n, \ldots$, respectively, such that

$$|T(x,\mu)-T(x,\mu')|<6\eta$$

in $G(\eta_n)$, for $\mu>l_n$, $\mu'>l_n$, and for $n=1, 2, 3, \ldots$.

All the sets $G(\eta_n)$ have in common a set W of measure greater than $1-2\sum_{n=1}^{\infty}\eta_n$, or than $1-\xi$. In this set W we have

$$\lim_{\mu\to\infty,\,\mu'\to\infty}|T(x,\mu)-T(x,\mu')|=0,$$

since

$$\lim_{n\to\infty}\eta_n=0.$$

And, since ξ is any prescribed positive number, it follows that

$$\lim_{\mu\to\infty,\,\mu'\to\infty}\{T(x,\mu)-T(x,\mu')\}=0$$

almost everywhere.

Therefore

$$\lim_{n\to\infty}T(x,n)=\lim_{m\to\infty}T(x,2^m)=0$$

almost everywhere; *i.e.*,

$$\lim_{n\to\infty}\frac{(s_1-f)^2+(s_2-f)^2+\ldots+(s_n-f)^2}{n}=0$$

almost everywhere.

Apply the well known inequality

$$\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^2\leqslant\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}$$

to the above result. Then it immediately follows that

$$\lim_{n\to\infty}\frac{|s_1-f|+|s_2-f|+\ldots+|s_n-f|}{n}=0$$

almost everywhere.

Thus we have proved the theorem :—

If (1) $\{\phi_n(x)\}$ *be a sequence of normal orthogonal functions for the interval* (0, 1),

(2) $\{c_n\}$ *be a sequence of given numbers, such that the series*

$$c_1^2 \log 1 + c_2^2 \log 2 + \ldots + c_n^2 \log n + \ldots$$

is convergent,

(3) $f(x)$ *be a summable function, whose square is also summable in the interval* (0, 1), *and to which the series*

$$c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) + \ldots$$

converges on the average in the interval (0, 1),

(4) $$s_n = c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x),$$

then $$\lim_{n\to\infty} \frac{(s_1-f)^2 + (s_2-f)^2 + \ldots + (s_n-f)^2}{n} = 0$$

and $$\lim \frac{|s_1-f| + |s_2-f| + \ldots + |s_n-f|}{n} = 0$$

almost everywhere.

Now evidently we have

$$\left| \frac{(s_1-f) + (s_2-f) + \ldots + (s_n-f)}{n} \right| \leqslant \frac{|s_1-f| + |s_2-f| + \ldots + |s_n-f|}{n}.$$

Therefore, under the same hypotheses as in the above theorem, we have

$$\lim_{n\to\infty} \left| \frac{s_1+s_2+\ldots+s_n}{n} - f(x) \right| = 0$$

almost everywhere: *i.e.*,

$$\lim_{n\to\infty} \frac{s_1+s_2+\ldots+s_n}{n} = f(x)$$

almost everywhere.

Hence we have the following corollary:—

Cor. 1.—*If*

(1) $\{\phi_n(x)\}$ *be a sequence of normal orthogonal functions for the interval* (0, 1),

(2) $\{c_n\}$ *be a sequence of given numbers, such that the series*

$$c_1^2 \log 1 + c_2^2 \log 2 + \ldots + c_n^2 \log n + \ldots$$

is convergent,

(3) *$f(x)$ be a summable function, whose square is also summable, and to which the series*

$$(1) \qquad c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)+\ldots$$

converges on the average in the interval (0, 1),

then the series (1) *is summable* ($C1$) *to the sum $f(x)$ almost everywhere.*

This is nothing but the theorem of Weyl and Hobson mentioned in § 1.

If the sequence $\{\phi_n(x)\}$ form a *closed system* of normal orthogonal functions, then the function $f(x)$, determined by the system of equations

$$\int_0^1 f(x)\,\phi_n(x)\,dx = c_n \quad (n = 1, 2, 3, \ldots),$$

where $\sum_{n=1}^{\infty} c_n^2$ is convergent and $\{f(x)\}^2$ is to be summable, is unique.* Hence it follows that, in this case, the series (1) converges on the average to this function $f(x)$.

Hence we have the following corollary :—

COR. 2.—*If*

(1) *$\{\phi_n(x)\}$ be a closed system of normal orthogonal functions for the interval* (0, 1),

(2) *the series*

$$c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)+\ldots$$

be the general Fourier series corresponding to any summable function $f(x)$, whose square is also summable, and which satisfies the condition that the series

$$c_1^2\log 1+c_2^2\log 2+\ldots+c_n^2\log n+\ldots$$

is convergent, where

$$c_n = \int_0^1 f(x)\,\phi_n(x)\,dx \quad (n = 1, 2, 3, \ldots),$$

(3) $$s_n = c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x),$$

* Of course, here, sets of points with measure zero are neglected.

then
$$\lim_{n\to\infty} \frac{(s_1-f)^2+(s_2-f)^2+\ldots+(s_n-f)^2}{n} = 0,$$

$$\lim_{n\to\infty} \frac{|s_1-f|+|s_2-f|+\ldots+|s_n-f|}{n} = 0$$

and
$$\lim_{n\to\infty} \frac{s_1+s_2+\ldots+s_n}{n} = f(x)$$

almost everywhere.

Prof. E. W. Hobson and Mr. G. H. Hardy have kindly looked this paper through and have given me some advice. I take this opportunity to tender them my hearty thanks.

ON THE VIBRATIONS OF A SPECIAL TYPE OF DISSIPATIVE SYSTEM

By H. JEFFREYS.

[Received September 21st, 1915.—Read December 9th, 1915.]

IN a paper of mine recently published* I adopted the value of the viscosity of the earth of fluid type derived from the lag of the semidiurnal tide, and employed it to find the effect of such viscosity if it exists on the free vibration called the Eulerian nutation. It was suggested that the results obtained might be vitiated if the frictional forces were not linear functions of the velocities, as was assumed, but were proportional to higher powers. I stated, however, that if the frictional force was always in the opposite direction to the relative velocity, the tides and the Eulerian nutation might be affected in proportion to the magnitudes of the velocities; so that the ratio of the lag of the tides to the modulus of decay of the Eulerian nutation would be of the same order of magnitude for any power of the velocity, when the two vibrations are supposed to occur simultaneously. The present investigation is intended to illustrate this point by comparison with a simplified system.

Consider then a system whose kinetic energy is $\frac{1}{2}(\dot{x}^2+\dot{y}^2)$, and whose potential energy is $\frac{1}{2}(n^2x^2+k^2y^2)$. Further suppose that a disturbing force $E\cos pt$ acts on the coordinate x. The frictional force is λv^{2m+1}, where λ is a constant, v the velocity, and m a positive integer, which may be zero. Such a system could be realised in the case of a particle of unit mass sliding inside a rough ellipsoidal bowl.

In comparison with the original problem $2\pi/k$ would be the period of the Eulerian nutation, and $2\pi/p$ a lunar day. The amplitude of the motion in x is considerably greater than that in y, as the amplitude of the tide is eighteen times that in the Eulerian nutation. Further, λ is small, and its square is neglected.

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* *Monthly Notices of R.A.S.*, Vol. 75, 1915, p. 648.

The equations of motion are then

$$\ddot{x}+n^2x+\lambda(\dot{x}^2+\dot{y}^2)^m\dot{x}=E\cos pt, \tag{i}$$

$$\ddot{y}+k^2y+\lambda(\dot{x}^2+\dot{y}^2)^m\dot{y}=0. \tag{ii}$$

These must be solved by successive approximation.

In accordance with the conditions of the problem, we can neglect y in (i). We have to solve then

$$\ddot{x}+n^2x+\lambda\dot{x}^{2m+1}=E\cos pt. \tag{iii}$$

To a first approximation $\quad x=\dfrac{E\cos pt}{n^2-p^2}.$

Substitute this in the frictional term.

$$\ddot{x}+n^2x=E\cos pt+\frac{\lambda p^{2m+1}E^{2m+1}}{(n^2-p^2)^{2m+1}}\sin^{2m+1}pt$$

$$=E\cos pt+\lambda\left(\frac{pE}{n^2-p^2}\right)^{2m+1}\frac{1}{2^{2m}}$$

$$\left\{\frac{(2m+1)_m}{m!}\sin pt-\frac{(2m+1)_{m-1}}{(m-1)!}\sin 3pt+\frac{(2m+1)_{m-2}}{(m-2)!}\sin 5pt-\dots\right\},$$

where $(2m+1)_r$ denotes $(2m+1)(2m)(2m-1)\dots(2m-r+2)$. Hence

$$x=\frac{E\cos pt}{n^2-p^2}+\lambda\left(\frac{pE}{n^2-p^2}\right)^{2m+1}\frac{1}{2^{2m}}\left\{\frac{(2m+1)_m}{m!}\frac{\sin pt}{n^2-p^2}-\frac{(2m+1)_{m-1}}{(m-1)!}\frac{\sin 3pt}{n^2-9p^2}\right.$$

$$\left.+\frac{(2m+1)_{m-2}}{(m-2)!}\frac{\sin 5pt}{n^2-25p^2}-\dots\right\}$$

$$=\alpha \text{ say.} \tag{iv}$$

This is a particular solution of equation (iii). It would vanish when $pt=\frac{1}{2}\pi$ if there was no friction. The observed lag ϵ, being determined by harmonic analysis, must involve only the terms in $\cos pt$ and $\sin pt$. It is found to be given by

$$\tan\epsilon=\lambda\frac{(2m+1)_m}{m!}\left(\frac{p}{n^2-p^2}\right)^{2m+1}\left(\frac{E}{2}\right)^{2m}. \tag{v}$$

The actual tide is therefore retarded owing to the fact that n is greater than p. $2\pi/n$ is a few hours for most oceanic vibrations.

Now to get an approximation to a general integral, put

$$x=\alpha+\xi,$$

where ξ^2 is to be neglected. Then (iii) becomes

$$\ddot{\xi}+n^2\xi+\lambda(2m+1)\,a^{2m}\,\dot{\xi}=0. \qquad \text{(vi)}$$

(ii) also becomes $\ddot{y}+k^2y+\lambda\dot{a}^{2m}\,\dot{y}=0.$

Let us treat the latter first.

Substituting for a and neglecting λ^2, we have

$$\ddot{y}+k^2y+\lambda\dot{y}\left(\frac{Ep}{n^2-p^2}\right)^{2m}\frac{1}{2^{2m-1}}\left\{\frac{(2m)_m}{m!}-\frac{(2m)_{m-1}}{(m-1)!}\cos 2pt\right.$$
$$\left.+\frac{(2m)_{m-2}}{(m-2)!}\cos 4pt-\ldots\right\}=0. \qquad \text{(vii)}$$

If λ was zero, the solutions would be $e^{\pm\iota kt}$. Try then

$$y=Ae^{\iota gkt}\begin{pmatrix}1+a_1\cos 2pt+a_2\cos 4pt+\ldots\\ +b_1\sin 2pt+b_2\sin 4pt+\ldots\end{pmatrix},$$

where g is nearly unity, and a_1, a_2, ... are small of the order of λ. The equation will have a solution of this form, as it belongs to the class of linear differential equations with periodic coefficients.

Write (vii) in the form

$$\ddot{y}+k^2y+\mu(1+f_1\cos 2pt+f_2\cos 4pt+\ldots)\,\dot{y}=0.$$

On substituting we see that the only terms involving $e^{\iota gnt}$ and not of the second order in μ give $\qquad -g^2k^2+k^2+\mu\iota kg=0,$

whence $\qquad g=\pm 1+\frac{\mu\iota}{2k}.$

It follows that the vibration will die down with a modulus of decay $\frac{1}{2}\mu$. If then the modulus of decay of the vibrations in y be κ, we have

$$\kappa=\lambda\left(\frac{Ep}{2(n^2-p^2)}\right)^{2m}\frac{(2m)_m}{m!}.$$

Further, from (v),

$$\epsilon=\lambda\left(\frac{p}{n^2-p^2}\right)^{2m+1}\left(\frac{E}{2}\right)^{2m}\frac{(2m+1)_m}{m!};$$

and therefore $\qquad \frac{\kappa}{\epsilon}=\frac{n^2-p^2}{p}\,\frac{m+1}{2m+1},$

so that the order of magnitude of this ratio is independent of both E and A.

The displacement in the coordinate x will oscillate about the particular solution $x = a$, while the variations from it will die down with a modulus of decay $(2m+1)\kappa$.

If then we determine the coefficient of the frictional terms from the lag, as if the frictional force was simply proportional to the velocity, and from this find κ' the modulus of decay of the y vibrations for the case of $m = 0$, this will be found to satisfy

$$\frac{\kappa'}{\epsilon} = \frac{n^2 - p^2}{p}.$$

Hence

$$\frac{\kappa}{\kappa'} = \frac{m+1}{2m+1},$$

so that the order of magnitude of the moduli of decay is the same in the two cases, and to this extent the assumption of linearity is therefore justified.

SOME APPLICATIONS OF THE TWO-THREE BIRATIONAL SPACE TRANSFORMATION

By T. L. WREN.

[Received November 11th, 1915.—Read December 9th, 1915.]

IN the following paper certain properties of the configuration of n straight lines meeting a common transversal have been established with the aid of the two-three birational space transformation. This transformation, first investigated by Cayley*, has been used by Noëther† in mapping certain types of surface point for point upon a plane.

The present paper consists of four parts. In Part I an account is given of the general properties of the transformation used in the subsequent work. All the results are derived from first principles without any reference to the properties of birational transformation in general.

Part II is devoted to the establishment of certain existence theorems for surfaces of order n with $(n-1)$-ple or $(n-2)$-ple straight line, and for the rational space curve of order n with an $(n-1)$-point secant. The results of sections 6–9 may be summarised as follows, the word "line" being understood to mean "straight line" throughout the paper.

(i) A surface of order n with $(n-2)$-ple line is uniquely determined by $2n-1$ lines meeting the multiple line.

(ii) A surface of order n with $(n-1)$-ple line is uniquely determined by p lines meeting the multiple line and $3n-2p$ points.

(iii) A curve of order n with an $(n-1)$-point secant is determined uniquely by p points and $3n+1-2p$ lines meeting the multiple secant.

The proof in each case is by induction, the transformation being used to reduce the order of the surface or curve while preserving its type. The results just related are thus deduced from the corresponding theorems for the quadric surface and straight line. In none of these cases does the

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* *Collected Papers*, Vol. VII, p. 230.

† *Math. Ann.*, Vol. III, p. 559.

proof involve consideration of elements of either space corresponding to fundamental elements of the other. In section 10 are discussed some relations between ruled surfaces and rational curves of the type already considered.

The method of sections (6–9) is really equivalent to a succession of applications of the two-three transformation, the principal fundamental line remaining the same throughout. The transformation resulting from such a succession of operations is dealt with in Part III, where it is shewn to lead to the plane constructions used by Noëther* in classifying the curves upon the surfaces of the types here considered.

The results of Parts II and III are essentially given by Noëther in the paper just referred to. They were collected by the present writer with a view to using the two-three transformation in an attempt to extend the line configuration theorem of Part IV indefinitely. For this purpose it was desirable to establish the uniqueness of the surfaces and curves as determined by the given conditions in sections 6–9. Noëther shews that in each case the given conditions determine at least one construct of the type considered, but does not prove definitely that this is the only one. For example, consider a surface of order n having a given $(n-2)$-ple line. Its equation contains $6n-3$ independent constants, so that the surface may be made to pass through $6n-3$ arbitrary points. If these $6n-3$ points be taken in threes on $2n-1$ lines meeting the given multiple line it appears that there will always be at least one surface of the required type containing the $2n-1$ lines. There may, however, be a linear system of such surfaces, as we have no justification for assuming that the $6n-3$ linear equations to determine the coefficients are linearly independent when the $6n-3$ points lie by threes on straight lines. Noëther appears to assume that this is the case. The proof by elementary methods used in sections 6–9 of this paper establishes with certainty not only the existence, but also the uniqueness of the constructs as determined by the configurations considered. It is interesting to note the mutual relations between the curves and surfaces discussed in sections 10, 12.

Part IV deals with a theorem of line configurations. If a straight line l be taken, any four lines meeting l determine another line which meets all four. Again, if we take five lines meeting l, every four of the five lines determines a new line as above. Then, by the well known theorem of the double-six (incidentally established anew in section 15) these five new lines are all met by a straight line, which we may regard as determined by the original set of five. The double-six plays an important part

* *Math. Ann.*, Vol. III, pp. 183, 195.

in the geometry of the cubic surface. It has also been discussed in relation to a certain quadric by Mr. G. T. Bennett,* and an elementary proof not involving the cubic surface has been given by Prof. Baker.†

Now suppose we start with six lines meeting l; every five determine in the above manner a new line. The question arises whether these six new lines are all met by a common transversal; and if so is there a similar theorem for seven lines, and so on universally. In section 15 we use the two-three transformation to deduce the five-line theorem from the four-line theorem; and in the succeeding sections 16, 17 we establish the six-line theorem with a single application of the transformation. For the six lines we get in the course of the construction a configuration consisting of two sets of 12 and 32 lines respectively; such that every line of the 32 is met by 6 of the 12, and every line of the 12 by 16 of the 32.

For the next case, that of seven lines with a common transversal, I have so far been unable to establish the theorem. The figure obtained for the construction of the seven new lines determined by every six of the original seven involves 64 lines, and the figures resulting from one or two applications of our transformation are extremely complicated and difficult to visualise.

The theorem for six lines meeting a common transversal is a particular case of the corresponding theorem for six linear complexes having a line in common. Any four of these complexes have another line in common; hence from a set of five of the six complexes we get five lines which determine a linear complex. From the six original complexes we thus get six sets of five, and so obtain six new complexes, which have one common line. This has been proved by Mr. J. H. Grace‡ by considerations of cubic threefolds in four dimensional space. Mr. Grace tells me that he has also considered the cases of seven and of eight lines.

In conclusion, I wish to express my thanks to Prof. Baker for his kindness in reading through the paper and in suggesting improvements.

I. *The Two-Three Transformation.*

1. Consider in a space (X) the system of an arbitrary straight line G and three arbitrary points 1, 2, 3. Through these pass four linearly independent quadric surfaces Θ, Φ, Ψ, F. Any quadric through the system will

* *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 336.

† *Proc. Roy. Soc.*, A, Vol. 84, p. 597.

‡ "Circles, Spheres, and Linear Complexes," *Camb. Phil. Trans.*, Vol. XVI, 1898, p. 163.

then be represented by an equation of form

$$l\Theta+m\Phi+n\Psi+pF=0.$$

Any two of these quadrics intersect, besides the line G, in a cubic curve passing through the points 1, 2, 3 and having G for a chord. A third quadric of the family meets the cubic curve in six points of which two are on G and three at 1, 2, 3. Thus three of these quadrics intersect in one point other than the fundamental elements. If then we put

$$\frac{x}{\Theta}=\frac{y}{\Phi}=\frac{z}{\Psi}=\frac{t}{F}$$

to any set of ratios $x:y:z:t$ corresponds one point of the X-space other than the fundamental elements. We shall thus be able to deduce from these equations a set of equations of the form

$$\frac{X}{\varsigma}=\frac{Y}{\phi}=\frac{Z}{\psi}=\frac{T}{f},$$

where ς, ϕ, ψ, f are polynomials homogeneous in x, y, z, t. The degree of these polynomials will be the number of common solutions of three equations of the form

$$\begin{aligned} a\varsigma+b\phi+c\psi+df &= 0,\\ a'x+b'y+c'z+d't &= 0,\\ a''x+b''y+c''z+d''t &= 0, \end{aligned}$$

that is, of

$$\begin{aligned} aX+bY+cZ+dT &= 0,\\ a'\Theta+b'\Phi+c'\Psi+d'F &= 0,\\ a''\Theta+b''\Phi+c''\Psi+d''F &= 0; \end{aligned}$$

and therefore the order of the residual curve of intersection of any two of the original quadrics, that is 3. Hence ς, ϕ, ψ, f represent cubic surfaces, such that any three intersect, beyond certain fundamental elements common to all surfaces of the system

$$a\varsigma+b\phi+c\psi+df=0,$$

in a single point. To find these fundamental elements we must specify the form of Θ, Φ, Ψ, F in more detail. We may choose the tetrahedron

of reference in the X-space so that the fundamental elements are

$$\text{the line } G \qquad X = Y = 0,$$

$$\text{the points } \begin{cases} 1 & X = Z = T = 0, \\ 2 & Y = Z = T = 0, \\ 3 & X-Z = Y-Z = T = 0. \end{cases}$$

Then the plane-pairs

$$XT = 0, \quad YT = 0, \quad X(Y-Z) = 0, \quad Y(X-Z) = 0$$

are four linearly independent quadrics through the fundamental elements, and we may take these to be the quadrics Θ, Φ, Ψ, F, putting

$$x : y : z : t = XT : YT : X(Y-Z) : Y(X-Z).$$

We may then readily deduce the reciprocal relations

$$X : Y : Z : T = x(xt-yz) : y(xt-yz) : xy(t-z) : xy(x-y). \qquad \text{(I)}$$

The fundamental elements of the x-space are then seen to be the four straight lines

$$g \qquad x = y = 0,$$

$$(1) \qquad x = z = 0,$$

$$(2) \qquad y = t = 0,$$

$$(3) \qquad x-y = t-z = 0.$$

The surfaces of the system $a\vartheta+b\phi+c\psi+df = 0$ are then seen to be cubic surfaces having g for double line, and (1), (2), (3) for generating lines. Any two of these surfaces meet besides the fundamental elements in a conic meeting g, (1), (2), (3). A third surface of the family meets this conic in a single point not lying on any of the fundamental lines, and this is the point x, y, z, t corresponding to the point X, Y, Z, T of intersection of the corresponding planes in the X-space. In fact the sets of equations (I) determine a birational transformation between the spaces (X), (x); we now proceed to consider the properties of this transformation.

The transformation is such that to the planes of x correspond the quadric surfaces in X which have G for a generator and pass through the points 1, 2, 3: while to the planes of X correspond in x cubic scrolls having g for double line and (1), (2), (3) for generators.

In general, to a point of either space corresponds a unique point of

the other: to an arbitrary straight line in x corresponds a space cubic in X, having G for chord and passing through the points 1, 2, 3: to an arbitrary straight line in X corresponds in x a conic meeting each of the fundamental lines g, (1), (2), (3) in a single point.

We have now to consider what elements of each space may be regarded as corresponding to the fundamental elements of the other, so as to be able to determine the construct in the former corresponding to a construct meeting or containing fundamental elements of the latter. To find the elements of x corresponding to the fundamental elements of X we replace X, Y, Z, T by $X+\epsilon\xi, Y+\epsilon\eta, Z+\epsilon\zeta, T+\epsilon\tau$ in the first set of relations (I): thus obtaining the set

$$\left.\begin{aligned} \rho x &= XT+\epsilon(\xi T+\tau X)+\epsilon^2\xi\tau, \\ \rho y &= YT+\epsilon(\eta T+\tau Y)+\epsilon^2\eta\tau, \\ \rho z &= X(Y-Z)+\epsilon\{\xi(Y-Z)+(\eta-\zeta)X\}+\epsilon^2\xi(\eta-\zeta), \\ \rho t &= Y(X-Z)+\epsilon\{\eta(X-Z)+(\xi-\zeta)Y\}+\epsilon^2\eta(\xi-\zeta). \end{aligned}\right\} \quad \text{(II)}$$

In this set give X, Y, Z, T values corresponding to a point on a fundamental construct, and then let ϵ tend to zero. The limiting ratios of $x : y : z : t$ give the coordinates of a point which may be considered to correspond to the point in question when approached in a definite direction determined by the ratios $\xi : \eta : \zeta : \tau$. Thus we find that

(i) To a given point ($Z = \mu T$) of G corresponds that generator of the quadric $xt-yz = 0$ determined by the equations

$$z+\mu x = t+\mu y = 0.$$

The point in which this generator is met by the plane $x = \lambda y$ corresponds to all directions of approach in the plane $X = \lambda Y$.

(ii) To every direction through the point 1 corresponds a point of the plane $x = 0$ given by $x : y : z : t = 0 : \tau : \xi : \xi-\zeta$.

(iii) To a direction through 2 corresponds a point of $y = 0$ given by $x : y : z : t = \tau : 0 : \eta-\zeta : \eta$.

(iv) To a direction through 3 corresponds a point of $x-y = 0$ given by $x : y : z : t = \tau : \tau : \eta-\zeta : \xi-\zeta$.

Proceeding similarly for the fundamental elements of the x-space, we

replace the second set of relations (I) by

$$\begin{aligned}
\rho X &= x(xt-yz)+\epsilon[\xi(xt-yz)+x(x\tau+t\xi-y\zeta-z\eta)] \\
&\quad +\epsilon^2[\xi(x\tau+t\xi-y\zeta-z\eta)+x(\xi\tau-\eta\zeta)]+\epsilon^3\xi(\xi t-\eta\zeta), \\
\rho Y &= y(xt-yz)+\epsilon[\eta(xt-yz)+y(x\tau+t\xi-y\zeta-z\eta)] \\
&\quad +\epsilon^2[\eta(x\tau+t\xi-y\zeta-z\eta)+y(\xi\tau-\eta\zeta)]+\epsilon^3\eta(\xi\tau-\eta\zeta), \\
\rho Z &= xy(t-z)+\epsilon[xy(\tau-\zeta)+(t-z)(x\eta+y\xi)] \\
&\quad +\epsilon^2[(t-z)\xi\eta+(x\eta+y\xi)(\tau-\zeta)]+\epsilon^3\xi\eta(\tau-\zeta), \\
\rho T &= xy(x-y)+\epsilon[xy(\xi-\eta)+(x-y)(x\eta+y\xi)] \\
&\quad +\epsilon^2[(x-y)\xi\eta+(x\eta+y\xi)(\xi-\eta)]+\epsilon^3\xi\eta(\xi-\eta).
\end{aligned} \qquad \text{(III)}$$

It appears then that we must regard as corresponding—

(v) To the point $z=\mu t$ of g, the conic

$$\mu YZ-XZ-(\mu-1)XY=0, \quad T=0:$$

so that to every direction of approach in the plane $x=\lambda y$ corresponds the point where the conic is met again by the plane $X=\lambda Y$.

(vi) To the point $y=\mu t$ of (1), the line $X=T+\mu Z=0$.

(vii) ,, $x=\mu z$,, (2), ,, $Y=T+Z\mu=0$.

(viii) ,, $t=\mu x$,, (3), ,, $X=Y=Z+\mu T$.

(ix) ,, $\{g.1\}$, $\{g.2\}$, $\{g.3\}$, respectively the straight lines [2.3], [3.1], [1.2].

2. From the original form of the equations of transformation we see at once that to the pencil of planes through G corresponds the pencil of planes through g: and to the pencils of planes through (1), (2), (3) correspond respectively the pencils through [2.3], [3.1], [1.2].

Thus to a straight line meeting the line (1) corresponds a conic passing through 2, 3 and meeting g: and similarly for lines meeting (2) or (3).

Again to a straight line meeting both (1) and (2) corresponds a straight line through the point 3.

Finally, to a straight line meeting all three lines (1), (2), (3) corresponds a definite point of G [see (i) above].

Further, to an arbitrary line $ay+bz+ct=0$ in the plane $x=\lambda y$, corresponds a straight line $AY+BZ+CT=0$ in the plane $X=\lambda Y$; where $A:B:C=(c+b)\lambda:-(c+b\lambda):a$. Hence it appears that to a

given curve in the first plane corresponds in general a curve of the same order in the second; such that to every singularity of the first corresponds a singularity of the same type on the second curve.

II. *Existence Theorems.*

3. *The most general surface of order n having an $(n-2)$-ple straight line contains $3n-4$ pairs of straight lines lying in planes through the multiple line.*

Taking $x = y = 0$ as the multiple line, the equation to the surface may be expressed in the form

$$t^2\alpha + zt\beta + z^2\gamma + t\delta + z\epsilon + \eta = 0, \tag{IV}$$

where α, β, γ, δ, ϵ, η are homogeneous polynomials in x, y of orders $n-2$, $n-2$, $n-2$, $n-1$, $n-1$, n respectively. Thus the surface appears as a locus of conics in planes $x = \lambda y$, and $3n-4$ of these break up into line-pairs, namely those for which λ satisfies the equation

$$\begin{vmatrix} 2\alpha & \beta & \delta \\ \beta & 2\gamma & \epsilon \\ \delta & \epsilon & 2\eta \end{vmatrix} = 0,$$

where x, y are replaced by λ, 1 respectively.

4. Take three lines, one from each of three of these $3n-4$ pairs as the lines (1), (2), (3) of the x-space, as defined in section 1. The equation to the surface may now be written in the form

$$t^2xa + tzb + z^2yc + txd + zye + xyf = 0, \tag{V}$$

where a, b, c, d, e, f are homogeneous polynomials in x, y of orders $n-3$, $n-2$, $n-3$, $n-2$, $n-2$, $n-2$ respectively, such that

$$f = e + d = a + b + c = 0,$$

when $x = y$. These conditions are equivalent to

$$\left.\begin{aligned} f &\equiv (x-y)A, \\ e+d &\equiv (y-x)B, \\ ye+xd &\equiv (x-y)D, \\ ya+b+xc &\equiv (x-y)C, \\ xa+b+yc &\equiv (x-y)F, \\ 2xy(a+c)+(x+y)b &\equiv (y-x)E, \end{aligned}\right\} \tag{VI}$$

where $A, B, \ldots, F$ are polynomials homogeneous in x, y; A, B, D being perfectly general, while C, E, F are subject to the single condition that $C+E+F=0$, when $x=y$.

Applying the transformation (I) to (V), we get, in virtue of (VI), after removing common factors,

$$T^2A+ZTB+Z^2C+TD+ZE+XYF=0. \qquad \text{(VII)}$$

This is the most general form of equation to a surface of order $n-1$ having G for $(n-3)$-ple line, and passing through the points 1, 2, 3. This surface will have $3n-7$ pairs of straight lines in planes through G, corresponding to the remaining $3n-7$ pairs on the original surface of order n.

Thus it appears that *the transformation* (I) *reduces the most general surface of order n having g for* $(n-2)$*-ple line and containing the lines* (1), (2), (3) *to the most general surface of order* $n-1$ *with G for* $(n-3)$*-ple line and passing through the points* 1, 2, 3.

5. If in the preceding section all the coefficients in the polynomials a, b, c be supposed zero, we obtain in place of (V), the equation

$$txd+yze+f=0, \qquad \text{(VIII)}$$

where $f=e+d=0$, when $x=y$. This represents a general scroll of order n with g as $(n-1)$-ple line, and (1), (2), (3) as generators. A factor T can now be removed from (VII), leaving

$$TA+ZB+D=0. \qquad \text{(IX)}$$

This is the most general equation to a scroll of order $n-2$ with G as $(n-3)$-ple line, not in general passing through the points 1, 2, 3. Thus *the transformation reduces the scroll of order n with g for* $(n-1)$*-ple line, and* (1), (2), (3) *as generators to a scroll of order* $n-2$ *with G as* $(n-3)$*-ple line.*

From the results of this and the preceding section, we derive the property referred to by Noëther, *Math. Ann.*, III, p. 561; namely, that by successive applications of the transformation the surface of the former type may be reduced to a quadric, and the surface of the latter type to a quadric or a plane according as n is even or odd.

6. We are now in a position to prove that—*One and only one surface of order n exists, having a given* $(n-2)$*-ple line G and containing* $2n-1$ *arbitrary lines which meet G but not one another.*

The form of equation (IV) shews that when the multiple line is given the surface may be made to satisfy $6n-3$ independent conditions. Now a line meeting the multiple line will lie wholly on the surface if the equation to the surface be satisfied by the coordinates of some three points on the line, giving three linear relations between the coefficients. Thus given a system of $2n-1$ lines meeting the line G there will always be at least one surface of the type required containing these lines: and only one such surface provided the $6n-3$ conditions for the $2n-1$ lines to lie upon the surface are linearly independent.

Suppose that for some value $n = N$ this is the case, so that our theorem is true for $n = N$. On replacing the three conditions that one of the $2N-1$ lines $A, B, C, \ldots$ should lie on the surface by the three conditions, also linear in the coefficients, for the surface to pass through three arbitrary points there can be no loss of independence among the $6N-3$ relations which determine the coefficients. Thus we see that a system of $2N-2$ lines $A, B, C, \ldots$ meeting G together with three arbitrary points also determine a unique surface of order N with G for $(N-2)$-ple line.

Take now at random a system of $2N+1$ lines (1), (2), (3), $a, b, c, \ldots$ all meeting a line g. There are then ∞^2 ways of choosing the coordinate system, so that the equations to g, (1), (2), (3) are those given in section 1.* Applying the transformation, to any surface of order $N+1$ with g as $(N-1)$-ple line and containing the $2N+1$ lines (1), (2), (3), $a, b, c, \ldots$, there corresponds in the X-space a surface of order N with G as $(N-2)$-ple line on which lie the points 1, 2, 3 and the $2N-2$ lines $A, B, C, \ldots$. These $2N-2$ lines correspond respectively to $a, b, c, \ldots$, and therefore have no special relations of position among themselves or with respect to 1, 2, 3; for the lines $a, b, c, \ldots$ were only subject to the condition of meeting g. Hence, on our hypothesis, there is in the X-space only one surface of order N satisfying the required conditions: and consequently in the x-space but one surface of order $N+1$ of the type considered.

Thus, if our theorem holds for $n = N$, it is true also for $n = N+1$. But it is well known to be true for $n = 3$. Indeed this result may be deduced as above from the fact that a quadric surface is uniquely determined by two non-intersecting generators and three points. Hence *the theorem is true generally for all values of n.*

7. The analogous theorem for the scroll of order n with $(n-1)$-ple line takes different forms according to the parity of n. The form of

* $z = t = 0$ may be any line meeting (1), (2), and (3), and the unit point any point of (3).

equation (IX) shews that, the multiple line G being given, the surface may be made to satisfy $3n$ independent conditions. A line meeting G will lie wholly on the surface if the equation be satisfied by the coordinates of any two of its points, giving two linear relations between the coefficients.

(i) When n is even we have—*A unique scroll of order n exists having a given $(n-1)$-ple line G and $3n/2$ arbitrary lines meeting G as generators.*

(ii) When n is odd—*A unique scroll of order n exists having a given $(n-1)$-ple line G, containing $(3n-1)/2$ arbitrary lines meeting G, and passing also through a single arbitrary point.*

In either case the proof is on the lines of the preceding section: first, that there is at least one surface satisfying the required conditions, since these involve relations linear in the coefficients: secondly, the uniqueness of this surface is established by induction. Case (i) is true when $n = 2$, since a unique quadric is determined by three non-intersecting lines; Case (ii) is true for $n = 1$, since a plane is uniquely determined by a straight line and a point.

Notice that on replacing the conditions for the surface to contain one of the lines by the conditions that it may pass through two arbitrary points, there is no loss of independence among the $3n$ linear equations which determine the coefficients. We may then combine (i) and (ii) into a single theorem which holds for all values of n, odd or even—

A surface of order n having a given $(n-1)$-ple line G is uniquely determined by p generators meeting G and $3n-2p$ arbitrary points $(p < 3n/2)$.

8. We shall now establish analogous existence theorems for the rational space curve of order n having $n-1$ points on a given straight line g. *In general there is one and only one such curve meeting each of $3n+1$ arbitrary lines which intersect g.*

For, with the notation of section 1 the most general rational C_n having $n-1$ points on g and meeting (1), (2), (3), can be represented by the equations

$$x : y : z : t = L_1 P_{n-1} : M_1 P_{n-1} : L_1 M_1 R_{n-2} - L_1 Q_{n-1} : L_1 M_1 R_{n-2} - M_1 Q_{n-1},$$

where L_1, M_1, P_{n-1}, Q_{n-1}, R_{n-2} are polynomials in a single parameter λ, of degrees denoted by their suffixes, and such that no two vanish for the same value of λ. The transformation gives for the corresponding curve

in the X-space the equations

$$X : Y : Z : T = L_1 R_{n-2} : M_1 R_{n-2} : Q_{n-1} : P_{n-1}.$$

These represent the most general curve of order $n-1$, having $n-2$ points on G (and *not* in general passing through the points 1, 2, 3).

Suppose that the theorem is true for a particular value $n = N$, and take a system of $3N+4$ arbitrary lines (1), (2), (3), a, b, c, ... meeting a line g. Then the tetrahedron of reference may be chosen in one of ∞^2 ways so that the equations to g, (1), (2), (3) are those of section 1. If there is any curve of order $N+1$ with N points on g and meeting (1), (2), (3), a, b, c, ... the corresponding construct in the X-space will be a curve of order N with $N-1$ points on G. The latter curve must also meet each of the $3N+1$ lines A, B, C, ... which correspond to a, b, c, ... These lines A, B, C, ... are subject to no further relations of position beyond that of meeting G. Hence in the X-space there is, according to our hypothesis, a unique curve of order N satisfying the required conditions. There will thus be at least one curve of order $N+1$ with N points on g and meeting each of the original $3N+4$ lines. This will indeed be the only such curve: for, although there might be others which by reason of special relations with the coordinate-tetrahedron give simpler corresponding constructs in the X-space, by a change of coordinates one or more of these could be generalised without bringing the first into any special relationship. We should then have more than one curve C_N of type considered in the X-space, which is contrary to the hypothesis that the theorem holds for $n = N$. Thus, if the theorem is true for $n = N$, it is also true for $n = N+1$.

It is well known that a system of four skew straight lines possesses two common transversals. Starting with one of these we may regard the other as uniquely determined by the system of four lines meeting a fifth. This is the form of our theorem for $n = 1$. Hence it is true for $n = 2$ (for an independent proof of this case see Clebsch, *Math. Ann.*, I, p. 258), and so on generally.

9. The theorem of the preceding section may now be used to establish the more general theorem—*Given p arbitrary points and $3n+1-2p$ arbitrary straight lines meeting a line G, there is in general a unique curve of order n with $n-1$ points on G passing through each of the p points and meeting each of the $3n+1-2p$ lines.*

The most general curve of order $n+2$ in the X-space having $n+1$

points on G and passing through the points 1, 2, 3 is represented by

$$X : Y : Z : T = L_1(L_1P_n - M_1Q_n) : M_1(L_1P_n - M_1Q_n) : L_1M_1(P_n - Q_n) : L_1M_1(L_1 - M_1)R_{n-1},$$

where L_1, M_1, P_n, Q_n, R_{n-1} are polynomials in a single parameter λ, and no two vanish for the same value of λ. The corresponding curve in the x-space is then given by

$$x : y : z : t = L_1R_{n-1} : M_1R_{n-1} : Q_n : P_n,$$

which represent the most general curve of order n with $n-1$ points on g.

From this result and the analogous property used in the preceding section it follows as before that if the theorem is true for N, P it is also true for $N+2$, $P+3$ and for $N+1$, P. Thus we may reduce our theorem to depend upon those cases for which p and $3n+1-2p$ are each less than 3 (the theorem for $p = 0$ having already been established independently). We have then to consider the cases—

(i) $p = 1$, $n = 1$. The theorem holds for this case since there is a unique line through a given point meeting each of two given lines.

(ii) $p = 2$, $n = 1$. The theorem is true here since a unique straight line passes through two given points.

Since the theorem is true for these simple cases it will be true generally.

10. By comparison between the theorems of the last three sections, we may deduce some characteristic properties of the type of ruled surface of the type considered in section 7. These surfaces fall into two distinct classes according as they are of odd or even order. In this section we shall use S_n to denote a scroll of order n having a given line g as $(n-1)$-ple line; and C_n a rational curve of order n having $n-1$ points on the line g. These are the only types of surface and curve with which we shall here be concerned.

(i) Start with an arbitrary system of one point and $3m+1$ lines meeting g. This system determines a unique S_{2m+1}, while the $3m+1$ lines alone determine a unique C_m, which lies wholly on the S_{2m+1}, since they have already $2m(m-1)+3m+1 = m(2m+1)+1$ common points. The curve C_m then meets every generator of the surface S_{2m+1}. *It is thus the only curve C_m on the surface*, for any $3m+1$ generators are independent as not sufficing to fix the surface.

Again, any $3m$ generators of S_{2m+1} form an independent system, and so determine a unique S_{2m} which also contains the curve C_m, having common with it $(2m-1)(m-1)+3m = 2m^2+1$ points. *The curve C_m thus appears as the residual intersection of the surface S_{2m+1} by every surface S_{2m} having $3m$ generators common with it.*

There are on the surface S_{2m+1} no curves of type C_n of order lower than m: for such a curve would have to meet all generators, and therefore every one of an independent set of $3m+1$, which is contrary to the theorem that a curve C_p is determined uniquely by $3p+1$ independent lines meeting g.

For $m = 1$ we have the cubic scroll, all whose generators meet the double line and one other straight line, usually called the directrix. In the general case the curve C_m may be regarded as the directrix of S_{2m+1}.

(ii) Start now with a system of $3m$ straight lines meeting g. This system determines a unique S_{2m}.

Taking $3m-1$ of the $3m$ lines and adding some one point of the remaining line, we have a perfectly independent system which determines a unique C_m lying wholly on the surface S_{2m}. Thus through every point of a generator of S_{2m} passes one such C_m, and there is a pencil of these curves on the surface. For $m = 1$, we have for this pencil the opposite system of generators of the quadric.

Again, any $3m-2$ generators of S_{2m} taken with an arbitrary point P of the surface constitute an independent system, and so determine a unique S_{2m-1} whose residual intersection with S_{2m} is the curve C_m through P. *The curves C_m thus appear as residual intersections of S_{2m} by the surfaces S_{2m-1} having common $3m-2$ generators of S_{2m}.*

Just as before there are on the surface S_{2m} no curves of this type of order lower than m.

III. *Repeated Application of the Two-Three Transformation.*

11. Let us now consider the properties of the birational transformation obtained by N successive applications of the transformation of section 1; at each step the line g_k being taken to be the line G_{k-1} of the last transformation, and the lines $(1)_k$, $(2)_k$, $(3)_k$ being any three meeting g_k.

To the properties of the transformation already noted must be added the following:—*To a surface of order n with $(n-1)$-ple line g for which* (1), (2), (3) *are not generators corresponds in the X-space a surface of order $n+1$ with n-ple line G passing through the points* 1, 2, 3. For the

scroll of order n in the x-space is represented by

$$tA_{n-1}+zB_{n-1}+C_n = 0,$$

where A_{n-1}, B_{n-1}, C_n are polynomials homogeneous in x, y. The equation to the equivalent construct in the X-space is then

$$Y(X-Z)A_{n-1}+X(Y-Z)B_{n-1}+TC_n = 0,$$

where A, B, C are now expressed in terms of X, Y. This is the most general equation to a scroll of order $n+1$ with n-ple line G, passing through 1, 2, 3.

In virtue of the results already obtained, we find that the result of N successive transformations is a transformation between spaces x_1, X_N, such that—

(i) To planes of x_1 correspond in X_N surfaces of order $N+1$ with N-ple line G_N and passing through $3N$ fundamental points a_i.

(ii) To straight lines of x_1 correspond space curves of order $2N+1$ with $2N$ points on G_N and passing through the $3N$ points a_i.

(iii) To planes of X_N correspond in x_1 surfaces of order $2N+1$ with $2N$-ple line g_1, on which lie $3N$ fundamental lines A_i.

(iv) To straight lines of X_N correspond space curves of order $N+1$ with N points on g_1 meeting each of the lines A_i.

By combining N transformations taken directly with N' taken inversely, so that $G_N __ G_{N'}$, but the points a_i all different from the points a_i', we obtain a more general transformation in which there correspond—

(i) To planes of x_1, surfaces in x_1' of order $N+2N'+1$ with $(N+2N')$-ple line g_1', passing through $3N$ fundamental points β_i' and containing $3N'$ fundamental lines B_i' which meet g_1'.

(ii) To planes of x_1', surfaces in x_1 of order $2N+N'+1$ with $(2N+N')$-ple line g_1, passing through $3N'$ points β_i and containing $3N$ lines B_i meeting g_1.

This transformation gives at once a point for point representation on a plane of the surface of order n with $(n-1)$-ple line—

(i) For odd $n = 2N+1$, taking $N' = 0$, we get the simplest representation, in which plane sections of the surface are represented by curves of order $N+1$ with fixed N-ple point.

(ii) For even $n = 2N+2$, taking $N' = 1$, plane sections of the surface are represented by curves of order $N+3$ with a fixed $(N+2)$-ple point and three ordinary points fixed. By a Cremona transformation of the plane this can be reduced to a system of curves of order $(N+2)$, having common an $(N+1)$-ple and one ordinary point.

These are the plane representations used by Noëther* in classifying the curves on the surface.

12. For the surface of order n with $(n-2)$-ple straight line a point for point representation upon a quadric is obtained by means of $n-2$ successive transformations. To plane sections of the original surface correspond the curves in which the quadric is cut by surfaces of order $n-1$ with $(n-2)$-ple line G_{n-2} which pass also through the $3(n-2)$ fundamental points a_i. These curves, of order $2(n-1)$, have $n-2$ branches through each of the points A, B in which the quadric is met by G_{n-2}. They have $n-1$ points on a generator of either system, since the lines of the space X_{n-2} correspond to space curves of order $n-1$ in the x_1-space.

Now project points of the quadric from A on to an arbitrary plane. The curves in question project into plane curves of order n having fixed an $(n-2)$-ple and $3n-4$ simple points (the sections of the two generators through A and the projections of the $3n-6$ points a_i). This is the simplest form of plane representation of the surface, as used by Noëther.†

The characteristic property from which Noëther obtains the plane representation may be stated as follows :—*Every set of* $3n-5$ *lines taken one out of each of* $3n-5$ *of the* $3n-4$ *line-pairs on the surface determines on the surface a unique curve of order* $n-2$ *with* $n-3$ *points of intersection with* G_{n-2}. *This curve meets each of the* $3n-5$ *lines and one line of the remaining pair. There are in all* 2^{3n-5} *such curves on the surface.* This may be deduced from the property of the general cubic surface by the method of induction used in sections 6–8.

Assuming the property to be true for $n = N$, consider a surface of order $N+1$ with g for $(N-1)$-ple line, and (1), (2), (3) members of three of its $3N-1$ line-pairs. The corresponding surface in the X-space is of order N with $(N-2)$-ple line G: its $3N-4$ line pairs correspond to the $3N-4$ on the original surface remaining after those containing (1), (2), (3) are removed. We have assumed that some $3N-5$ lines taken one from each of $3N-5$ of the line-pairs on the surface of order N determine a

* *Math. Ann.*, III, p. 195.

† *Math. Ann.*, III, p. 183.

unique curve of order $N-2$ meeting G in $N-3$ points and lying on the surface. To this corresponds in the x-space a definite curve of order $N-1$ with $N-2$ points on g, intersecting each of the corresponding $3N-5$ lines and the three lines (1), (2), (3); and conversely. It appears then that the curve of order $N-1$ is determined uniquely by the system of $3N-2$ lines on the surface of order $N+1$, and the theorem is true for $N+1$ if it is for N.

In the case of the cubic surface $n = 3$, the theorem is known to be true. Through a chosen line of the surface pass five planes cutting it again in line-pairs; a set of four lines taken one from each of four pairs determines a unique straight line which lies wholly on the surface, and therefore meets one of the lines of the remaining pair. Hence the theorem is true for $n = 4$, and so on universally.

We have already seen (section 8) that a perfectly arbitrary system of $3n-5$ lines meeting a common transversal g determines a unique curve of order $n-2$ having $n-3$ points on g. If now the $3n-5$ lines be not perfectly independent, but lie all on the surface of order n with $(n-2)$-ple line g which is determined by some $2n-1$ of them, the last theorem shews that they still determine a unique curve of order $n-2$, which however lies on the surface.*

IV. *Straight Line Configurations—an Extension of the Theorem of the Double-Six.*

13. The theorem of the double-six may be conveniently enunciated in the following form:—Suppose we are given an arbitrary system of five straight lines a_1, a_2, a_3, a_4, a_5, all meeting a straight line g. Then every set of four out of the five lines a is met by another straight line b_k, where the suffix k is that of the fifth line a. The five lines b thus determined, all meet a further straight line d which does not meet g; the position of the line d being completely determined when the original system of g and the five lines a is given.

We now proceed to establish the analogous theorem for a system of six arbitrary lines a_1, a_2, a_3, a_4, a_5, a_6 meeting a line g. Every set of five out of the six lines a determine as above a line d_k, the suffix k being that of the sixth line. It will be proved that there exists a straight line meeting all the six lines d thus obtained.

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* See Prof. Baker, "Some Recent Advances in the Theory of Algebraic Surfaces," *Proc. London Math. Soc.*, Ser. 2, Vol. 12, footnote on p. 36.

14. Consider a system of n arbitrary straight lines meeting a straight line g. Any three of these may be taken to be the lines (1), (2), (3) of section 1. The transformation then gives, corresponding to the remaining $n-3$ lines $a, b, c, \ldots$, a system of $n-3$ straight lines $A, B, C, \ldots$, subject only to the condition of meeting G. Every set of four of the n lines of the first set is met besides g by a second transversal, which may be regarded as determined by the four lines; and similarly for the $n-3$ lines of the second set. The following notation will be adopted:—

(i) $\{abcd\}$ to denote the second common transversal determined by the four lines a, b, c, d.

(ii) $\{P.ab\}$ the straight line through a point P meeting both a and b.

(iii) $\{g.a\}$ the point where a meets g.

Then, referring to section 2, we may tabulate the elements of the X-space which correspond to transversals of four of the n lines in the x-space according to the types—

(i) To a line $\{(1)(2)(3)a\}$ corresponds the point $\{G.A\}$.

(ii) ,, $\{(1)(2)ab\}$,, line $\{3.AB\}$.

(iii) ,, $\{(1)abc\}$,, conic passing through the points 2, 3, and meeting lines G, A, B, C.

(iv) ,, $\{abcd\}$ corresponds the space cubic passing through the points 1, 2, 3, meeting A, B, C, D, and having two points on G. (Section 1, p. 149.)

15. Take now the system of five straight lines $a_1 \equiv (1)$, $a_2 \equiv (2)$, $a_3 \equiv (3)$, a_4, a_5, all meeting g. Applying the transformation the elements corresponding to the five straight lines $b_1, b_2, \ldots, b_5$ are—

$$b_1 \equiv \{(2)(3)a_4a_5\} \quad \text{straight line} \quad B_1 \equiv \{1.A_4A_5\},$$
$$b_2 \equiv \{(1)(3)a_4a_5\} \quad \text{,,} \quad B_2 \equiv \{2.A_4A_5\},$$
$$b_3 \equiv \{(1)(2)a_4a_5\} \quad \text{,,} \quad B_3 \equiv \{3.A_4A_5\},$$
$$b_4 \equiv \{(1)(2)(3)a_5\} \quad \text{point} \quad \beta_4 \equiv \{G.A_5\},$$
$$b_5 \equiv \{(1)(2)(3)a_4\} \quad \text{,,} \quad \beta_5 \equiv \{G.A_4\}.$$

Now the lines B_1, B_2, B_3 determine a unique quadric surface Q which passes through the points 1, 2, 3, β_4, β_5. Taking on this quadric any generator l of the same system as B_1, a pencil of quadrics can be drawn

to pass through 1, 2, 3, β_4, β_5, and have l as common generator. Each of these quadrics meets Q besides l in a space cubic passing through 1, 2, 3, β_4, β_5, and meeting B_1, B_2, B_3 each in a second point. Conversely, any space cubic satisfying these conditions has eight points in common with Q and therefore lies entirely on Q. It will therefore meet every generator of the same system as B_1 in two points: and consequently lies entirely on every quadric of the pencil determined by the generator l and the five points 1, 2, 3, β_4, β_5. There is then one and only one such space cubic. To this curve corresponds in the x-space a straight line d which meets each of the five lines b_1, b_2, b_3, b_4, b_5. *We have thus incidentally established the theorem of the double-six.*

16. Let us now start with a set of six arbitrary straight lines $a_1, a_2, a_3, a_4, a_5, a_6$ meeting the line g. Every set of five out of the six lines a determines a line d, and we shall find that the six lines d thus obtained are all met by a single straight line. The proof depends upon the fact that there are 15 straight lines c_{ij},* such that, for example, c_{12} meets $a_1, a_2, d_3, d_4, d_5, d_6$.

Consider the three sets of five lines obtained by omitting respectively a_4, a_5, a_6. These determine three lines d_4, d_5, d_6 to which correspond in the X-space the three space cubics D_4, D_5, D_6: where D_4 is the unique space cubic passing through the points 1, 2, 3, $\{G.A_5\}$, $\{G.A_6\}$, and having the lines $\{1.A_5A_6\}$, $\{2.A_5A_6\}$, $\{3.A_5A_6\}$ as chords; and similarly for D_5, D_6 by cyclic permutation of the suffixes 4, 5, 6. These curves D_4, D_5, D_6 are projected from the point 3 by quadric cones K_4, K_5, K_6 determined respectively by the sets of five generators—

K_4: [3.1], [3.2], [3.$\{G.A_5\}$], [3.$\{G.A_6\}$], $\{3.A_5A_6\}$,

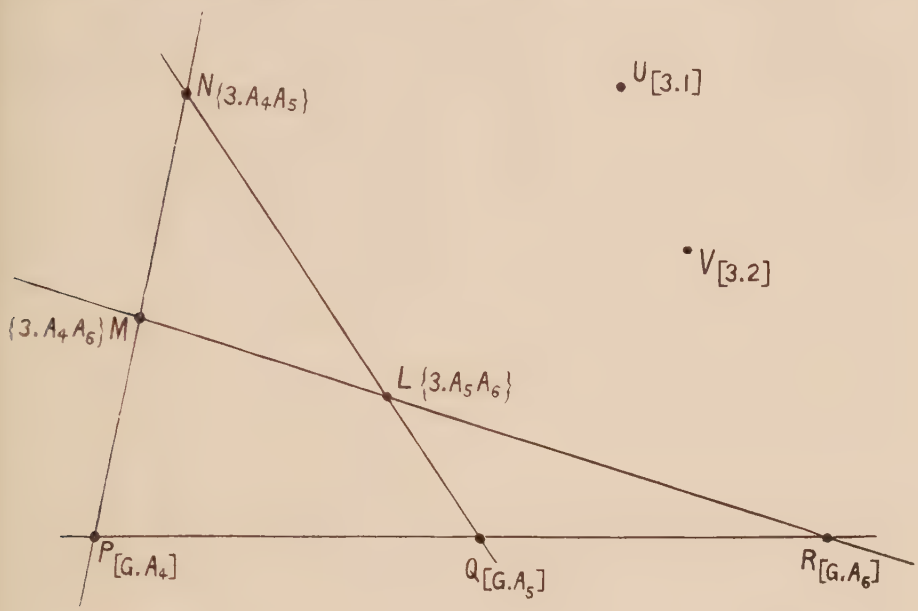
K_5: [3.1], [3.2], [3.$\{G.A_6\}$], [3.$\{G.A_4\}$], $\{3.A_6A_4\}$,

K_6: [3.1], [3.2], [3.$\{G.A_4\}$], [3.$\{G.A_5\}$], $\{3.A_4A_5\}$.

Projecting from 3 on to an arbitrary plane through G, the lines A_4, A_5, A_6 project into three straight lines PMN, QNL, RLM respectively, and the lines [3.1], [3.2] to the points U, V. Then the space cubics D_4, D_5, D_6 project into conics through U, V circumscribing the triangles QRL, RPM, PQN. These three conics have besides U, V a third common point W,

* These lines must not be confused with the lines denoted by c_{ij} in the usual notation for the twenty-seven lines on the general cubic surface.

through which passes also the conic through U, V, L, M, N.* The straight line $[3.W]$ is a chord of each of the space cubics D_4, D_5, D_6, and



therefore meets each in a further point besides 3. To this line there corresponds in the x-space a *straight line meeting each of the lines* (1), (2), d_4, d_5, d_6. It will be seen that *this line also meets* d_3. For the elements in the X-space corresponding to the five lines b_{31}, b_{32}, b_{34}, b_{35}, b_{36} which meet d_3 are—

b_{31}: the conic through 1, 3 meeting G, A_4, A_5, A_6,

b_{32}: ,, 2, 3 ,, G, A_4, A_5, A_6,

b_{34}: the straight line $\{3.A_5A_6\}$,

b_{35}: ,, $\{3.A_4A_6\}$,

b_{36}: ,, $\{3.A_4A_5\}$.

Thus to the line d_3 in general corresponds a definite space cubic D_3 passing through 1, 2, 3 and having G, $\{3.A_5A_6\}$, $\{3.A_4A_6\}$, $\{3.A_4A_5\}$ as

* If W be the fourth point common to the conics $UVQRL$, $UVRPM$, we have

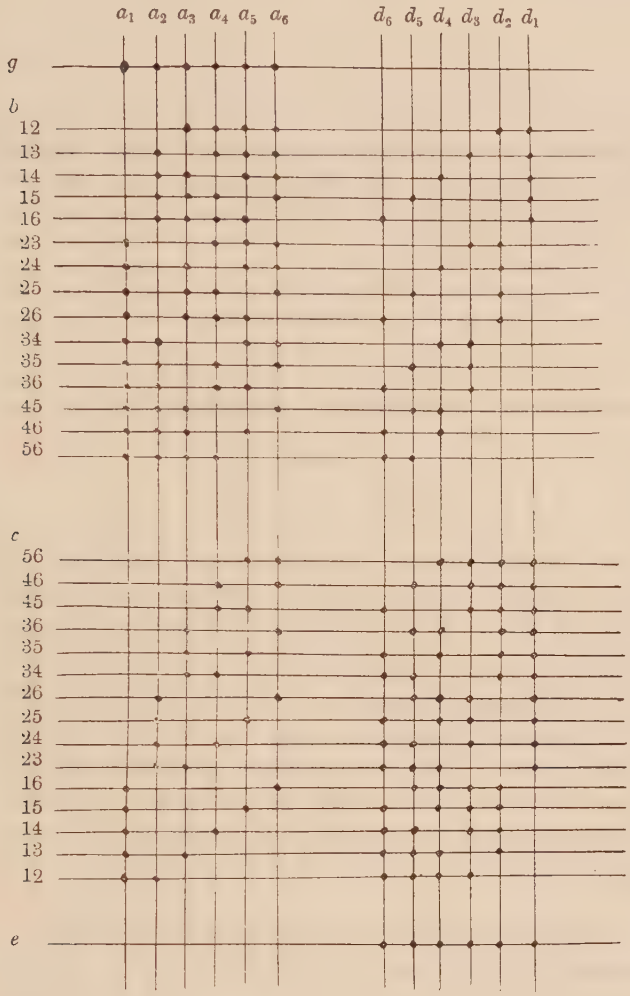
$$R(UVLW) = Q(UVLW) \quad \text{and} \quad R(UVMW) = P(UVMW).$$

But RLM, PMN, QNL are straight lines. Thus $P(UVNW) = Q(UVNW)$, and W lies also on the conic $UVPQN$. Similarly it may be shewn that the conic $UVLMN$ passes through W.

chords. This curve is therefore projected from 3 by the quadric cone K_3 determined by the generators $[3.1]$, $[3.2]$, $\{3.A_5A_6\}$, $\{3.A_4A_6\}$, $\{3.A_4A_5\}$. This cone projects the cubic into the conic $UVLMN$ which passes through W. Hence the line $[3.W]$ is a generator of K_3 and meets the space cubic D_3 in a further point besides 3. The corresponding line in the x-space consequently meets a_1, a_2, d_3, d_4, d_5, d_6; denote it by the symbol c_{12}. There will be in all 15 such lines c, one being determined by every set of four of the lines d. No two of the lines c intersect, and no line c is met by any of the 14 lines b not having the same suffix—for it is readily seen that if any two of the thirty lines having different suffixes are coplanar their plane must contain two or more of the lines a; which is contrary to the supposition that the six lines a were taken at random.

17. Consider now the system of five lines c_{12}, c_{13}, c_{14}, c_{15}, c_{16} which meet a_1. These are met in fours by d_2, d_3, d_4, d_5, d_6: there is then a straight line e meeting these last five. This line e must also meet d_1; otherwise, from the symmetry of the configuration there would be six lines of this type, different from any of the lines c, each meeting five of the lines d—in which case the lines d would all lie on a quadric surface on which the original six lines a would also lie. Thus *there exists a single straight line e meeting the six lines d.*

18. Figure 2 gives a diagrammatic representation of the configuration: the dots denote actual intersections, while the unmarked crossings correspond to apparent intersections. The symmetry of the configuration becomes at once evident. Each of the 32 lines g, b, c, e meets 6 of the 12 lines a, d: while every line of the latter group meets 16 lines of the former. It appears that the whole configuration may be obtained from any one of the 32 lines of the former group and the 6 lines of the latter that meet it—for example, starting from b_{56} and a_1, a_2, a_3, a_4, d_5, d_6 we finish with the line c_{56} meeting d_1, d_2, d_3, d_4, a_5, a_6. In fact, the 32 lines consist of 16 pairs, such that if we start from one member of a pair we finish with the other.


a_1 a_2 a_3 a_4 a_5 a_6
d_6 d_5 d_4 d_3 d_2 d_1
g
b
12
13
14
15
16
23
24
25
26
34
35
36
45
46
56
c
56
46
45
36
35
34
26
25
24
23
16
15
14
13
12
e

THE CONFORMAL REPRESENTATION OF THE VARIOUS TRIANGLES BOUNDED BY THE ARCS OF THREE INTERSECTING CIRCLES

By J. HODGKINSON.

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I. *Scope of the Paper.*

The conformal representation of a curvilinear triangle is discussed by Schwarz in his memoir "Ueber diejenigen Fälle in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt" (*Crelle*, LXXV, pp. 292–335). It is by this means that he obtains his celebrated fifteen cases of the hypergeometric equation which admit of algebraic integration. But of all the triangles bounded by the arcs of three given circles he considers effectively only that one the sum of whose angles is least, which he styles the "reduced triangle." Calling the other triangles bounded by arcs of the same circles the "associated triangles," I hope to establish the following results.

(1) The functions effecting the conformal representations of a reduced and an associated triangle are only connected by an algebraic equation when the reduced triangle is one of the fifteen cases of Schwarz.

(2) If α, β, γ are the elements of one of the fifteen algebraically integrable hypergeometric equations, the equation whose elements are $\alpha+n_1$, $\beta+n_2$, $\gamma+n_3$, where n_1, n_2, n_3 are integers, is also algebraically integrable.

(3) The function z effecting the representation upon a half-plane of a triangle in a w-plane has a finite number of branches in the following cases:—

(a) It is known to be uniform when the angles are sub-multiples of π.

(*b*) It has a finite number of branches when it is one of Schwarz's cases or an associated triangle.

(*c*) When the triangle is reduced, and the sum of its angles is less than π, the number of branches is finite in certain cases which can be discriminated by a test involving only a finite number of constructions. The fundamental triangle by means of which and its repetitions such a triangle is constructed is always right-angled.

For the purpose in hand I have first given the briefest possible survey of the argument of Schwarz. In addition to his memoir, a detailed account is conveniently found in Forsyth, *Theory of Functions*, Ch. XX.

II. *Survey of the Position.*

Three circles intersect in three pairs of points, A, A'; B, B'; C, C'. These can be associated in threes in eight ways to give curvilinear triangles, viz., ABC, $A'BC$, $AB'C$, ABC', $AB'C'$, $A'BC'$, $A'B'C$, $A'B'C'$. These eight triangles, however, may be regarded as four only, for they exhibit the same angles in pairs, viz., ABC, $A'B'C'$; $A'BC$, $AB'C'$; $AB'C$, $A'BC'$; ABC', $A'B'C$. This is at once evident on reference to a figure. (The triangle $A'B'C'$ consists of the part of the plane exterior to the circles.)

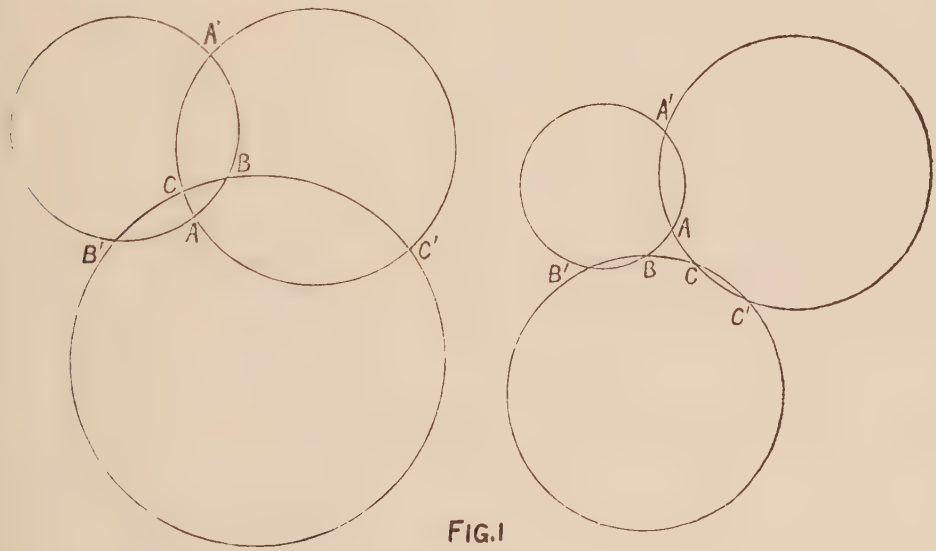

FIG. 1

In the above figure I show the cases when the three circles have not or have a real common orthogonal circle by distinct figures.

This, however, is not all, for the three points ABC determine not one triangle, but sixteen, which have the same vertices. These are shown in the following figures.

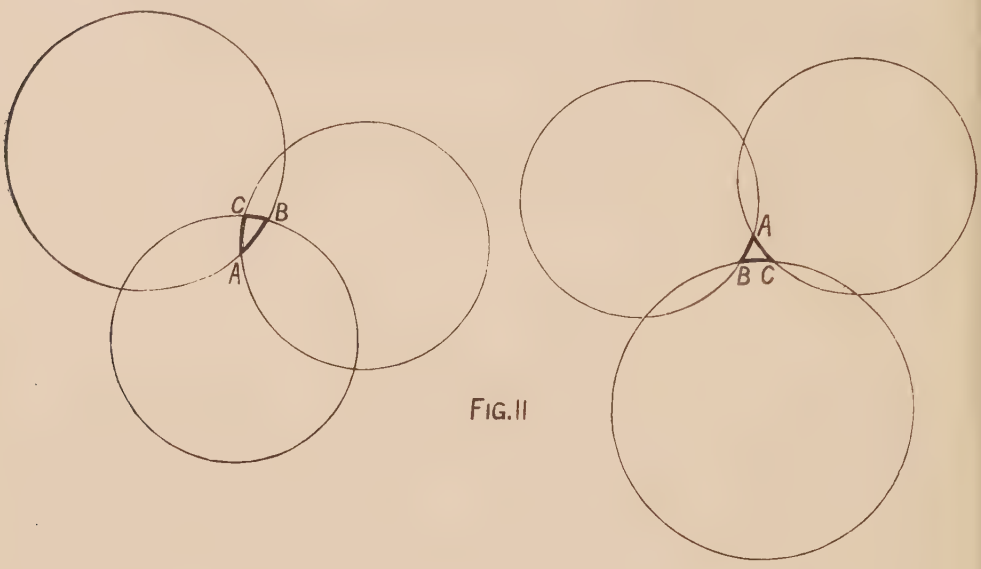

FIG. II

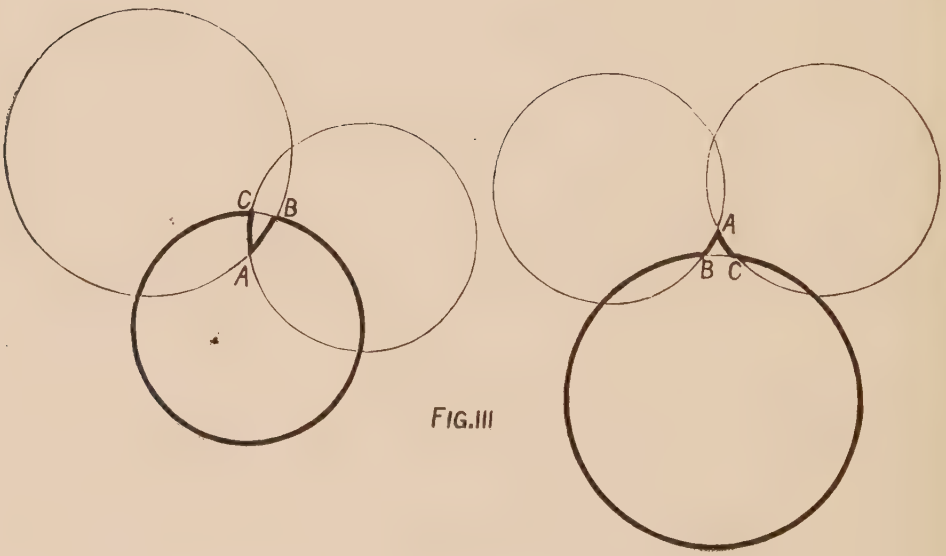

FIG. III

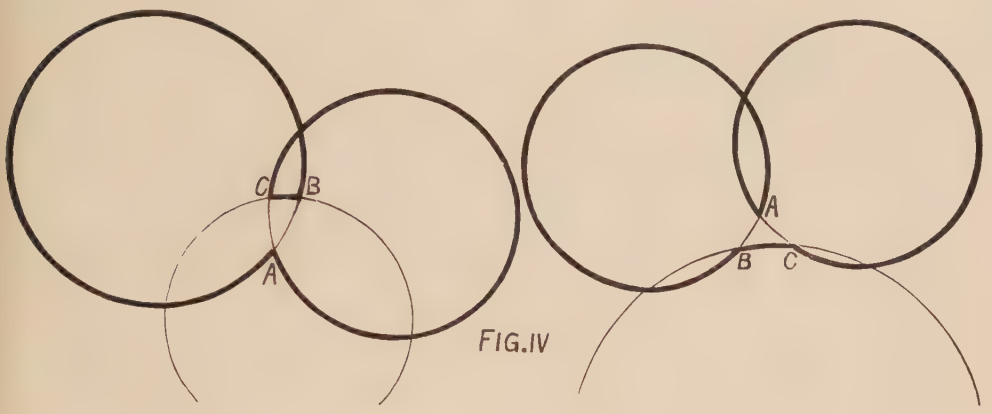

FIG. IV

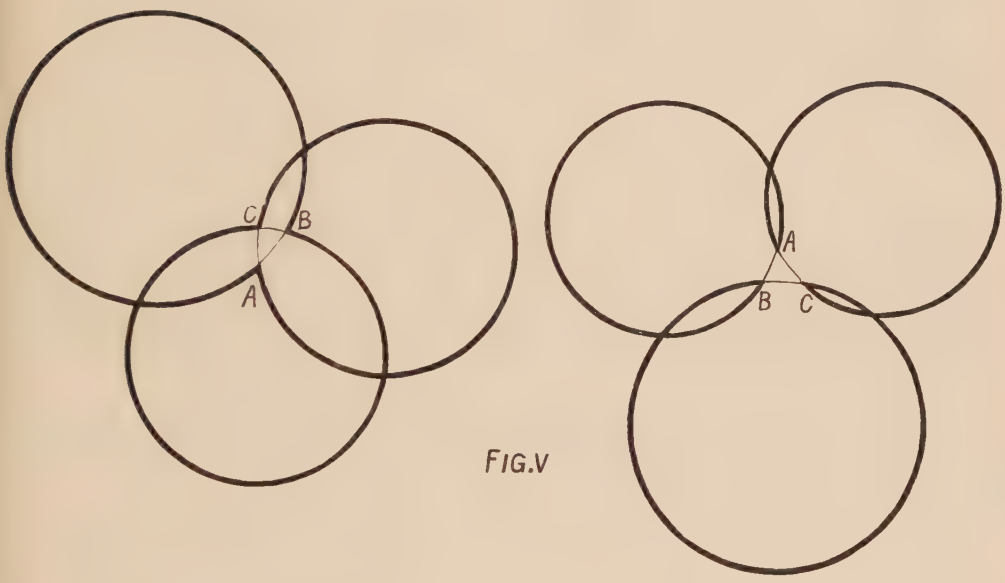

FIG. V

Of the triangles formed by three circles, Fig. II shows the reduced triangle. Let its angles be $\lambda\pi$, $\mu\pi$, $\nu\pi$. The sides are described in the figure in the counter-clockwise direction, the enclosed area being, as usual, to the left. If, however, the sides be described in the reverse order, the area enclosed consists of the remainder of the surface of the plane, and the angles of the triangle are $(2-\lambda)\pi$, $(2-\mu)\pi$, $(2-\nu)\pi$.

Fig. III shows two triangles whose angles are respectively $(2-\lambda)\pi$, $(1-\mu)\pi$, $(1-\nu)\pi$, and $\lambda\pi$, $(1+\mu)\pi$, $(1+\nu)\pi$. There are also four similarly obtained triangles.

Fig. IV shows two triangles whose angles are respectively $(2-\lambda)\pi$,

$(1-\mu)\pi$, $(1-\nu)\pi$, and $\lambda\pi$, $(1+\mu)\pi$, $(1+\nu)\pi$. There are also four similarly obtained triangles.

Fig. V shows two triangles whose angles are respectively $(2-\lambda)\pi$, $(2-\mu)\pi$, $(2-\nu)\pi$, and $\lambda\pi$, $\mu\pi$, $\nu\pi$.

It is at once observed that the triangles obtained from Figs. IV and V have the same angles as those obtained from Figs. II and III. Inasmuch, however, as the sides of these eight triangles cross one another, they are unsuitable for conformal representation on a half-plane and we drop them from our consideration.

We are thus left with our original triangle ABC, the three supplemental triangles $A'BC$, $AB'C$, ABC', each giving rise to a set of eight triangles, making in all thirty-two, which we describe as the reduced triangle and its "associated triangles."

If now the plane of the triangles be the plane of a variable w, a representation of the triangle whose angles are $\lambda\pi$, $\mu\pi$, $\nu\pi$, is obtained on a z half-plane by means of the relation

$$\tfrac{1}{2}\{w, z\} = \tfrac{1}{4}\left\{\frac{1-\lambda^2}{z^2}+\frac{1-\nu^2}{(z-1)^2}+\frac{\lambda^2-\mu^2+\nu^2-1}{z(z-1)}\right\}.$$

The different branches of the function w give the conformal representation of the triangles, called by Schwarz the "symmetrical repetitions" of the original triangle, obtained by a series of inversions of that triangle with respect to the three circles which enclose it. It is plain that the same inversions will be operative for the associated triangles.

The solution of the equation above can be derived from that of the hypergeometric equation, and conversely, where the hypergeometric elements α, β, γ are given by

$$\lambda^2 = (1-\gamma)^2, \quad \mu^2 = (\alpha-\beta)^2, \quad \nu^2 = (\gamma-\alpha-\beta)^2.$$

We take, say, $\quad \alpha-\beta = \mu, \quad \alpha+\beta-\gamma = \nu, \quad \gamma = 1+\lambda.$

It can be seen without difficulty that, if α', β', γ' are the corresponding elements for an associated triangle, solutions of

$$\lambda'^2 = (1-\gamma')^2, \quad \mu'^2 = (\alpha'-\beta')^2, \quad \nu'^2 = (\gamma'-\alpha'-\beta')$$

can always be found that exhibit the relation that $\alpha'-\alpha$, $\beta'-\beta$, $\gamma'-\gamma$ are integers.

It is at this stage that Schwarz dismisses the associated triangles in these words (*l.c.*, p. 317),

"The consideration of the circles to which the sides of the curvilinear triangle S and its symmetrical repetitions belong, shows the same circles

which appear in the construction of the symmetrical repetitions of the triangle S, also present themselves for the associated reduced triangle S'' and its symmetrical repetitions. It is therefore permissible, if none of the three numbers λ, μ, ν be an integer, to limit our investigation, for the end in view, to the consideration of the associated reduced triangle.

"The possibility of such a reduction is connected with the theorem that between three hypergeometric series whose first three elements differ only by integers, subsists an identical relation by virtue of which it is possible as a rule to express any one of the three series in terms of the others with rational functions of x as coefficients."

In the former, however, of these two paragraphs no indication is made of the relation of the solution of the problem of the associated triangle to that of the reduced triangle, while the latter appears rather to indicate that the complete problem can be regarded as solved only when two of the equations have been integrated, viz., that of the reduced triangle, and that of one other associated triangle.

It is the nature of this relationship that we are first to investigate. We shall use the symbols Z and z for the functions effecting the representations of the reduced and associated triangles respectively.

II. *Proof that the Relation of the Functions by means of which a Reduced and an Associated Triangle are represented is Algebraic only in certain well known cases.*

We shall consider separately, as is usual, the two cases in which the three circles have or have not a real common orthogonal circle. In either case we remember that Z and z as functions of w are automorphic for the same group of homographic transformations, viz., that group of which any member is equivalent to an even number of inversions with respect to the circles bounding the original triangle.

In the former case any associated triangle contains a finite length, in some cases the whole length of the circumference of the common orthogonal triangle. Now in the neighbourhood of the common orthogonal circle, the transformations of the reduced triangle become infinitely small, and it follows that any associated triangle will contain an infinite number of transformations of the reduced triangle.

Suppose we take any value of Z determining a point in its half plane. To this point will correspond a point in the reduced triangle and one in each of its transformations. Taking only those transformations which lie in the associated triangle under consideration, we have nevertheless an

infinite number of such points within the associated triangle. To each of these points corresponds a separate point in the z plane. Thus we see that a single value of Z determines an infinite number of values of z. Thus the relation between Z, z is transcendental. It thus appears that the representation of the associated triangle is by no means solved by that of the reduced triangle.

The argument is somewhat simpler in the case when the common orthogonal circle is imaginary. In this case the triangle is conveniently represented as a spherical triangle by means of a stereographic projection, and the symmetric repetitions are all equal in area to the original triangle. Suppose now we take a value of Z. As before we determine a point in the reduced triangle and in each of its symmetrical repetitions. A certain number of these will lie in the associated triangle under consideration. If now the group of transformations be infinite, the repetitions of the reduced triangle will cover the surface of the sphere an infinite number of times. Hence an infinite number of the points taken will lie in the associated triangle. The relation between Z, z will be transcendental. If, however, the group of transformations be finite, only a finite number of those points will lie in the associated triangle. Hence a single value of Z determines a finite number of values of z, and the converse follows in the same manner. In this case the relation between Z, z is algebraic.

Now the number and composition of such finite groups is known, and are those associated with the regular solids (Klein, *Ikosahedron*, p. 115). Hence we find that only in certain cases can the function effecting the representation of the associated triangle be obtained by an algebraic process from the corresponding function for the reduced triangle. These are the cases by means of which Schwarz obtained his fifteen cases of the hypergeometric equation which are algebraically integrable.

III. *Two Examples giving the Function for the Associated Triangle.*

In the cases where the function for the associated triangle can be worked out algebraically, a process has been given by Klein (*Math. Ann.*, xii, p. 171), by means of which it can be performed without intrinsic difficulty, although the actual algebra proves laborious.

(A) In the reduced triangle

$$\lambda = \tfrac{1}{2}, \quad \mu = \tfrac{1}{2}, \quad \nu = \frac{1}{n}.$$

Consider the associated triangle whose angles are

$$(1-\lambda)\pi, \quad \mu\pi, \quad (1-\nu)\pi.$$

A solution of the equation

$$\tfrac{1}{2}\{w,z\} = \tfrac{1}{4}\left\{\frac{1-\frac{1}{4}}{z^2}+\frac{1-\left(1-\frac{1}{n}\right)^2}{(z-1)^2}+\frac{\left(1-\frac{1}{n}\right)^2-1}{z(z-1)}\right\}$$

is indicated by Forsyth, *Theory of Functions*, p. 668, namely,

$$z = \left(\frac{w^{n/(n-1)}+1}{w^{n/(n-1)}-1}\right)^2.$$

(B) Take the reduced triangle

$$\lambda = \tfrac{1}{2}, \quad \mu = \tfrac{1}{3}, \quad \nu = \tfrac{1}{3}.$$

For the associated triangle

$$1-\lambda = \tfrac{1}{2}, \quad 1-\mu = \tfrac{2}{3}, \quad \nu = \tfrac{1}{3}.$$

Consulting the figures on Forsyth, p. 659, we see that this triangle contains three repetitions of the reduced triangle. Hence the relation between z, Z is of the third degree in z. The equations for the conformal representations are

$$\tfrac{1}{2}\{w,Z\} = \tfrac{1}{4}\left\{\frac{1-\frac{1}{4}}{Z^2}+\frac{1-\frac{1}{9}}{(Z-1)^2}+\frac{\frac{1}{4}-\frac{1}{9}+\frac{1}{9}-1}{Z(Z-1)}\right\},$$

and

$$\tfrac{1}{2}\{w,z\} = \tfrac{1}{4}\left\{\frac{1-\frac{1}{4}}{z^2}+\frac{1-\frac{1}{9}}{(z-1)^2}+\frac{\frac{1}{4}-\frac{4}{9}+\frac{1}{9}-1}{z(z-1)}\right\}.$$

If Z as a function of z have zeros $a\ldots$ of multiplicities $\alpha\ldots$, poles $c\ldots$ of multiplicities $\gamma\ldots$, and $Z-1$ have zeros $b\ldots$ of multiplicities $\beta\ldots$, so that

$$Z = k\frac{\Pi(z-a)^\alpha}{\Pi(z-c)^\gamma}, \qquad Z-1 = k'\frac{\Pi(z-b)^\beta}{\Pi(z-c)^\gamma},$$

where k, k' are constants, we obtain by means of the transformation

$$\{w,z\} = \{w,Z\}\left(\frac{dZ}{dz}\right)^2+\{Z,z\},$$

$$\tfrac{1}{2}\{w,z\} = \tfrac{1}{4}\left\{\Sigma\frac{1-\frac{\alpha^2}{4}}{(z-a)^2}+\Sigma\frac{1-\frac{\beta^2}{9}}{(z-b)^2}+\Sigma\frac{1-\frac{\gamma^2}{9}}{(z-c)^2}\right\}$$

$$+\text{terms of higher order in } z-a,\ z-b,\ z-c$$

$$+\text{terms arising from the zeros of } \frac{dZ}{dz}.$$

Identifying this with $\frac{1}{4}\left\{\frac{1-\frac{1}{4}}{z^2}+\frac{1-\frac{1}{9}}{(z-1)^2}\right\}$, we get $a=1$, or 2, or $a=\infty$;

$\beta = 1$, or 3, or $b = \infty$; $\gamma = 3$ or $c = \infty$, and by the geometry of the figure $Z = 0$, $z = 0$, $Z = 1$, $z = 1$ or ∞, are corresponding values.

This suggests $$\frac{Z}{Z-1} = \frac{mz(z-a)^2}{z-1},$$

where m, a are yet to be determined. This satisfies the first pair of conditions. To satisfy the third it is necessary that the equation

$$z-1 = mz\,(z-a)^2,$$

should have three equal roots; and thus, finally,

$$Z = \frac{z(8z-9)^2}{(4z-3)^3}, \qquad Z-1 = \frac{27(1-z)}{(4z-3)^3}.$$

With considerable elementary algebraic reduction I have verified that this does effect the transformation. The equation in w, Z is known to admit the integral

$$1-Z = \left\{\frac{w^4+2w^2\sqrt{3}-1}{w^4-2w^2\sqrt{3}-1}\right\}^3.$$

Hence the equation in w, z admits the integral

$$\frac{27(z-1)}{(4z-3)^3} = \left\{\frac{w^4+2w^2\sqrt{3}-1}{w^4-2w^2\sqrt{3}-1}\right\}^3.$$

IV. *Additional Cases of the Hypergeometric Equation which are Algebraically Integrable.*

From the fifteen cases of the reduced triangle which are conformally represented by means of an algebraic function, Schwarz obtained his fifteen cases of the hypergeometric equation which are algebraically integrable. But in exactly the same manner there is a hypergeometric equation connected with each associated triangle, and since any such associated triangle is conformally represented by means of an algebraic function, it follows that all these hypergeometric equations are also algebraically integrable.

This, however, is not all. Schwarz in the paragraph quoted refers to an identical relation between hypergeometric series whose elements differ by integers. We shall look into this more closely.

We have the following identities

$$(\alpha+1)(1-z)\,F(\alpha+2, \beta, \gamma, z) + \{z(\alpha-\beta+1)-(2\alpha-\gamma+2)\}F(\alpha+1, \beta, \gamma, z) + (\alpha-\gamma+1)\,F(\alpha, \beta, \gamma, z) = 0, \quad \text{(A)}$$

$$(\gamma-\alpha+1)(\gamma-\beta+1)zF(\alpha,\beta,\gamma+2,z)$$
$$+(\gamma+1)\{\gamma+(\alpha+\beta-2\gamma-1)z\}F(\alpha,\beta,\gamma+1,z)$$
$$-\gamma(\gamma+1)(1-z)F(\alpha,\beta,\gamma,z)=0, \quad (B)$$

$$\alpha F(\alpha+1,\beta,\gamma,z)-\beta F(\alpha,\beta+1,\gamma,z)-(\alpha-\beta)F(\alpha,\beta,\gamma,z)=0, \quad (C)$$

$$\alpha\gamma(1-z)F(\alpha+1,\beta,\gamma,z)-(\alpha-\gamma)(\beta-\gamma)zF(\alpha,\beta,\gamma+1,z)$$
$$-\gamma\{(\beta-\gamma)z+\alpha\}F(\alpha,\beta,\gamma,z)=0, \quad (D)$$

$$\beta(1-z)F(\alpha+1,\beta+1,\gamma,z)-(\alpha+\beta-\gamma+1)F(\alpha+1,\beta,\gamma,z)$$
$$+(\alpha-\gamma+1)F(\alpha,\beta,\gamma,z)=0, \quad (E)$$

$$(\beta-\gamma)zF(\alpha+1,\beta,\gamma+1,z)-\gamma(1-z)F(\alpha+1,\beta,\gamma,z)$$
$$+\gamma F(\alpha,\beta,\gamma,z)=0. \quad (F)$$

By means of the repetition of (A), we get an identity (G) connecting

$$F(\alpha,\beta,\gamma,z), \quad F(\alpha+1,\beta,\gamma,z), \quad \text{and} \quad F(\alpha+m_1,\beta,\gamma,z),$$

where m_1 is any integer. In the same way we get an identity (H) connecting

$$F(\alpha+m_1,\beta,\gamma,z), \quad F(\alpha+m_1,\beta+1,\gamma,z), \quad \text{and} \quad F(\alpha+m_1,\beta+m_2,\gamma,z),$$

and another (K) connecting

$$F(\alpha,\beta+1,\gamma,z), \quad F(\alpha+1,\beta+1,\gamma,z), \quad F(\alpha+m_1,\beta+1,\gamma,z).$$

Now, from the identities (G), (H), (K), (C), (E), we eliminate

$$F(\alpha+m_1,\beta,\gamma,z),\ F(\alpha+m_1,\beta+1,\gamma,z),\ F(\alpha,\beta+1,\gamma,z),\ F(\alpha+1,\beta+1,\gamma,z),$$

and we have remaining an identity connecting

$$F(\alpha,\beta,\gamma,z), \quad F(\alpha+1,\beta,\gamma,z), \quad \text{and} \quad F(\alpha+m_1,\beta+m_2,\gamma,z).$$

In a similar way we proceed until we obtain an identity connecting

$$F(\alpha,\beta,\gamma,z), \quad F(\alpha+1,\beta,\gamma,z), \quad \text{and} \quad F(\alpha+m_1,\beta+m_2,\gamma+m_3,z).$$

Yet, again, we have a further identity connecting

$$F(\alpha,\beta,\gamma,z), \quad F(\alpha+1,\beta,\gamma,z), \quad \text{and} \quad F(\alpha+n_1,\beta+n_2,\gamma+n_3,z),$$

and, finally, by the elimination of $F(\alpha+1,\beta,\gamma,z)$, we obtain a relation

$$PF(\alpha,\beta,\gamma,z)+QF(\alpha+m_1,\beta+m_2,\gamma+m_3,z)$$
$$+RF(\alpha+n_1,\beta+n_1,\gamma+n_1,z)=0, \quad (L)$$

where P, Q, R are rational integral functions of z.

If now α, β, γ are the elements for one of the fifteen cases of Schwarz, we take $\alpha+m_1$, $\beta+m_2$, $\gamma+m_3$ as the elements of the equation connected with any of the triangles associated with the same reduced triangle, and the identity (L) shows that $F(\alpha+n_1, \beta+n_2, \gamma+n_3, z)$ is also an algebraic function of z.

The second solution of the hypergeometric equation can be treated in like manner, and we finally see that if α, β, γ be the elements of one of Schwarz's fifteen cases, the hypergeometric equation whose elements are $\alpha+n_1$, $\beta+n_2$, $\gamma+n_3$, where n_1, n_2, n_3 are any integers, is also algebraically integrable.

V. *Investigation of the Cases in which the Function effecting the Representation of a Triangle has a Finite Number of Branches.*

We know that when, for the reduced triangle, λ, μ, ν are the reciprocals of integers, z is a uniform function of w. We have also obtained some cases in which z has a finite number of branches. In one of these, however, which I have worked out [above III (B)], the degree of the equation in z does not agree with the estimate given by Forsyth, *Theory of Functions*, pp. 654, 670, where we find—

"When $\lambda+\mu+\nu<1$, and λ, μ, ν, have the forms of fractions in their lowest terms, and if N be the least common multiple of their numerators, the integral relation between w and z equivalent to the differential relation

$$\{w, z\} = 2I(z)$$

is an algebraical equation of degree N in z, the coefficients of which are transcendental functions of z," and

"In all other cases in which λ, μ, ν are commensurable, this integral relation is rational in z and transcendental in w."

In both these cases the argument ostensibly refers to the reduced triangle only, but the restriction is immaterial to the method employed. This method, however, only establishes the necessity that λ, μ, ν be commensurable. I hope to show its insufficiency.

We deal first with those triangles which can be represented by spherical triangles. Suppose we take a definite value of w. This determines a point on the sphere. It will lie in one or more transformations of the original triangle. For each transformation we take the particular branch of w which gives the representation on the z half-plane, and corresponding to this chosen point we obtain from each a point in the z plane. The number of values of z thus corresponding to one value of w will be

finite or infinite according as the group of transformations is finite or infinite.

Thus for all cases except those already found, z is a function of w of infinite multiplicity of value, or, in other words, has an infinite number of branches.

Next we take those triangles which have a real common orthogonal circle.

We deal first with the associated triangles. We divide them into two classes, viz., (a) those that extend to infinity, (b) those of finite area.

For the former it is sufficiently obvious that, the group of transformations being infinite, the surface of the w plane is covered an infinite number of times by the repetitions of the triangle. Hence, if we take a definite value of w, the point so determined lies in an infinite number of transformations of the original triangle, and as in the earlier case we find that z must have an infinite number of branches.

For the class (b) we remark that in the neighbourhood of the orthogonal circle the transformations, not only of the reduced triangle, but also of such an associated triangle, become infinitely small. Hence an associated triangle of this class will contain an infinite number of transformations of itself, and thus a point can be taken to lie in an infinite number of transformations of the original triangle. The result then follows exactly as before.

There remains simply the reduced triangle when the three circles have a real common orthogonal circle, or, what amounts to the same thing, when

$$\lambda+\mu+\nu<1.$$

We know that when λ, μ, ν are the reciprocals of integers, Z is a uniform function of w. Suppose the triangle under consideration is built up of repetitions of such a triangle. In such a case when we fix a value of Z, we get a value of w corresponding to a point in each such repetition, and therefore a finite number of corresponding points in the z half plane upon which the triangle is represented. Hence the relation between Z, z is algebraic, and since Z is a uniform function of w, z as a function of w has finite multiformity.

That such cases occur is readily seen from a simple example. Take a triangle ABC whose angles are $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{7}$. The triangle formed by ABC and its inversion in AC is one of the type indicated. We shall next show that triangles of this type are the only ones which have the required property. (This is also true when $\lambda+\mu+\nu>1$. It can be at once verified that all Schwarz's remaining cases, and all the associated tri-

angles, can be built up of repetitions of the triangles I, II, IV, VI. For the numbering see Forsyth, p. 658.)

Let us for simplicity transform our plane of w by a homographic transformation, so that the orthogonal circle becomes the axis of real quantities; then the centres of all the sides of triangles which appear lie on this axis.

It is known that, if two circles intersect at an angle θ in the points A, A', inversion first with respect to one circle and then with respect to the other, is equivalent to an elliptic transformation whose common points are AA', and whose multiplier is $e^{2i\theta}$.

The transformation whose common points are C, C' and multiplier e^{2iC} is equivalent to inversion in AC followed by inversion in BC, *i.e.*, inversion in AC, inversion in AB, inversion in AB, and inversion in BC; *i.e.*, the transformation whose common points are AA' and multiplier e^{2iA} followed by the transformation whose common points are BB' and

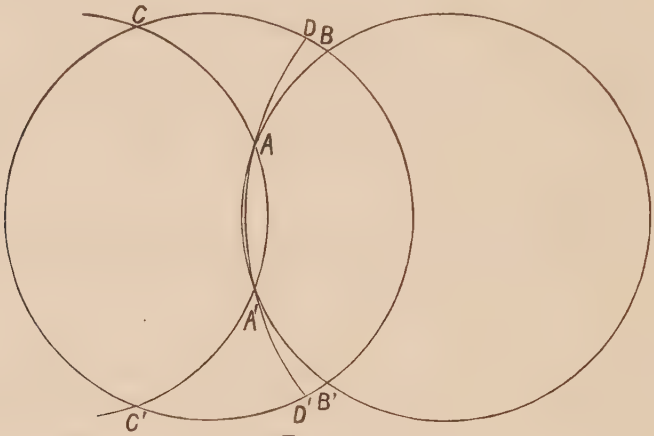

FIG. VI

multiplier e^{2iB}. Now all the transformations of the original triangle are given by the series of inversions in the three sides of that triangle. The two transformations whose common points are AA', and BB', and multipliers e^{2iA}, e^{2iB} respectively, will therefore determine the whole group.

Suppose $A = l'\pi/l$, where l'/l is in its lowest terms. The transformation whose multiplier is $e^{2i\pi l'/l}$ can be replaced by one whose multiplier is $e^{2i\pi/l}$ with the same common points, *i.e.*, the circle AC may be replaced by the circle AD, which makes an angle π/l with AB. Since its centre lies in the real axis, the circle is unique. Since also it intersects both the circles AB, AC in the points A, A', it must cross out of the triangle again by the side BC. Let its intersection with BC be D. Then, as be-

fore, we may replace our transformations with common points BB' by the two transformations with common points AA', DD'. These two transformations can also be regarded as fundamental for the group. It follows that if the group be discrete, the ratio of the angle D to π must be commensurable. Also AD being a transformation of AC, it follows that D, being the intersection of two circles of the system which give the complete set of triangles, is a transformation of one of the angular points ABC.

If the angles of D are not right angles, at least one of them may be divided in the same way as A by a circle meeting the opposite side of the triangle containing it, and we may proceed in this manner to subdivide the original triangle indefinitely unless it happens that after a finite number of repetitions of the process we obtain a circle cutting the opposite side at right angles.

The former of these two hypotheses asserts that it is possible to obtain an infinite concentration of transformations of the angular points of the original triangle within its own area. Hence the transformations of this triangle must cover the w plane an infinite number of times, and, following the same line of reasoning that we have already used so often, z as a function of w must have an infinite number of branches. If, then, we require z as a function of w to have a finite number of branches, we must impose upon the triangle ABC the necessity of its being capable of composition of transformations of a triangle whose angles are $\frac{\pi}{2}$, $\frac{\pi}{l}$, $\frac{\pi}{m}$, where l, m, n are integers.

A similar argument can be applied in the case where $\lambda+\mu+\nu>1$. Here, however, the convenient transformation of the w plane is by stereographic projection upon the surface of a sphere, and the necessary change is "great circles on the sphere" for "circles with their centres on the axis of real quantities." We have, however, obtained our results otherwise.

VI. *Test for the Resolubility of a Reduced Triangle.*

We have already seen that in the case where $\lambda+\mu+\nu>1$, all reduced triangles except the fifteen cases of Schwarz give a function z of w which is infinitely multiform. I am not able to explore the corresponding case where $\lambda+\mu+\nu<1$ so thoroughly, but I have obtained a test, which even if in any actual case it proves extremely lengthy, will decide whether or not z has a finite number of branches.

We first require a simple theorem, viz., that the sum of the angles of

any composite triangle is less than the sum of the angles of the fundamental triangle. [The corresponding property, when $\lambda+\mu+\nu > 1$, is that the sum of the angles of a composite triangle is greater than the sum of the angles of a reduced triangle. Remembering that the area of a spherical triangle $ABC = A+B+C-\pi$, this is sufficiently obvious.]

Suppose the w plane to be transformed homographically, so that AB, AC become straight lines. These will now meet in the centre of the common orthogonal circle, so that any other circle through A which occurs in the complete system also becomes a straight line.

Since the triangle ABC lies wholly within the orthogonal circle, BC is convex towards A. In the process of constructing a fundamental triangle we draw AD, as before, meeting BC in D.

Now CB being convex to A, the tangent at D must meet CA (not CA produced) in a point E, and the tangent CF must meet DE (not DE produced) in a point F. We then have

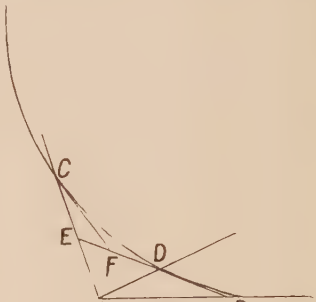

FIG. VII

$$\text{angle } ADB = \text{angle } DAE + \text{angle } DEA,$$

i.e.,
$$\text{angle } ADB > \text{angle } DAE + \text{angle } ECF,$$

whence the sum of the angles of the triangle $ADB >$ the sum of the angles of the triangle ACB. In the same way it can be shown that each subdivision of the angles in the process of forming a fundamental triangle increases the sum of the angles of a component triangle. Hence the fundamental triangle has the sum of its angles greater than that of any composite triangle.

Take now
$$\lambda = \tfrac{2}{5}, \quad \mu = \tfrac{1}{4}, \quad \nu = \tfrac{1}{5},$$
$$\lambda+\mu+\nu = \tfrac{17}{20}.$$

Such a triangle might conceivably be built up from repetitions of one whose angles are $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{5}$; or $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{10}$; or $\frac{\pi}{2}, \frac{\pi}{5}, \frac{\pi}{8}$; and so on.

Now
$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{5} = \tfrac{19}{20},$$
$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{10} = \tfrac{17}{20},$$
$$\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{8} = \tfrac{33}{40}.$$

Hence it appears that the only fundamental triangle possible is that with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{5}$. In connection with the paper to which I have so often referred, Schwarz gives the figure showing the division of the orthogonal circle in this case, and on inspection we see that we are unable to find a composite triangle of angles $\frac{2\pi}{5}, \frac{\pi}{4}, \frac{\pi}{5}$.

This property of the sum of the angles of a composite triangle limits the number of figures which it is necessary to construct to determine whether any particular reduced triangle is capable of resolution in the manner desired.

When the resolution, if possible, has been effected, we can express Z as a function of z by the method of Klein already adopted above [III, (B)], Z, z being the functions effecting the representations of the fundamental and composite triangles respectively.

THE TRANSITION FROM VAPOUR TO LIQUID WHEN THE RANGE OF THE MOLECULAR ATTRACTION IS SENSIBLE

By Sir JOSEPH LARMOR.

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THE physical properties of fluid media, as regards change of state, and also as regards capillary phenomena, have been closely illustrated in theory by consideration of a model medium, subject to internal expansive pressure, of kinetic or other origin, which is counteracted by the contractive effect of mutual attraction between its molecules—the latter force extending through much the greater, though for ordinary purposes still insensible, range. For liquids, the difference between these two much larger quantities, of different types, constitutes the transmitted hydrostatic pressure.

Suppose the potential (reckoned as energy) of forces arising from a plane layer of molecules of density ρ and thickness δx is, at distance x from it, equal to $f(x)\,\rho\,\delta x$, so that the attraction is $f'(x)\,\rho\,\delta x$. Let the homogeneous medium of uniform density ρ_0 be disturbed so that x becomes $x+\xi$, and therefore ρ_0 becomes $\rho_0\left[1-\frac{d\xi}{dx}+\left(\frac{d\xi}{dx}\right)^2+\ldots\right]$. The potential of the whole mass at the origin of measurement of x is then

$$V=\int_{-\infty}^{\infty} f(x+\xi-\xi_0)\,dx\,\rho_0\left[1-\frac{d\xi}{dx}+\left(\frac{d\xi}{dx}\right)^2+\ldots\right],$$

wherein, the suffixes denoting values at the origin of x,

$$f(x+\xi-\xi_0)=f(x)+(\xi-\xi_0)f'(x)+\tfrac{1}{2}(\xi-\xi_0)^2 f''(x)+\ldots,$$

while

$$\xi-\xi_0=x\left(\frac{d\xi}{dx}\right)_0+\tfrac{1}{2}x^2\left(\frac{d^2\xi}{dx^2}\right)_0+\ldots,$$

and

$$\frac{d\xi}{dx}=\left(\frac{d\xi}{dx}\right)_0+x\left(\frac{d^2\xi}{dx^2}\right)_0+\ldots.$$

The displacement ξ from the condition of uniform density, and its gradients,

can be regarded as very small, so that, for the purpose of forming equations of disturbance from the potential energy, terms higher than squares or products of such quantities are negligible. On this understanding,

$$V = \int_{-\infty}^{\infty} f(x)\,dx\,\rho_0\left[1-\frac{d\xi}{dx}+\left(\frac{d\xi}{dx}\right)^2-\frac{d^2\xi}{dx^2}x-\tfrac{1}{2}\frac{d^3\xi}{dx^3}x^2-\ldots\right]$$

$$+\int_{-\infty}^{\infty}\left\{f'(x)\rho_0 dx\left[\frac{d\xi}{dx}x-\left(\frac{d\xi}{dx}\right)^2 x+\tfrac{1}{2}\frac{d^2\xi}{dx^2}x^2+\ldots\right]+f''(x)\rho_0 dx\left(\frac{d\xi}{dx}\right)^2 x^2\right\},$$

wherein the gradients are now to be understood as having the constant values belonging to the origin of x.

The second part of this expression can be integrated by parts. The terms at the limits vanish, so far as we shall require to go, that being, in fact, a condition necessary to our system being defined at each point by a local constitution. This part thus becomes

$$-\int_{-\infty}^{\infty} f(x)\,\rho_0 dx\left[\frac{d\xi}{dx}-2\left(\frac{d\xi}{dx}\right)^2+\frac{d^2\xi}{dx^2}x+\tfrac{1}{2}\frac{d^3\xi}{dx^3}x^2+\ldots\right].$$

Hence $V=\int_{-\infty}^{\infty} f(x)\,\rho_0 dx\left[1-2\frac{d\xi}{dx}+3\left(\frac{d\xi}{dx}\right)^2-2\frac{d^2\xi}{dx^2}x-\frac{d^3\xi}{dx^3}x^2-\ldots\right].$

If then $A_0=\int_{-\infty}^{\infty} f(x)\,dx,\ \ldots,\ A_r=\int_{-\infty}^{\infty} x^r f(x)\,dx,\ \ldots,$

so that A_r is null by symmetry when r is odd, we have for any point x of the medium

$$V=A_0\rho_0\left[1-2\frac{d\xi}{dx}+3\left(\frac{d\xi}{dx}\right)^2\right]-A_2\rho_0\frac{d^3\xi}{dx^3}\left(1-\frac{d\xi}{dx}\right)-\ldots.$$

This expresses the potential of the attractions in terms of the strain $d\xi/dx$ and its gradients, with coefficients that are physical constants of the medium.

To obtain, from this value of the potential, the force tending to increase the displacement ξ (which is not the gradient of V), we must first form in the Lagrangian manner the potential energy of the system

$$W=\tfrac{1}{2}\int V\rho\,dx, \quad \text{where} \quad \rho=\rho_0\left[1-\frac{d\xi}{dx}+\left(\frac{d\xi}{dx}\right)^2+\ldots\right].$$

Thus $W = \frac{1}{2}A_0\rho_0^2\int\left\{1-3\frac{d\xi}{dx}+6\left(\frac{d\xi}{dx}\right)^2\right\}dx$

$$-\frac{1}{2}A_2\rho_0^2\int\frac{d^3\xi}{dx^3}\left(1-2\frac{d\xi}{dx}+\ldots\right)dx+\ldots$$

$$= 3A_0\rho_0^2\int\left(\frac{d\xi}{dx}\right)^2dx-A_2\rho_0^2\int\left(\frac{d^2\xi}{dx^2}\right)^2dx+\ldots,$$

on omitting all perfect differentials, which affect only the dynamical equilibria at boundaries. Thus

$$\delta W = 6A_0\rho_0^2\int\frac{d\xi}{dx}\frac{d\delta\xi}{dx}dx-2A_2\rho_0^2\int\frac{d^2\xi}{dx^2}\frac{d^2\delta\xi}{dx^2}dx+\ldots$$

$$= [\ldots]-6A_0\rho_0^2\int\frac{d^2\xi}{dx^2}\delta\xi\,dx-2A_2\rho_0^2\int\frac{d^4\xi}{dx^4}\delta\xi\,dx+\ldots.$$

The force acting on the element at dx to alter its displacement ξ is therefore $X\,dx$, where

$$X = 6A_0\rho_0^2\frac{d^2\xi}{dx^2}+2A_2\rho_0^2\frac{d^4\xi}{dx^4}+\ldots.$$

If ϖ represent internal pressure arising from sources other than these attractions, and we assume that it is determined by a relation $\varpi = F(\rho)$, the equation of slight disturbance of the elements of the medium is

$$\rho\frac{d^2\xi}{dt^2} = -\frac{d\varpi}{dx}+X.$$

Now, here $$-\frac{1}{\rho}\frac{d\varpi}{dx} = -\frac{1}{\rho}F'(\rho)\frac{d\rho}{dx},$$

where $$\rho = \rho_0\left(1-\frac{d\xi}{dx}+\ldots\right).$$

The equation of motion therefore becomes

$$\frac{d^2\xi}{dt^2} = F'(\rho_0)\frac{d^2\xi}{dx^2}+6A_0\rho_0\frac{d^2\xi}{dx^2}+2A_2\rho_0\frac{d^4\xi}{dx^4}+\ldots,$$

or, now omitting subscripts, it is

$$\frac{d^2\xi}{dt^2} = \frac{d}{d\rho}\left\{F(\rho)+3A_0\rho^2\right\}\frac{d^2\xi}{dx^2}+2A_2\rho_0\frac{d^4\xi}{dx^4}+\ldots.$$

For the type of simple undulatory disturbance represented by

$$\xi \propto e^{-i\kappa t - imx},$$

we have therefore $$\kappa^2 = \frac{dp_0}{d\rho} m^2 - 2A_2\rho m^4 + \ldots,$$

where $$p_0 = \varpi + 3A_0\rho^2.$$

Thus very short wave-lengths (m great) will exhibit instability unless $-A_2$, and other such quantities, are positive constants. We must stop in this series at or before $A_{2r}\left(=\int x^{2r} f(x)\,dx\right)$, the last of them that is a convergent integral, and, as such, depends only on the state of affairs around the origin to which it belongs. Thus if A_4 is to be included, $f(x)$ would have to fall away with distance after the manner of x^{-5} or a higher inverse power. The higher divergent constants of this series are to be regarded as excluded from operation by the ultimate molecular character of the medium, which they do not sufficiently represent.

Long waves are propagated with velocity $(dp_0/d\rho)^{\frac{1}{2}}$, where

$$p_0 = \varpi + 3A_0\rho^2;$$

shorter ones with greater velocity $(dp_0/d\rho - 2A_2\rho m^2)^{\frac{1}{2}}$. As $dp_0/d\rho$ diminishes, stability will decrease: and the medium of uniform density ρ will become unstable at the value of ρ for which $dp_0/d\rho$ is about to change sign by vanishing.

If the internal forces of the medium are effectively cohesive and not repulsive, as they must be in an ordinary liquid, $A_0\left(=\int f(x)\,dx\right)$ must be negative. As the potential energy must be taken to be zero at infinite distance, this requires that the graph of $f(x)$ should be in the main below the axis of x.

If the range of the attraction is considerable, disturbances of very short wave length can be contemplated in this continuous analysis; and these, as we have seen, would be unstable unless $A_2\left(=\int x^2 f(x)\,dx\right)$ is negative. Thus if the graph of $f(x)$ is undulating, so that the force (its gradient) may change from attractive to repulsive with change of distance, the regions of repulsion must be in the main for small values of x, and they must be limited by this condition. We might introduce such repulsion of small range into the argument instead of the kinetic momentum-pressure: and we could thus contemplate, after the manner of Young, the relations

of surface tension, and of change of state as well, as established for a medium, liable to variation of density, with its properties determined solely by a single law of attraction of this type, viz., an attraction on the whole, counteracted by repulsion at shorter range.

We have now to determine what is the value of the pressure transmitted into the medium, in compression experiments such as the classical investigations of Andrews on change of state. In the first place, let us imagine a plane of separation in the medium, at the origin of x, and calculate the total force per unit area exerted by the medium situated to the left of this plane on the medium situated to the right. If distance x is measured to the right of the plane and distance X to the left, this force is equal to ϖ diminished by mutual attractions which amount to

$$P = \int_0^\infty \rho_x dx \int_0^\infty \rho_X dX f'(X+x),$$

where, when the disturbance of ρ is very gradual,

$$\rho_x = \rho + \frac{d\rho}{dx} x + \tfrac{1}{2} \frac{d^2\rho}{dx^2} x^2 + \ldots,$$

$$\rho_X = \rho - \frac{d\rho}{dx} X + \tfrac{1}{2} \frac{d^2\rho}{dx^2} X^2 - \ldots,$$

the values of ρ and its gradients after substitution being here the constant values belonging to the origin. This gives

$$\begin{aligned} P &= \int_0^\infty \rho_x dx \left\{ \rho \int_0^\infty f'(X+x)\, dX - \frac{d\rho}{dx} \int_0^\infty X f'(X+x)\, dX \right\} + \ldots \\ &= \int_0^\infty \left(\rho + \frac{d\rho}{dx} x + \ldots \right) dx \left\{ -\rho f(x) - \frac{d\rho}{dx} f_1(x) \right\} + \ldots \\ &= -\rho^2 \int f(x)\, dx - \rho \frac{d\rho}{dx} \int \{ x f(x) + f_1(x) \}\, dx - \left(\frac{d\rho}{dx} \right)^2 \int x f_1(x)\, dx + \ldots \\ &= -\rho^2 . \tfrac{1}{2} A_0 - \rho \frac{d\rho}{dx} \{ \tfrac{1}{2} A_1 - f_{11}(0) \} + \left(\frac{d\rho}{dx} \right)^2 f_{111}(0) + \ldots, \end{aligned}$$

where $A_0 = -f_1(0)$, and A_1 is null.

If the interface is situated at a place where the medium is uniform, the pressural force $\varpi - P$ transmitted across it is thus of intensity

$$\varpi + \tfrac{1}{2} A_0 \rho^2,$$

where, as we have seen, A_0 must be negative for attractive forces.

Now consider the equilibrium of a finite composite slice bounded on one side by this interface and on the other side by an interface situated in the solid piston which applies the pressure. A transition layer, of very rapid variation of density, between the fluid and the solid, will be included in the slice, but the forces in it are compensating mutual forces which will not affect the equilibrium of the whole. Thus p, equal to $\varpi+\frac{1}{2}A_0\rho^2$, is equal to the pressure transmitted into the fluid as measured on a hydrostatic piston.*

If $dp/d\rho$ were negative, compression would diminish the statical applied pressure; that is, the fluid of uniform density ρ would be an unstable system for applied disturbances even when its density is constrained to remain uniform.

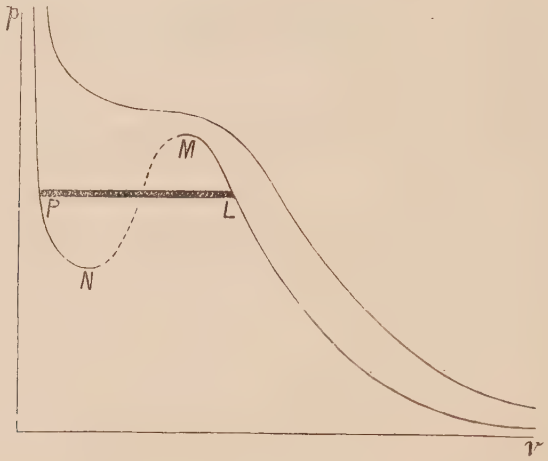
But if our previous calculation is correct, instability for internal disturbance—*i.e.*, owing to irregularity in density—sets in earlier than this, viz., when $dp_0/d\rho$ vanishes, where $p_0 = p+\frac{5}{2}A_0\rho^2$. For $\frac{1}{2}A_0$ is negative, as above noticed: it is in fact $-a$ if the equation of van der Waals is adopted. Generally, $p = \varpi - P$, where ϖ represents the repulsive effect of molecular motion, and P the attractive effect $-\frac{1}{2}A_0\rho^2$ of molecular forces. Thus the limit of internal stability is when $dp_0/d\rho$ vanishes, not $dp/d\rho$; where $p_0 = p-5P$, in which p is the hydrostatic pressure, and P the internal cohesion of the mutual molecular attractions.† It is naturally to be expected that these limits will be narrower than would obtain when the density is kept uniform.

Below the critical temperature, on the Andrews-Thomson (p, v) diagram of isotherms representing the vapour-liquid state, this distinction is not practically important, because at such temperatures gaseous and liquid phases definitely separate out, at a level on the diagram which is determined by the Maxwell-Clausius rule of equal areas cut off above and below by the horizontal line representing vapour and liquid in contact. But above the critical temperature there will be a region in which the single phase will become internally unstable, while yet no such vapour-liquid line can be drawn. For, as $\rho = v^{-1}$, and so $p_0 = p+\frac{5}{2}A_0v^{-2}$, where A_0 is negative, the point (dp_0/dv null) at which internal instability sets in is determined by $dp/dv = 5A_0v^{-3}$; viz., it begins when the downward slope with increasing v is less than the positive quantity $-5A_0v^{-3}$, and thus it obviously obtains even in a region above the critical point where

* In an important modern paper by Rayleigh on these topics, especially those of surface tension (*Phil. Mag.*, Vol. XXXIII, 1892; *Collected Papers*, Vol. III, p. 514), this result is obtained through the equation $p = \int \rho\, dV = |\, \rho V\, |_0 - \int V d\rho$.

† This quantity p_0, which plays the part of pressure as regards internal kinetic disturbances, will usually over large ranges be negative.

the (p, v) curve continually trends downwards so that no liquid phase can separate off.



If we sketched in along with the van der Waals graph of $p = \frac{RT}{v-b} - \frac{a}{v^2}$, the hyperbola $\varpi = \frac{RT}{v-b}$, then $p_0 = 6p - 5\varpi$, and the place at which $\frac{dp_0}{dv}$ vanishes would be seen on inspection to fall in general well below the horizontal vapour-liquid line. In case this did not obtain, there would be a region of instability before liquefaction, of the same kind as is here suggested on isothermals above the critical point.

Whether our scheme sufficiently represents actual fluid matter or not, we can certainly form an image of a medium possessing its properties. For such a medium there can be no separation of phases unless the straight line representing the constant pressure of liquid in presence of vapour can be drawn to cut off equal areas from the characteristic curve above and below it. This criterion assumes, on the Maxwell-Clausius principle,* that the work done in a cycle conducted at constant temperature, $LMNPL$ in the diagram, is null, even when part of the cycle is conducted through unstable, though possible states, those represented by the arc of dots in the diagram. The unstable arc can, in fact, be imagined, for the purposes of the argument, as maintained against very slight disturbance by very slight constraints; but these must be such as the internal motion constituting the constant temperature does not vitiate.

What then can happen to such an unstable medium if liquefaction is precluded? It has appeared that any slight change of density in the in-

* Cf. Maxwell's *Theory of Heat*.

terior of the medium will be augmented in unstable manner when the downward slope of the characteristic (p, v) curve becomes less than a definite limit, and that thus there can be internal instability at temperatures extending to a considerable range above the critical temperature, which is that for which dp/dv vanishes at a point on the curve and is the highest at which liquid can appear.*

If the precision and definiteness attained in the measures of critical temperatures are held to indicate that such a state of matters does not occur sensibly in actual fluids,† this result might possibly arise from the range of molecular attraction extending over so few molecules that they cannot be treated by the present continuous methods. Or it may be due to the intrinsic pressure, arising from the momentum of the molecules, having itself a range (which is of the order of the free path in gases) comparable with that of molecular attraction ; though it might seem that the greater freedom of a long free path would rather increase instability. In either case, the application of the Young-Laplace principles in illustration of capillary phenomena would in actuality be considerably limited. But we can face an alternative that is still open. We may assert that on a (p, v) isothermal not too far above the critical point there is an arc, in the neighbourhood of the critical volume, along which the state of physical aggregation must change as the volume increases, recovering the previous or more probably some other molecular constitution again after this arc, which would otherwise be unstable, has been passed. We would thus hold that there is a region of instability in the (p, v) field in crossing which the substance gradually changes its molecular state of adhesion, but without any tendency to divide into two phases, in which therefore our present (p, v) formula can no longer obtain. There seems to be nothing in the experimental facts to disprove such a view. Indeed the actual phenomena would not be very different : without an S bend in the (p, v) curve the phenomenon of abrupt liquefaction cannot occur ; and a definite sharp critical point must in any case be exhibited where the bend vanishes. Notwithstanding the aptness of the simple van der Waals formula, the gradual transition from the molecular freedom of perfect gas to the high molecular crowding of liquid can hardly be claimed to be accomplished without some sort of gradual change of aggregation,

* When the constraint to plane strata that is the basis of this analysis is released, the range of internal instability will naturally be greater than that here determined.

† There has, I find, been much discussion as to whether the phenomena of the critical point are really perfectly definite : an account with references is given by Graetz in Winkelmann's *Physik*, under the section "Wärme" : cf. also Poynting, in Poynting and Thomson's *Heat*, p. 192 : also for a complete account, Onnes and Keesom, "Die Zustandsgleichung," *Encyc. Math.* or *Leyden Reports*, 1912, cf. § 50.

especially where it is intensified just above the critical point.* The very remarkable flickering striae described and figured by Andrews,† when a liquid is heated in a confined space at a few degrees above the critical point, might be due partly to such a cause; they are ascribed by him to the very great changes of density there produced by small changes of pressure or temperature, and the sluggish recovery from disturbance of density that is involved; but that need not be their sole cause. "They are always a clear proof that the matter in the tube is homogeneous, and that we have not liquid and gas in presence of one another"; the latter state of affairs being indicated, as he states, by a fog or cloud in the tube, instead of by striation.

In the above the propagation of internal disturbance has been investigated on the basis of constant temperature. It may well be that adiabatic conditions would be more in keeping with the rapid periods, as in the propagation of waves of sound in gases. But the conclusions as to internal stability would hardly be affected; for, in accordance with Andrews' remark just quoted, when instability is approached the oscillations become very sluggish.

It is well known that a liquid like water can sustain very large negative pressure, or tension on the hydrostatic piston, provided it fills the reservoir without the presence of any bubble of vapour or other gas. This corresponds to the first descending arm (PN) of the isothermal (p, v) curve. If the temperature is far enough below the critical point, the lowest point N may fall below the axis of zero pressure; but the vapour-liquid line (PL) must, of course, be above the axis in all cases. Actually, as shown above, internal stability breaks up before N is reached; thus the condition of N situated on the axis gives too high a value to the limiting temperature beyond which negative pressure in the fluid is possible.

If we could employ the simple van der Waals equation

$$p = \frac{RT}{v-b} - \frac{a}{v^2},$$

where $-a$ represents the $\frac{1}{2}A_0$ above, so far from the critical point T_1, there

* It is even to be expected, on any reasonable view, that the physical cohesion of encountering molecules, which at lower temperatures is frequent enough to establish change of state, should still occur less persistently at temperatures somewhat above the critical point, and in fact be a main cause of the large changes of volume that there accompany slight differences of pressure.

† *Scientific Papers*, p. 337. There is nothing to contradict this in the more recent explorations of the critical isotherm.

would be two real values of v for which p vanishes so long as

$$RT < \frac{a}{4b}.$$

Only at temperatures below this limit (say T_0) could the pressure become negative.

The usual determination of the critical point T_1 by the conditions $dp/dv = 0$ and $d^2p/dv^2 = 0$ gives

$$RT_1 = \frac{8a}{27b}.$$

Thus we would have $$T_0 = \frac{27}{32} T_1$$

for all substances. For example, the critical temperature for water is given as 365 absolute: thus the equation would indicate that water could not sustain negative pressure at temperatures above 308 absolute, or 35° C., and the conclusions drawn above would show that this limit is in excess. While this particular estimate carries very slight weight, the Andrews continuity principle makes it clear that a limit of this kind must exist; and it will have a bearing on certain theories of the suspension of sap in high trees,* and on other phenomena in which intervention of liquid tension has been invoked.

* Cf. J. Joly and H. H. Dixon, *Phil. Trans.*, 1895 B; also, on the Continuity of the Phenomena from Positive into Negative Pressures, Worthington, *Phil. Trans.*, 1892 A.

THE AVERAGE ORDER OF THE ARITHMETICAL FUNCTIONS $P(x)$ AND $\Delta(x)$

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I.

INTRODUCTION.

1. One of the most important unsolved problems in the analytic theory of numbers is that of the determination of the real order of the functions $P(x)$ and $\Delta(x)$ defined by the equations

$$R(x) = \sum_{n \leqslant x} r(n) = \pi x + P(x), \tag{1.1}$$

$$D(x) = \sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x), \tag{1.2}$$

where γ is Euler's constant, and $r(n)$ and $d(n)$ are the number of representations of n as the sum of two squares and the number of its divisors. It is known* that

$$P(x) = O(x^{\frac{1}{3}}), \tag{1.31}$$

$$P(x) \neq o\{x^{\frac{1}{4}}(\log x)^{\frac{1}{4}}\}, \tag{1.32}$$

$$\Delta(x) = O(x^{\frac{1}{3}} \log x), \tag{1.41}$$

$$\Delta(x) \neq o\{x^{\frac{1}{4}} (\log x)^{\frac{1}{4}} \log \log x\}\ ; \tag{1.42}$$

* See Hardy, "On the expression of a number as the sum of two squares", *Quarterly Journal*, Vol. 46, 1915, pp. 263–283, and "On Dirichlet's divisor problem", *Proc. London Math. Soc.*, Ser. 2, Vol. 15, 1916, pp. 1–25. These papers contain numerous references to the literature of the subject: to the papers referred to I can now add three others, viz., E. Landau, "Über Dirichlets Teilerproblem", *Münchener Sitzungsberichte*, 1915, pp. 317–328; E. Landau, "Über die Anzahl der Gitterpunkte in gewissen Bereichen (Zweite Mitteilung)", *Göttinger Nachrichten*, 1915; and a report by Landau on Wigert's memoir "Sur quelques fonctions arithmétiques" (*Acta Mathematica*, Vol. 37, 1914, pp. 113–140), published in the *Göttingischer gelehrte Anzeigen*, 1915, Nos. 7 and 8.

and it is not unlikely that

$$P(x) = O(x^{\frac{1}{4}+\epsilon}), \tag{1.5}$$

$$\Delta(x) = O(x^{\frac{1}{4}+\epsilon}), \tag{1.6}$$

for all positive values of ϵ, though this has never been proved.

My object in this paper is to prove that both (1.5) and (1.6) are true *on the average*, that is to say that

$$\frac{1}{x}\int_1^x |P(\tau)|\,d\tau = O(x^{\frac{1}{4}+\epsilon}), \tag{1.7}$$

and

$$\frac{1}{x}\int_1^x |\Delta(\tau)|\,d\tau = O(x^{\frac{1}{4}+\epsilon}).* \tag{1.8}$$

The proofs of (1.7) and (1.8) proceed on quite similar lines. I shall give a full proof of (1.7), and then indicate shortly the modifications in the argument required for the proof of (1.8).

II.

THE AVERAGE ORDER OF $P(x)$.

2.1. THEOREM 2.1.—*If*

$$R(x) = \sum_{n\leqslant x} r(n) = \pi x + P(x),$$

then

$$\int_1^x \{P(\tau)\}^2 d\tau = O(x^{\frac{3}{2}+\epsilon}), \tag{2.11}$$

and

$$\int_1^x |P(\tau)|\,d\tau = O(x^{\frac{1}{2}+\epsilon}), \tag{2.12}$$

for all positive values of ϵ.

It will be enough to prove (2.11), as (2.12) then follows immediately by a direct application of "Schwarz's inequality".

* I stated the theorem expressed by (1.8), without proof, in my paper "On Dirichlet's divisor problem" already referred to.

Preliminary lemmas.

2.21. I shall begin by proving a series of lemmas.

LEMMA 2.21.—*Suppose that*

$$f(z) = z^2 \log z \; g(z) + h(z),$$

where $\log z$ *has its principal value, and* $g(z)$ *and* $h(z)$ *are analytic functions of* z *regular at the origin. Suppose further that* $f(0)$ *is, by definition, equal to* $h(0)$. *Then numbers* δ *and* K *exist such that*

$$|f(z)-f(Z)| < K|z-Z|,$$

if z *and* Z *both lie within the domain* $D(\delta)$ *defined by*

$$0 \leqslant |z| \leqslant \delta, \quad 0 \leqslant |Z| \leqslant \delta, \quad \Re(z) \geqslant 0, \quad \Re(Z) \geqslant 0.$$

We have

$$f'(z) = (2z \log z + z)\, g(z) + z^2 \log z \; g'(z) + h'(z),$$

so that $$f'(z) \to h'(0),$$

when $z \to 0$ in $D(\delta)$. If we attribute to $f'(z)$ the value $h'(0)$ when $z = 0$, we can find δ and K so that

$$|f'(z)| < K$$

in $D(\delta)$; and $$|f(z)-f(Z)| = \left| \int_Z^z f'(\zeta)\, d\zeta \right| < K|z-Z|,$$

the path of integration being the straight line from Z to z.

2.22. LEMMA 2.221.—*If* ϵ *is any positive number, then*

$$r(n) = O(n^\epsilon).$$

This is well known.*

* See, for example, Landau, "Über einen Satz des Herrn Sierpinski", *Giornale di Matematiche di Battaglini*, Vol. 51, 1913, pp. 73–81 (p. 75).

LEMMA 2.222.—*Let $F(s)$, $G_1(s)$, and $G_2(s)$ be the functions of $s=\sigma+it$ defined, when $\sigma>0$, by the Dirichlet's series*

$$F(s) = \Sigma \frac{r(n)}{n^{\frac{3}{4}}} e^{-s\sqrt{n}},$$

$$G_1(s) = \Sigma \frac{e^{-s\sqrt{n}}}{n},$$

$$G_2(s) = \Sigma \frac{e^{-s\sqrt{n}}}{n^{\frac{3}{4}}}.$$

Then $F(s)$ is regular at all points of the imaginary axis except the origin and the points

$$s = \pm s_q = \pm it_q = \pm 2\pi i\sqrt{q},$$

and is, near the point $s=s_q$, of the form

$$\log(s-s_q)G(s)+H(s),$$

where $G(s)$ and $H(s)$ are regular for $s=s_q$. And constants A and B exist such that

$$F(s)-AG_1(s-s_q)-BG_2(s-s_q)$$

is, near $s=s_q$, of the form

$$(s-s_q)^2\log(s-s_q)\,g(s)+h(s),$$

where $g(s)$ and $h(s)$ are regular for $s=s_q$.

The proofs of the first two propositions are contained in my paper "On the expression of a number as the sum of two squares" already referred to. It is also proved there that

$$G_1(s-s_q) = -2\log(s-s_q)+H_1(s),$$

where $H_1(s)$ is regular at $s=s_q$. It follows that

$$G_2(s-s_q) = 2(s-s_q)\log(s-s_q)+H_2(s),$$

where $H_2(s)$ also is regular at $s=s_q$. Thus

$$F(s)-AG_1(s)-BG_2(s)$$

is of the form prescribed by the lemma if

$$A = -\tfrac{1}{2}G(s_q), \qquad B = \tfrac{1}{2}G'(s_q).$$

The value of A is given by

$$A = \frac{e^{-\frac{1}{4}\pi i}r(q)}{2q^{\frac{1}{4}}}. \tag{2.221}$$

* "On the expression &c.", p. 280, equation (3.74).

2.23. LEMMA 2.231.—*If A_n is the sum of the first n terms of the series*

$$\Sigma \frac{r(n)}{n^{\frac{3}{4}}} e^{-it\sqrt{n}}, \tag{2.2311}$$

then

$$A_n = O(\log n), \tag{2.2312}$$

uniformly throughout any interval

$$0 < T_0 \leqslant t \leqslant T_1.$$

The series is uniformly convergent throughout any such interval which includes no point $t = t_q = 2\pi\sqrt{q}$.*

It is therefore sufficient to prove that (2.2312) holds uniformly throughout any small interval of the type $(t_q-\delta,\ t_q+\delta)$.

Let $$s-s_q = z,$$

$$\phi(s) = F(s)-AG_1(s-s_q)-BG_2(s-s_q) = F(s_q+z)-AG_1(z)-BG_2(z) = f(z),$$

say. Then $f(z)$ satisfies the conditions of Lemma 2.21.

Suppose now that δ and K are determined as in Lemma 2.21, that $\delta_1 < \frac{1}{2}\delta$, that

$$Z = S-s_q = i(T-t_q),$$

where $$|T-t_q| \leqslant \delta_1,$$

and that $$|z-Z| \leqslant \delta_1, \qquad \Re(z) \geqslant 0.$$

Then both z and Z lie in $D(\delta)$, and so

$$|f(z)-f(Z)| < K|z-Z|,$$

or $$|\phi(s)-\phi(iT)| < K|s-iT|.$$

In other words, $\phi(s)$ satisfies (to the right of the imaginary axis) a "Lipschitz condition" of order unity with respect to every point iT, and uniformly throughout the interval

$$-\delta_1 \leqslant T-t_q \leqslant \delta_1.$$

It follows, by one of the theorems of Riesz of which I have made such constant use throughout this series of researches, that the Dirichlet's

* "On the expression &c.", pp. 277, 278.

series which represents $\phi(s)$ to the right of the imaginary axis, viz.,

$$\Sigma \left\{ \frac{r(n)}{n^{\frac{3}{4}}} - \frac{A}{n} e^{s_q\sqrt{n}} - \frac{B}{n^{\frac{3}{4}}} e^{s_q\sqrt{n}} \right\} e^{-s\sqrt{n}}$$

is uniformly convergent for

$$\sigma = 0, \qquad t_q - \delta_1 \leqslant t \leqslant t_q + \delta_1.$$

And from this it follows that

$$A_n = O\left(\sum_1^n \frac{1}{\nu}\right) = O(\log n),$$

uniformly throughout the interval $(t_q - \delta_1,\ t_q + \delta_1)$. This completes the proof of the lemma.

LEMMA 2.232.—*The series*

$$\Sigma \frac{r(n)}{n^{\beta}} e^{-it\sqrt{n}},$$

where $\beta > \frac{3}{4}$, *is uniformly convergent throughout any interval of the type*

$$0 < T_0 \leqslant t \leqslant T_1.$$

This is an immediate corollary of Lemma 2.231.

LEMMA 2.233.—*The series*

$$\Sigma \frac{r(n)}{n^{\frac{3}{4}}} \cos(t\sqrt{n} + \tfrac{1}{4}\pi)$$

is boundedly convergent throughout any interval of the type*

$$0 < T_0 \leqslant t \leqslant T_1.$$

We may suppose, without loss of generality, that the interval includes just one point t_q. The series

$$\Sigma \left\{ \frac{r(n)}{n^{\frac{3}{4}}} - \frac{A}{n} e^{it_q\sqrt{n}} - \frac{B}{n^{\frac{3}{4}}} e^{it_q\sqrt{n}} \right\} e^{-it\sqrt{n}}$$

is uniformly convergent throughout any such interval. Hence the series

$$\Sigma \Re\left[\left\{ \frac{r(n)}{n^{\frac{3}{4}}} - \frac{A}{n} e^{it_q\sqrt{n}} \right\} e^{-it\sqrt{n} - \frac{1}{4}\pi i} \right]$$

* A series $\Sigma u_n(x)$ is said to be boundedly convergent throughout an interval if it is convergent throughout the interval and

$$\left| \sum_1^n u_\nu(x) \right| < K,$$

where K is independent of both n and x.

is also uniformly convergent. It is therefore sufficient to prove that

$$\Sigma \Re \left\{ -\frac{ir(q)}{2q^{\frac{1}{2}}n} e^{-i(t-t_q)\sqrt{n}} \right\}$$

is boundedly convergent; or that

$$\Sigma \frac{\sin(t\sqrt{n})}{n} \tag{2.2331}$$

is boundedly convergent throughout an interval which includes $t = 0$, say the interval $0 \leqslant t \leqslant 1$. Now

$$\sum_1^{n-1} \frac{\sin t\sqrt{\nu}}{\nu} = \int_1^n \frac{\sin t\sqrt{x}}{x}\, dx - \sum_1^{n-1} \int_\nu^{\nu+1} \left(\frac{\sin t\sqrt{x}}{x} - \frac{\sin t\sqrt{\nu}}{\nu} \right) dx = S_1 + S_2,$$

where

$$|S_1| = 2 \left| \int_1^{\sqrt{n}} \frac{\sin tu}{u}\, du \right| < 2\pi,$$

$$|S_2| < \sum_1^{n-1} \left(\frac{t}{2\nu^{\frac{3}{2}}} + \frac{1}{\nu^2} \right) < \tfrac{1}{2} \sum_1^{\infty} \frac{1}{n^{\frac{3}{2}}} + \sum_1^{\infty} \frac{1}{n^2}.$$

Hence (2.2331) is boundedly convergent, and the lemma is proved.

LEMMA 2.234.—*The series*

$$\Sigma \frac{r(n)}{\sqrt{n}} J_1\{2\pi\sqrt{(nx)}\}$$

is boundedly convergent throughout any interval

$$0 < x_0 \leqslant x \leqslant x_1.$$

This follows at once from Lemma 2.233 and the formula

$$J_1\{2\pi\sqrt{(nx)}\} = -\frac{\cos\{2\pi\sqrt{(nx)} + \frac{1}{4}\pi\}}{\pi(nx)^{\frac{1}{4}}} + O\{(nx)^{-\frac{3}{4}}\}.$$

2.24. LEMMA 2.241.—*The series*

$$\Sigma \frac{1}{(pq)^\beta(\sqrt{p}+\sqrt{q})}$$

and

$$\sum_{p \neq q} \frac{1}{(pq)^\beta |\sqrt{p}-\sqrt{q}|},$$

where $\beta > \frac{3}{4}$, and the summations extend over all positive integral values of p and q, are convergent.

We need only consider the second series, as the terms of the first

(except those for which $p = q$, which obviously form a convergent series) are less than those of the second.

It is enough to prove that

$$\sum_{p>q} \frac{1}{(pq)^\beta(\sqrt{p}-\sqrt{q})} = \sum_p \frac{1}{p^\beta} \sum_{1\leqslant q\leqslant p-1} \frac{1}{q^\beta(\sqrt{p}-\sqrt{q})}$$

is convergent. Now

$$\sum_{1\leqslant q\leqslant p-1} \frac{1}{q^\beta(\sqrt{p}-\sqrt{q})} = \sum_{1\leqslant q\leqslant p-1} \frac{\sqrt{p}+\sqrt{q}}{q^\beta(p-q)} < 2\sqrt{p} \sum_{1\leqslant q\leqslant p-1} \frac{1}{q^\beta(p-q)}$$

$$= 2\sqrt{p}\left(\sum_{1\leqslant q\leqslant \frac{1}{2}p} + \sum_{\frac{1}{2}p<q\leqslant p-1}\right)\frac{1}{q^\beta(p-q)}$$

$$< \frac{4}{\sqrt{p}} \sum_{1\leqslant q\leqslant \frac{1}{2}p} \frac{1}{q^\beta} + 2^{1+\beta}p^{\frac{1}{2}-\beta} \sum_{\frac{1}{2}p<q\leqslant p-1} \frac{1}{p-q}$$

$$= O(p^{\frac{1}{2}-\beta}) + O(p^{\frac{1}{2}-\beta}\log p).$$

And the series $$\Sigma p^{\frac{1}{2}-2\beta}\log p$$

is convergent.

It is important to observe that the series are *not* convergent when $\beta = \frac{3}{4}$. The convergence of the first series, for example, would involve that of

$$\sum_n \sum_{nq\leqslant p<(n+1)q} \frac{1}{(pq)^{\frac{3}{4}}(\sqrt{p}+\sqrt{q})},$$

and so that of $$\sum_{q,n} \frac{1}{(n+1)^{\frac{3}{4}}\{1+\sqrt{(n+1)}\}q};$$

and this series is not convergent.

LEMMA 2.242.—*The series*

$$\Sigma \frac{r(p)\,r(q)}{(pq)^\beta(\sqrt{p}+\sqrt{q})}$$

and

$$\sum_{p\neq q} \frac{r(p)\,r(q)}{(pq)^\beta\,|\sqrt{p}-\sqrt{q}\,|},$$

where $\beta > \frac{3}{4}$, are convergent.

This is an immediate corollary of Lemmas 2.221 and 2.241.

2.25. LEMMA 2.25.—*If*

$$S_\alpha(x) = \sum_{n \leqslant x} (x-n)^\alpha a_n,$$

where $0 < \alpha < 1$,* *and*

$$S(x) = S_0(x);$$

and

$$\sum_1^n \{S_\alpha(\nu)\}^2 = O(n^k),$$

where $k > 0$; *then*

$$\sum_1^n \{S(\nu)\}^2 = O(n^k).$$

From

$$\sum_1^n \{S_\alpha(\nu)\}^2 = O(n^k),$$

it follows that

$$S_\alpha(n) = O(n^{\frac{1}{2}k}),$$

or†

$$\alpha \int_0^n (n-\tau)^{\alpha-1} S(\tau)\, d\tau = O(n^{\frac{1}{2}k}).$$

Hence‡

$$\alpha \int_{n-1}^n (n-\tau)^{\alpha-1} S(\tau)\, d\tau = O(n^{\frac{1}{2}k}),$$

or

$$S(n-1) = O(n^{\frac{1}{2}k});$$

and so

$$a_n = O(n^{\frac{1}{2}k}).$$

It follows that the series $\Sigma a_n z^n$ is convergent for $|z| < 1$.

Now let

$$f(z) = f(re^{i\theta}) = \sum_1^\infty a_n z^n,$$

where $0 \leqslant r < 1$, and

$$\phi_\alpha(z) = \sum_1^\infty n^\alpha z^n.$$

Then

$$f(z)\, \phi_\alpha(z) = \sum_2^\infty S_\alpha(n) z^n,$$

and

$$(2.251) \qquad \frac{1}{2\pi}\int_0^{2\pi} |f(z)\, \phi_\alpha(z)|^2\, d\theta = \Sigma \{S_\alpha(n)\}^2 r^{2n} = O\{(1-r)^{-k}\}.$$

---

* The restriction that $\alpha < 1$ is unnecessary for the truth of the lemma, but enables us to simplify the proof a little.

The lemma asserts, to put it roughly, that the average order of $S(x)$ is not greater than that of $S_\alpha(x)$. It is an analogue for "average orders" of the following simpler theorem:—*if* $S_\alpha(x) = O(x^k)$, *then* $S(x) = O(x^k)$. This theorem is a special case of Theorem 21 (p. 36) of Riesz's and my tract "The general theory of Dirichlet's series", the case in which $\lambda_n = n$.

† Hardy and Riesz, *l.c.*, p. 21.

‡ Hardy and Riesz, *l.c.*, p. 28 (Lemma 7).

Also

$$(2.252)\qquad \Sigma\{S(n)\}^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(z)}{1-z}\right|^2 d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}|f(z)\,\phi_a(z)|^2\left|\frac{1}{(1-z)\,\phi_a(z)}\right|^2 d\theta.$$

Suppose now for the moment that we have proved the existence of a constant K such that

$$(2.253)\qquad |(1-z)\,\phi_a(z)| > K,$$

for all values of z whose modulus is near enough to unity. Then it follows from (2.252), (2.253), and (2.251), that

$$\Sigma\{S(n)\}^2 r^{2n} < \frac{1}{2\pi K^2}\int_0^{2\pi}|f(z)\,\phi_a(z)|^2\,d\theta = O\{(1-r)^{-k}\}.$$

Taking $$r = 1-\frac{1}{m},$$

we obtain

$$\sum_1^m\{S(n)\}^2 < \left(1-\frac{1}{m}\right)^{-2m}\sum_1^m\{S(n)\}^2 r^{2n} < \left(1-\frac{1}{m}\right)^{-2m}\Sigma\{S(n)\}^2 r^{2n}$$

$$= O(m^k);$$

and the lemma is proved.

It remains to prove (2.253). The function $\phi_a(z)$ has a singularity at the point $z=1$ of the type $(1-z)^{-1-a}$.* We can therefore draw a circle with its centre at $z=1$, such that (2.253) is satisfied at all of the points inside both this circle and the circle of convergence. And there is a positive number δ such that

$$1-r\cos\theta > \delta,$$

at all other points of the latter circle.

Suppose now that z lies inside the circle of convergence, but outside

* See, for example, Hardy, "A method for determining the behaviour of certain classes of power series near a singular point on the circle of convergence", *Proc. London Math. Soc.*, Ser. 2, Vol. 3, 1905, pp. 381–389.

the circle with its centre at $z = 1$. Then

$$g(z) = \frac{(1-z)\,\phi_a(z)}{z} = 1^a + (2^a - 1^a)\,z + (3^a - 2^a)\,z^2 + \ldots$$

$$= c_0 + c_1 z + c_2 z^2 + \ldots,$$

say, and $c_{n+1} < c_n$ for all values of n. Also

$$\begin{aligned} |\,(1-z)\,g(z)\,| &= |\,c_0 + (c_1 - c_0)\,z + (c_2 - c_1)\,z^2 + \ldots\,| \\ &\geqslant \mathfrak{R}\,|\,c_0 + (c_1 - c_0)\,z + (c_2 - c_1)\,z^2 + \ldots\,| \\ &= c_0 - (c_0 - c_1)\,r\cos\theta - (c_1 - c_2)\,r^2\cos 2\theta - \ldots \\ &= (c_0 - c_1)(1 - r\cos\theta) + c_1 - (c_1 - c_2)\,r^2\cos 2\theta - \ldots \\ &\geqslant (c_0 - c_1)(1 - r\cos\theta) + c_1 - (c_1 - c_2) - \ldots \\ &= (c_0 - c_1)(1 - r\cos\theta) > \delta\,(2 - 2^a). \end{aligned}$$

Hence $$|\,(1-z)\,\phi_a(z)\,| = |\,zg(z)\,| > \frac{\delta(2-2^a)}{2\,|\,1-z\,|} > \frac{\delta(2-2^a)}{4},$$

if $\frac{1}{2} < r < 1$. Thus (2.253) is satisfied at all points of the annulus $\frac{1}{2} < r < 1$, whether they do or do not lie within the circle whose centre is at $z = 1$.

This completes the proof of the lemma.

The explicit formula for $R_a(x) = \sum_{n \leqslant x} (x-n)^a\, r(n)$.

2.3. In one of my papers already quoted* I proved that

$$R(x) = \pi x - 1 + \sqrt{x} \sum_1^\infty \frac{r(n)}{\sqrt{n}}\, J_1\{2\pi\sqrt{(nx)}\}, \tag{2.31}$$

if x is positive but not integral. I shall now deduce from this formula the corresponding formula for the function

$$R_a(x) = \sum_{n \leqslant} (x-n)^a\, r(n) \quad (a > 0). \tag{2.32}$$

I shall assume that $0 < a < 1$: the formula, when once proved for such

* "On the expression &c.", pp. 272–282. It would of course be possible to construct a direct proof on similar lines of the formula which we require here.

values of α, is easily extended to larger values by the help of the equation

$$\frac{dR_\alpha(x)}{dx} = \alpha R_{\alpha-1}(x) \quad (\alpha \geqslant 1).$$

It follows from Lemma 2.233 that

$$(2.33) \quad \int_\epsilon^x (x-\tau)^{\alpha-1}\sqrt{\tau}\,\Sigma\frac{r(n)}{\sqrt{n}}J_1\{2\pi\sqrt{(n\tau)}\}\,d\tau$$

$$= \Sigma\frac{r(n)}{\sqrt{n}}\int_\epsilon^x (x-\tau)^{\alpha-1}\sqrt{\tau}\,J_1\{2\pi\sqrt{(n\tau)}\}\,d\tau,$$

if $\alpha > 0$, $0 < \epsilon < x$.* I shall now prove that we may write 0 for ϵ in (2.33). In order to prove this it is necessary to show that the series

$$(2.331) \quad s(\epsilon) = \Sigma\frac{r(n)}{\sqrt{n}}\int_0^\epsilon (x-\tau)^{\alpha-1}\sqrt{\tau}\,J_1\{2\pi\sqrt{(n\tau)}\}\,d\tau$$

is convergent and tends to zero with ϵ. I shall prove, in fact, that

$$(2.332) \quad s(\epsilon) = O(\epsilon).\dagger$$

I write

$$(2.333) \quad s(\epsilon) = \sum_{n\leqslant 1/\epsilon} + \sum_{n>1/\epsilon} = s_1(\epsilon)+s_2(\epsilon),$$

assuming for the moment the convergence of s_2. The discussion of s_1 is easy; for

$$|J_1(u)| < Ku,$$

for $0 < u < 2\pi$, and so

$$|s_1| < K\sum_{n\leqslant 1/\epsilon} r(n)\int_0^\epsilon \tau\,d\tau < K\epsilon^2\sum_{n\leqslant 1/\epsilon} r(n) < K\epsilon,$$

$$(2.334) \quad s_1(\epsilon) = O(\epsilon).$$

* See, for example, W. H. Young, "On the theory of the application of expansions to definite integrals", *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 463–485 (Theorem 2, p. 468).

† This is the result to be expected, since the sum of the series

$$\Sigma\frac{r(n)}{\sqrt{n}}J_1\{2\pi\sqrt{(n\tau)}\}$$

has an infinity of order $\frac{1}{2}$ for $\tau = 0$. The result seems rather more difficult to prove than might be anticipated on the grounds of its crudity.

It remains to discuss s_2. I write

$$(2.335)\quad s_2 = \sum_{n>1/\epsilon} \frac{r(n)}{\sqrt{n}} \int_0^{\epsilon} = \sum_{n>1/\epsilon} \frac{r(n)}{\sqrt{n}} \int_0^{1/n} + \sum_{n>1/\epsilon} \frac{r(n)}{\sqrt{n}} \int_{1/n}^{\epsilon} = s_{2,1} + s_{2,2},$$

assuming provisionally the convergence of the series. It is clear that

$$\left| \frac{r(n)}{\sqrt{n}} \int_0^{1/n} \right| < K r(n) \int_0^{1/n} \tau \, d\tau < K \frac{r(n)}{n^2}.$$

Hence $s_{2,1}$ is convergent, and

$$|s_{2,1}| < K \sum_{n>1/\epsilon} r(n) \int_0^{1/n} \tau \, d\tau < K \sum_{n>1/\epsilon} \frac{r(n)}{n^2}$$

$$= K \sum_{n>1/\epsilon} \frac{R(n)-R(n-1)}{n^2} < K \sum_{n>1/\epsilon} R(n) \left\{ \frac{1}{n^2} - \frac{1}{(n+1)^2} \right\}$$

$$= K \sum_{n>1/\epsilon} O(n) \, O\left(\frac{1}{n^3}\right) < K\epsilon,$$

$$(2.336)\qquad s_{2,1}(\epsilon) = O(\epsilon).$$

The discussion of $s_{2,2}$ is a little more troublesome. Since $(x-\tau)^{\alpha-1}$ increases from $\tau = 1/n$ to $\tau = \epsilon$, we have

$$\int_{1/n}^{\epsilon} = (x-\epsilon)^{\alpha-1} \int_{\epsilon'}^{\epsilon} \sqrt{\tau} \, J_1\{2\pi\sqrt{(n\tau)}\} \, d\tau,$$

where $1/n < \epsilon' < \epsilon$. Hence

$$\left| \int_{1/n}^{\epsilon} \right| < K \left| \int_{\epsilon'}^{\epsilon} \sqrt{\tau} \, J_1\{2\pi\sqrt{(n\tau)}\} \, d\tau \right|$$

$$< K \left| \int_{\sqrt{\epsilon'}}^{\sqrt{\epsilon}} u^2 J_1(2\pi u\sqrt{n}) \, du \right|$$

$$< \frac{K}{n^{\frac{3}{2}}} \left| \int_{2\pi\sqrt{(n\epsilon')}}^{2\pi\sqrt{(n\epsilon)}} w^2 J_1(w) \, dw \right|$$

$$= \frac{K}{\sqrt{n}} \left[\epsilon J_2\{2\pi\sqrt{(n\epsilon)}\} - \epsilon' J_2\{2\pi\sqrt{(n\epsilon')}\} \right]$$

$$< \frac{K\epsilon}{\sqrt{n}} |J_2\{2\pi\sqrt{(n\epsilon)}\}| + \frac{K\epsilon'}{\sqrt{n}} |J_2\{2\pi\sqrt{(n\epsilon')}\}|$$

$$< \frac{K\epsilon^{\frac{3}{4}}}{n^{\frac{3}{4}}} + \frac{K\epsilon'^{\frac{3}{4}}}{n^{\frac{3}{4}}} * < \frac{K\epsilon^{\frac{3}{4}}}{n^{\frac{3}{4}}}.$$

---

* Since

$$|J_2(u)| < \frac{K}{\sqrt{u}},$$

for $u > 2\pi$.

It follows that $s_{2,2}$ is convergent, and that

$$s_{2,2}| < K\epsilon^{\frac{3}{4}} \sum_{n>1/\epsilon} \frac{r(n)}{n^{\frac{3}{4}}};$$

from which it follows, by an argument similar to that used in proving (2.336), that

$$|s_{2,2}| < K\epsilon^{\frac{3}{4}}\epsilon^{\frac{1}{4}} = K\epsilon,$$

$$s_{2,2}(\epsilon) = O(\epsilon). \tag{2.337}$$

The truth of (2.332) now follows from (2.333)–(2.337). We have therefore

$$\int_0^x (x-\tau)^{\alpha-1}\sqrt{\tau}\,\Sigma\frac{r(n)}{\sqrt{n}}\,J_1\{2\pi\sqrt{(n\tau)}\}\,d\tau = \Sigma\frac{r(n)}{\sqrt{n}}\int_0^x (x-\tau)^{\alpha-1}\sqrt{\tau}\,J_1\{2\pi\sqrt{(n\tau)}\}\,d\tau. \tag{2.34}$$

2.4. Now*

$$\begin{aligned} R_\alpha(x) &= \alpha\int_0^x (x-\tau)^{\alpha-1}R(\tau)\,d\tau \\ &= \alpha\int_0^x (x-\tau)^{\alpha-1}\Big[\pi\tau-1+\sqrt{\tau}\,\Sigma\frac{r(n)}{\sqrt{n}}\,J_1\{2\pi\sqrt{(n\tau)}\}\Big]\,d\tau \\ &= \frac{\pi x^{1+\alpha}}{1+\alpha}-x^\alpha+\alpha\,\Sigma\frac{r(n)}{\sqrt{n}}\int_0^x (x-\tau)^{\alpha-1}\sqrt{\tau}\,J_1\{2\pi\sqrt{(n\tau)}\}\,d\tau. \end{aligned}$$

Also

$$\begin{aligned} &\int_0^x (x-\tau)^{\alpha-1}\sqrt{\tau}\,J_1\{2\pi\sqrt{(n\tau)}\}\,d\tau \\ &= 2\int_0^{\sqrt{x}} (x-u^2)^{\alpha-1}u^2 J_1(2\pi u\sqrt{n})\,du \\ &= 2x^{\frac{1}{2}+\alpha}\int_0^{\frac{1}{2}\pi} (\cos\theta)^{2\alpha-1}\sin^2\theta\,J_1\{2\pi\sin\theta\sqrt{(nx)}\}\,d\theta \\ &= \pi^{-\alpha}\Gamma(\alpha)\,x^{\frac{1}{2}+\frac{1}{2}\alpha}J_{1+\alpha}\{2\pi\sqrt{(nx)}\}.\dagger \end{aligned}$$

Hence, finally,

$$R_\alpha(x) = \frac{\pi x^{1+\alpha}}{1+\alpha}-x^\alpha+\pi^{-\alpha}\Gamma(1+\alpha)x^{\frac{1}{2}+\frac{1}{2}\alpha}\Sigma\frac{r(n)}{n^{\frac{1}{2}+\frac{1}{2}\alpha}}J_{1+\alpha}\{2\pi\sqrt{(nx)}\}. \tag{2.41}$$

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* See p. 21 of the tract of Riesz and myself already quoted.

† Nielsen, *Cylinderfunktionen*, p. 181.

We may remark in passing that the series in (2.41) is absolutely convergent if $\alpha > \frac{1}{2}$, so that

$$R_\alpha(x) = \frac{\pi x^{1+\alpha}}{1+\alpha} - x^\alpha + O(x^{\frac{1}{4}+\frac{1}{2}\alpha}), \tag{2.42}$$

if $\alpha > \frac{1}{2}$. There is accordingly no difficulty in the discussion of the order of the error term in the expression for $R_\alpha(x)$ when $\alpha > \frac{1}{2}$, and in what follows I shall suppose that $0 < \alpha \leqslant \frac{1}{2}$.

The average order of the error term in the formula for $R_\alpha(x)$.

2.5. Let $$r_n = r(n) - \pi,$$

and

$$S_\alpha(x) = \sum_{n \leqslant x} (x-n)^\alpha r_n, \tag{2.51}$$

$$S(x) = S_0(x). \tag{2.511}$$

Then
$$\begin{aligned} S_\alpha(x) &= R_\alpha(x) - \pi \sum_{n \leqslant x} (x-n)^\alpha \\ &= R_\alpha(x) - \alpha\pi \int_0^x (x-\tau)^{\alpha-1} [\tau]\, d\tau \\ &= R_\alpha(x) - \alpha\pi \int_0^x (x-\tau)^{\alpha-1} \tau\, d\tau + O(x^\alpha) \\ &= R_\alpha(x) - \frac{\pi x^{1+\alpha}}{1+\alpha} + O(x^\alpha). \end{aligned}$$

Thus $S_\alpha(x)$ is substantially the error term in $R_\alpha(x)$; and

$$S_\alpha(x) = \pi^{-\alpha}\, \Gamma(1+\alpha)\, x^{\frac{1}{2}+\frac{1}{2}\alpha}\, T_\alpha(x) + O(x^\alpha), \tag{2.52}$$

where
$$T_\alpha(x) = \Sigma \frac{r(n)}{n^{\frac{1}{2}+\frac{1}{2}\alpha}} J_{1+\alpha}\{2\pi\sqrt{(nx)}\}.$$

But
$$J_{1+\alpha}(u) = -\sqrt{\left(\frac{2}{\pi u}\right)} \cos\{u + \tfrac{1}{4}(1-2\alpha)\pi\} + O(u^{-\frac{3}{2}}),$$

so that

$$T_a(x) = -\frac{1}{\pi x^{\frac{1}{4}}} \Sigma \frac{r(n)}{n^{\frac{3}{4}+\frac{1}{2}a}} \cos\{2\pi\sqrt{(nx)} + \tfrac{1}{4}(1-2a)\pi\} + O\frac{1}{x^{\frac{3}{4}}} \Sigma \frac{r(n)}{n^{\frac{3}{4}+\frac{1}{2}a}}$$

$$= -\frac{1}{\pi x^{\frac{1}{4}}} \Sigma \frac{r(n)}{n^{\frac{3}{4}+\frac{1}{2}a}} \cos\{2\pi\sqrt{(nx)} + \tfrac{1}{4}(1-2a)\pi\} + O\left(\frac{1}{x^{\frac{3}{4}}}\right).$$

Hence

$$S_a(x) = \pi^{-1-a}\,\Gamma(1+a)\, x^{\frac{1}{4}+\frac{1}{2}a} U_a(x) + O(x^a), \tag{2.53}$$

where

$$U_a(x) = -\Sigma \frac{r(n)}{n^{\frac{3}{4}+\frac{1}{2}a}} \cos\{2\pi\sqrt{(nx)} + \tfrac{1}{4}(1-2a)\pi\}. \tag{2.54}$$

2.6. The next step in the proof is to show that

$$\int_1^x \{U_a(\tau)\}^2\, d\tau = O(x), \tag{2.61}$$

that is to say that the average order of $\{U_a(x)\}^2$, and *a fortiori* of $U_a(x)$, is $O(1)$.

I observe first that, in order to prove (2.61), it is sufficient to prove that

$$\int_1^x \{U_a(w^2)\}^2\, dw = O(x). \tag{2.62}$$

For suppose that (2.62) is true. Putting $w^2 = \tau$ and $x^2 = \xi$, we obtain

$$W(\xi) = \int_1^\xi \frac{\{U_a(\tau)\}^2}{\sqrt{\tau}}\, d\tau = O(\sqrt{\xi});$$

and so

$$\int_1^x \{U_a(\tau)\}^2 d\tau = \int_1^x \sqrt{\tau}\; W'(\tau)\, d\tau = \sqrt{x}\; W(x) - W(1) - \tfrac{1}{2}\int_1^x \frac{W(\tau)}{\sqrt{\tau}}\, d\tau$$

$$< \sqrt{x}\; W(x) = O(x).$$

Thus (2.61) is a corollary of (2.62).

Now

$$U_a(w^2) = -\Sigma \frac{r(n)}{n^{\frac{3}{4}+\frac{1}{2}a}} \cos\{2\pi w\sqrt{n} + \tfrac{1}{4}(1-2a)\pi\}. \tag{2.63}$$

This series is, by Lemma 2.232, uniformly convergent for $1 \leqslant w \leqslant x$.

Hence, writing ϕ for $\frac{1}{4}(1-2\alpha)\pi$, we have

$$\int_1^x \{U_\alpha(w^2)\}^2 dw$$

$$= \Sigma\Sigma \frac{r(p)\,r(q)}{(pq)^{\frac{3}{4}+\frac{1}{2}\alpha}} \int_1^x \cos(2\pi w\sqrt{p}+\phi)\cos(2\pi w\sqrt{q}+\phi)dw$$

$$= \tfrac{1}{2}(x-1)\,\Sigma \frac{\{r(p)\}^2}{p^{\frac{3}{2}+\alpha}}$$

$$-\tfrac{1}{2}\Sigma\Sigma\, r(p)\,r(q)\frac{\sin\{2\pi x(\sqrt{p}+\sqrt{q})+2\phi\}-\sin\{2\pi(\sqrt{p}+\sqrt{q})+2\phi\}}{(pq)^{\frac{3}{4}+\frac{1}{2}\alpha}(\sqrt{p}+\sqrt{q})}$$

$$+\tfrac{1}{2}\sum_{p\neq q}\Sigma\, r(p)\,r(q)\frac{\sin\{2\pi x(\sqrt{p}-\sqrt{q})\}-\sin\{2\pi(\sqrt{p}-\sqrt{q})\}}{(pq)^{\frac{3}{4}+\frac{1}{2}\alpha}(\sqrt{p}-\sqrt{q})}$$

$$= O(x)+O\,\Sigma\Sigma\frac{r(p)\,r(q)}{(pq)^{\frac{3}{4}+\frac{1}{2}\alpha}(\sqrt{p}+\sqrt{q})}+O\sum_{p\neq q}\Sigma\frac{r(p)\,r(q)}{(pq)^{\frac{3}{4}+\frac{1}{2}\alpha}|\sqrt{p}-\sqrt{q}|}$$

$$= O(x)+O(1)+O(1) = O(x).$$

Thus (2.62), and therefore (2.61), is proved.

2.7. From (2.53) and (2.61) it follows that

$$\int_1^x \{S_\alpha(\tau)\}^2 d\tau = \int_1^x [O\{\tau^{\frac{1}{4}+\frac{1}{2}\alpha}|U_\alpha(\tau)|\}+O(\tau^\alpha)]^2 d\tau \tag{2.71}$$

$$= O\left[x^{\frac{1}{2}+\alpha}\int_1^x \{U_\alpha(\tau)\}^2 d\tau\right]+O(x^{1+2\alpha})$$

$$= O(x^{\frac{3}{2}+\alpha})+O(x^{1+2\alpha})$$

$$= O(x^{\frac{3}{2}+\alpha}),$$

since $\alpha \leqslant \frac{1}{2}$. And, applying Schwarz's inequality, we obtain

$$\int_1^x |S_\alpha(\tau)|\,d\tau = O(x^{\frac{3}{4}+\frac{1}{2}\alpha}). \tag{2.72}$$

That is to say, *the average order of* $S_\alpha(x)$, *and therefore of the error term in* $R_\alpha(x)$, *is* $O(x^{\frac{1}{4}+\frac{1}{2}\alpha})$. This theorem has been proved on the assumption that $0 < \alpha \leqslant \frac{1}{2}$. When $\alpha > \frac{1}{2}$, the error term is actually of this order, in virtue of (2.42), and *a fortiori* of this average order. The theorem is therefore true for every positive α, but trivial if $\alpha > \frac{1}{2}$.

Proof of the theorem.

2.8. We are now in a position to complete the proof of Theorem 2.1. I suppose that $\alpha < \frac{1}{2}$, and that

$$0 < \epsilon < \tfrac{1}{4} - \tfrac{1}{2}\alpha. \tag{2.81}$$

Let $\nu = [x]$. Then

$$S_\alpha(x) - S_\alpha(\nu) = \sum_{\lambda \leqslant \nu} \{(x-\lambda)^\alpha - (\nu-\lambda)^\alpha\} r_\lambda$$

$$= \sum_{\lambda \leqslant \nu-1} \{(x-\lambda)^\alpha - (\nu-\lambda)^\alpha\} r_\lambda + O(x^\epsilon).$$

Also, if $\lambda \leqslant \nu-1$, we have

$$(x-\lambda)^\alpha - (\nu-\lambda)^\alpha = \frac{1}{\alpha}\int_\nu^x \frac{d\tau}{(\tau-\lambda)^{1-\alpha}} < \frac{1}{\alpha(\nu-\lambda)^{1-\alpha}}.$$

Hence $\sum_{\lambda \leqslant \nu-1} \{(x-\lambda)^\alpha - (\nu-\lambda)^\alpha\} r_\lambda$

$$= O\sum_1^{\nu-1} \frac{|r_\lambda|}{(\nu-\lambda)^{1-\alpha}} = O\sum_1^{\nu-1} \frac{\lambda^\epsilon}{(\nu-\lambda)^{1-\alpha}}$$

$$= O\left(\frac{1}{\nu^{1-\alpha}} \sum_{\lambda \leqslant \frac{1}{2}\nu} \lambda^\epsilon\right) + O\left\{\nu^\epsilon \sum_{\frac{1}{2}\nu < \lambda < \nu} \frac{1}{(\nu-\lambda)^{1-\alpha}}\right\}$$

$$= O(\nu^{\alpha+\epsilon});$$

and so

$$S_\alpha(x) - S_\alpha(\nu) = O(x^{\alpha+\epsilon}).$$

It follows that

$$\sum_1^n \{S_\alpha(\nu)\}^2 = \int_1^{n+1} \{S_\alpha(\nu)\}^2 dx = \int_1^{n+1} \{S_\alpha(x) + O(x^{\alpha+\epsilon})\}^2 dx \tag{2.82}$$

$$= O\left[\int_1^{n+1} \{S_\alpha(x)\}^2 dx\right] + O(n^{1+2\alpha+2\epsilon})$$

$$= O(n^{\frac{3}{2}+\alpha}) + O(n^{1+2\alpha+2\epsilon}) = O(n^{\frac{3}{2}+\alpha}),$$

since

$$1 + 2\alpha + 2\epsilon < \tfrac{3}{2} + \alpha.$$

If now we apply Lemma 2.25, we obtain

$$\sum_1^n \{S(\nu)\}^2 = O(n^{\frac{3}{2}+\alpha}), \tag{2.83}$$

and so

$$\int_1^{n+1} \{S(x)\}^2 dx = \int_1^{n+1} \{S(\nu) + O(x^\epsilon)\}^2 dx$$

$$= \sum_1^n \{S(\nu)\}^2 + O(n^{1+2\epsilon})$$

$$= O(n^{\frac{3}{2}+\alpha}).$$

Hence finally, since

$$P(x) = R(x) - \pi x = \sum_{n \leqslant x} r(n) - \pi x = S(x) + O(1),$$

we obtain

$$\begin{aligned} (2.84) \qquad \int_1^x \{P(\tau)\}^2 d\tau &= \int_1^x \{S(\tau) + O(1)\}^2 d\tau \\ &= O\left[\int_1^x \{S(\tau)\}^2 d\tau\right] + O(x) \\ &= O(x^{\frac{3}{2}+\alpha}). \end{aligned}$$

This being true for every positive α, the proof of Theorem 2.1 is completed.

The Average Order of $\Delta(x)$.

3.1. THEOREM 3.1.—*If*

$$D(x) = \sum_{n \leqslant x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x),$$

then

$$(3.11) \qquad \int_1^x \{\Delta(\tau)\}^2 d\tau = O(x^{\frac{3}{2}+\epsilon}),$$

and

$$(3.12) \qquad \int_1^x |\Delta(\tau)| d\tau = O(x^{\frac{3}{4}+\epsilon}),$$

for all positive values of ϵ.

It is known* that

(3.13)

$$D(x) = x \log x + (2\gamma - 1)x + \tfrac{1}{4} + \sqrt{x} \sum \frac{d(n)}{\sqrt{n}} \{H_1\{4\pi\sqrt{(nx)}\} - Y_1\{4\pi\sqrt{(nx)}\}],$$

where $Y_1(u)$ is the ordinary second solution† of Bessel's equation, and

$$H_1(u) = \frac{2}{\pi} \int_1^\infty \frac{we^{-uw}}{\sqrt{(w^2-1)}} dw.$$

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* Voronoï, "Sur une fonction transcendante et ses applications à la sommation de quelques séries", *Annales de l'École Normale*, Ser. 3, Vol. 21, 1904, pp. 207–268, 459–534. See also my paper "On Dirichlet's divisor problem" already quoted.

† In Hankel's notation, not Neumann's: see Whittaker and Watson, *Modern Analysis*, pp. 363–365.

From (3.13) we can, by an argument similar to that of 2.3 and 2.4, derive an explicit formula for the function

$$D_\alpha(x) = \sum_{n \leqslant x} (x-n)^\alpha d(n). \tag{3.14}$$

But it is more convenient in this case to proceed as follows.

3.2. Let

$$d_n = d(n) - \log n - 2\gamma,$$

$$\mathfrak{D}_\alpha(x) = \sum_{n \leqslant x} (x-n)^\alpha d_n,$$

$$\mathfrak{D}(x) = \mathfrak{D}_0(x).$$

Then

$$\mathfrak{D}(x) = D(x) - x\log x - (2\gamma-1)x + O(\log x) \tag{3.21}$$

$$= \sqrt{x}\sum \frac{d(n)}{\sqrt{n}}\left[H_1\{4\pi\sqrt{(nx)}\} - Y_1\{4\pi\sqrt{(nx)}\}\right] + O(\log x),$$

and

$$\Delta(x) = \mathfrak{D}(x) + O(\log x). \tag{3.22}$$

Now

$$H_1(u) = O\left(\frac{e^{-u}}{\sqrt{u}}\right),$$

$$\sqrt{x}\sum \frac{d(n)}{\sqrt{n}} H_1\{4\pi\sqrt{(nx)}\} = O\left\{x^{\frac{1}{4}}\sum \frac{d(n)}{n^{\frac{3}{4}}} e^{-4\pi\sqrt{(nx)}}\right\} = o(1);$$

and

$$Y_1(u) = -\sqrt{\left(\frac{2}{\pi u}\right)}\cos(u - \tfrac{1}{4}\pi).$$

Hence

$$\mathfrak{D}(x) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}}\sum \frac{d(n)}{n^{\frac{3}{4}}}\cos\{4\pi\sqrt{(nx)} - \tfrac{1}{4}\pi\} + O(\log x). \tag{3.23}$$

3.3. From (3.23) we deduce, by an argument similar to that of 2.3, that

$$\mathfrak{D}_\alpha(x) = \alpha\int_0^x (x-\tau)^{\alpha-1} D(\tau)\, d\tau \tag{3.31}$$

$$= \frac{\alpha}{\pi\sqrt{2}}\sum \frac{d(n)}{n^{\frac{3}{4}}}\int_0^x (x-\tau)^{\alpha-1}\tau^{\frac{1}{4}}\cos\{4\pi\sqrt{(n\tau)} - \tfrac{1}{4}\pi\}\, d\tau + O(x^\alpha \log x).$$

We now require an asymptotic formula for the function

$$(3.32)\qquad g(x) = a\int_0^x (x-\tau)^{a-1}\,\tau^{\frac{1}{4}}\cos\left\{4\pi\sqrt{(n\tau)}-\tfrac{1}{4}\pi\right\}d\tau$$

$$= 2ax^{\frac{1}{4}+a}\int_0^1 (1-u^2)^{a-1}\,u^{\frac{1}{2}}\cos\left\{4\pi u\sqrt{(nx)}-\tfrac{1}{4}\pi\right\}du$$

$$= 2ax^{\frac{1}{4}+a}\,h\left\{4\pi\sqrt{(nx)}\right\},$$

where

$$(3.33)\qquad h(y) = \int_0^1 (1-u^2)^{a-1}\,u^{\frac{1}{2}}\cos(yu-\tfrac{1}{4}\pi)\,du$$

$$= \Re\left\{e^{-\frac{1}{4}\pi i}\,k(y)\right\},$$

$$(3.34)\qquad k(y) = \int_0^1 (1-u^2)^{a-1}\,u^{\frac{1}{2}}\,e^{iyu}\,du,$$

and $y > 0$.

The behaviour of $k(y)$ for large positive values of y is easily determined by means of Cauchy's theorem. Integrating round the rectangle whose vertices are

$$0,\ 1,\ 1+\infty\, i,\ \infty\, i,$$

we find

$$k(y) = 2^{a-1}e^{i(y-\frac{1}{2}a\pi)}\int_0^\infty (\eta+\tfrac{1}{2}i\eta^2)^{a-1}\,e^{-y\eta}\,d\eta - e^{\frac{1}{4}\pi i}\int_0^\infty (1+\eta^2)^{a-1}\,\eta^{\frac{1}{2}}\,e^{-y\eta}\,d\eta$$

$$= 2^{a-1}\Gamma(a)\,y^{-a}e^{i(y-\frac{1}{2}a\pi)}\left\{1+O\left(\frac{1}{y}\right)\right\}+O\left(\frac{1}{y^{\frac{1}{2}}}\right).$$

Using (3.32)–(3.34), we obtain

$$g(x) = (2\pi)^{-a}\Gamma(1+a)n^{-\frac{1}{2}a}x^{\frac{1}{4}+\frac{1}{2}a}\cos\left\{4\pi\sqrt{(nx)}-\tfrac{1}{2}(a+\tfrac{1}{2})\pi\right\}+O(x^{-\frac{1}{4}+\frac{1}{2}a}\,n^{-\frac{1}{2}-\frac{1}{2}a}).$$

And, substituting in (3.31), we find

$$(3.35)\qquad \mathfrak{D}_a(x) = 2^{-\frac{1}{2}-a}\pi^{-1-a}\Gamma(1+a)\,x^{\frac{1}{4}+\frac{1}{2}a}\sum\frac{d(n)}{n^{\frac{3}{4}+\frac{1}{2}a}}\cos\left\{4\pi\sqrt{(nx)}-\tfrac{1}{2}(a+\tfrac{1}{2})\pi\right\}$$

$$+O\left\{x^{-\frac{1}{4}+\frac{1}{2}a}\sum\frac{d(n)}{n^{\frac{5}{4}+\frac{1}{2}a}}\right\}+O(x^a\log x)$$

$$= 2^{-\frac{1}{2}-a}\pi^{-1-a}\,\Gamma(1+a)\,x^{\frac{1}{4}+\frac{1}{2}a}\sum\frac{d(n)}{n^{\frac{3}{4}+\frac{1}{2}a}}\cos\left\{4\pi\sqrt{(nx)}-\tfrac{1}{2}(a+\tfrac{1}{2})\pi\right\}+O(x^a\log x).$$

3.4. The remainder of the proof follows the lines of section 2. We

suppose that $0 < \alpha < \frac{1}{2}$; and from (3 . 35) we deduce successively

$$\int_1^x \{\mathfrak{D}_\alpha(\tau)\}^2 d\tau = O(x^{\frac{3}{2}+\alpha}),$$

$$\sum_1^n \{\mathfrak{D}_\alpha(\nu)\}^2 = O(n^{\frac{3}{2}+\alpha}),$$

$$\sum_1^n \{\mathfrak{D}(\nu)\}^2 = O(n^{\frac{3}{2}+\alpha}),$$

$$\sum_1^n \{\Delta(\nu)\}^2 = O(n^{\frac{3}{2}+\alpha}),$$

$$\int_1^x \{\Delta(\tau)\}^2 d\tau = O(x^{\frac{3}{2}+\alpha}).$$

THE UNIFORM CONVERGENCE OF A CERTAIN CLASS OF TRIGONOMETRICAL SERIES

By T. W. CHAUNDY *and* A. E. JOLLIFFE.

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1. It is well-known that the series $\Sigma a_n \sin n\theta$, where (a_n) is a sequence of positive numbers decreasing steadily to zero, is convergent for all real values of θ. Since, moreover, by Abel's lemma,

$$|a_n \sin n\theta + \ldots + a_p \sin p\theta| < |a_n \operatorname{cosec} \tfrac{1}{2}\theta|,$$

the series is uniformly convergent throughout any interval which does not include zero or a multiple of 2π.

We shall show that the condition $na_n \to 0$ is both necessary and sufficient to secure the uniform convergence of the series throughout any interval whatever.

Since the series is unaltered, if we substitute $2k\pi+\theta$ for θ, and changed in sign only, if we substitute $2k\pi-\theta$ for θ, k being any integer, we can without loss of generality suppose θ to lie in the interval $(0, \pi)$.

2. We shall prove the two following propositions :—

(1) If p is taken sufficiently great, we can find a value of θ in the interval $(0, \pi)$ such that

$$|a_n \sin n\theta + \ldots + a_p \sin p\theta| > pa_p/\pi.$$

(2) If ra_r, where r is greater than or equal to n, is never greater than M, then

$$|a_n \sin n\theta + \ldots + a_p \sin p\theta| < (\pi+1)M,$$

for all values of p and θ.

The condition that the series $\Sigma a_n \sin n\theta$ should be uniformly conver-

gent throughout any interval is that, for any ϵ, we can find an integer N_ϵ, such that

$$| a_n \sin n\theta + \ldots + a_p \sin p\theta | < \epsilon,$$

for all values of θ in the interval, and for all values of p greater than n, provided only that $n \geqslant N_\epsilon$.

The first of the above propositions shows that, unless $na_n \to 0$, the interval of uniform convergence cannot include zero, or consequently any multiple of 2π.

The second shows that, if $na_n \to 0$, then there is uniform convergence throughout any interval whatever.

3. To prove the first proposition :—

We have

$$\sin n\theta + \ldots + \sin p\theta = \tfrac{1}{2} \{ \cos(n-\tfrac{1}{2})\theta - \cos(p+\tfrac{1}{2})\theta \} \operatorname{cosec} \tfrac{1}{2}\theta.$$

If we give θ the value $\pi/(2p+1)$, this sum becomes $\frac{1}{2} \cos(n-\frac{1}{2})\theta \operatorname{cosec} \frac{1}{2}\theta$, which is greater than $\{1-\frac{1}{2}(n-\frac{1}{2})^2 \theta^2\}/\theta$, which for this value of θ is certainly greater than p/π, if p is greater than $2n-1$.

Since $\sin n\theta$, ..., $\sin p\theta$ are all positive, and $a_n \geqslant a_{n+1} \ldots \geqslant a_p$,

$$a_n \sin n\theta + \ldots + a_p \sin p\theta > a_p (\sin n\theta + \ldots + \sin p\theta) > p a_p / \pi.$$

Hence the first proposition is proved.

4. To prove the second proposition :—

Since $(\sin \phi)/\phi$ decreases steadily as ϕ increases from 0 to $\pi/2$, it follows that, if $0 < \theta < \pi$, $\operatorname{cosec} \frac{1}{2}\theta < \pi/\theta$.

By Abel's lemma,

$$| a_n \sin n\theta + \ldots + a_p \sin p\theta | < a_n \operatorname{cosec} \tfrac{1}{2}\theta < \pi a_n/\theta < n a_n,$$

if $\theta \geqslant \pi/n$.

If $\theta < \pi/n$, let $\theta = \pi/(m+\alpha)$, where m is an integer $\geqslant n$, and $0 < \alpha \leqslant 1$.

If $p \leqslant m$, then no one of $n\theta$, $(n+1)\theta$, ..., $p\theta$ exceeds π, so that

$$\begin{aligned} a_n \sin n\theta + \ldots + a_p \sin p\theta &< \theta(n a_n + \ldots + p a_p) \\ &< (p-n+1) M\pi/(m+\alpha) < \pi M. \end{aligned}$$

If $p > m$,

$$| a_n \sin n\theta + \ldots + a_p \sin p\theta |$$

$$\leqslant | a_n \sin n\theta + \ldots + a_m \sin m\theta | + | a_{m+1} \sin (m+1)\theta + \ldots + a_p \sin p\theta |$$

$$< \pi M + a_{m+1} \operatorname{cosec} \tfrac{1}{2}\theta < \pi M + (m+a)\, a_{m+1} < (\pi+1)\, M.$$

Hence the second proposition is proved, and the result stated is established.

Note.—This paper replaces one communicated to the Society (January 13th, 1916) by Mr. T. W. Chaundy: he then showed that $na_n \to 0$ is a necessary condition for uniform convergence, and that $na_n \to 0$ *steadily* is a sufficient condition. But, as will be seen from the foregoing investigation, the condition $na_n \to 0$ is sufficient by itself.

ON SEMINVARIANTS OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

By E. B. STOUFFER.

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LET us consider the system of linear homogeneous differential equations

$$y_i^{(m)} + \sum_{l=0}^{m-1} \sum_{k=1}^{n} \binom{m}{l} p_{ikl} y_k^{(l)} = 0 \quad (i = 1, 2, \ldots, n), \tag{1}$$

where

$$y_k^{(l)} = \frac{d^l y_k}{dx^l},$$

and where p_{ikl} are functions of the independent variable x. Wilczynski* has shown that the most general transformation of the dependent variables which converts this system into another of the same form is given by the equations

$$y_\kappa = \sum_{\lambda=1}^{n} a_{\kappa\lambda}(x)\, \eta_\lambda \quad (\kappa = 1, 2, \ldots, n), \tag{2}$$

where $a_{\kappa\lambda}$ are arbitrary functions of x, and where the determinant

$$| a_{\kappa\lambda} | \quad (\kappa, \lambda = 1, 2, \ldots, n)$$

does not vanish identically. A function of the coefficients of (1) and of their derivatives which has the same value for (1) as for any system derived from (1) by transformations of the form (2) is called a seminvariant.

The transformations (2) form an infinite continuous group which, according to Lie's terminology, is defined by differential equations. Lie† has shown that the invariants of such a group may always be determined as the solutions of a complete system of partial differential equations which is obtained by equating to zero the symbol of the most general infinitesimal transformation of the group.

* Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, 1906, Chap. I.

† Lie, *Mathematische Annalen*, 1884, Vol. XXIV, p. 537.

Wilczynski* has calculated the seminvariants for the case $m = 2$, $n = 2$ by actually solving the systems of partial differential equations which arise. In the present paper the seminvariants for the system (1) are calculated under the single limitation $m = 2$. In order to do this, it suffices to solve a few simple systems of partial differential equations. The process of solving the more complicated systems is avoided by the use of certain operators.

The system (1) under the limitation $m = 2$, $n = 2$ leads to the study of non-developable ruled surfaces in ordinary space. Under the single limitation $m = 2$, it leads to the study of spreads generated by a single infinity of $n-1$ flats in space of $2n-1$ dimensions.

Let us write the system (1) for $m = 2$ in the form

$$(A) \qquad y_i'' + \sum_{k=1}^{n} (2p_{ik}y_k' + q_{ik}y_k) = 0 \quad (i = 1, 2, \ldots, n).$$

The infinitesimal transformations of the coefficients of (A) are given by the equations†

$$(3) \qquad \begin{cases} \dfrac{\delta p_{\lambda\mu}}{\delta t} = \sum\limits_{\kappa=1}^{n} (\phi_{\kappa\mu}\, p_{\lambda\kappa} - \phi_{\lambda\kappa}\, p_{\kappa\mu}) + \phi'_{\lambda\mu}, \\ \dfrac{\delta q_{\lambda\mu}}{\delta t} = \sum\limits_{\kappa=1}^{n} (\phi_{\kappa\mu}\, q_{\lambda\kappa} - \phi_{\lambda\kappa} q_{\kappa\mu} + 2\phi'_{\kappa\mu}\, p_{\lambda\kappa}) + \phi''_{\lambda\mu}. \end{cases}$$

The infinitesimal transformations of the derivatives of the coefficients can be obtained directly from these equations by differentiation.

The system of partial differential equations which any seminvariant involving only the variables p_{rs}, p'_{rs}, and q_{rs}, must satisfy is easily seen from (3) to be

$$(4) \qquad \left\{ \begin{array}{l} (a) \ \dfrac{\partial f}{\partial p'_{rs}} + \dfrac{\partial f}{\partial q_{rs}} = 0 \\ (b) \ \dfrac{\partial f}{\partial p_{rs}} + \sum\limits_{\lambda=1}^{n} \left(p_{\lambda r} \dfrac{\partial f}{\partial p'_{\lambda s}} - p_{s\lambda} \dfrac{\partial f}{\partial p'_{r\lambda}} + 2p_{\lambda r} \dfrac{\partial f}{\partial q_{\lambda s}} \right) = 0 \\ (c) \ \sum\limits_{\lambda=1}^{n} \left(p_{\lambda r} \dfrac{\partial f}{\partial p_{\lambda s}} - p_{s\lambda} \dfrac{\partial f}{\partial p_{r\lambda}} + p'_{\lambda r} \dfrac{\partial f}{\partial p'_{\lambda s}} - p'_{s\lambda} \dfrac{\partial f}{\partial p'_{r\lambda}} + q_{\lambda r} \dfrac{\partial f}{\partial q_{\lambda s}} - q_{s\lambda} \dfrac{\partial f}{\partial q_{r\lambda}} \right) = 0 \end{array} \right\}$$

$$(r, s = 1, 2, \ldots, n).$$

It is not difficult to verify that this is a complete system of equations. The sets (a) and (b) of $2n^2$ independent equations involve $3n^2$ variables.

* Wilczynski, *loc. cit.*, pp. 95–101.

† *Ibid.*, p. 95.

The n^2 solutions are easily found to be

$$u_{ik} = p'_{ik} - q_{ik} + \sum_{j=1}^{n} p_{ij} p_{jk} \quad (i, k = 1, 2, \ldots, n). \tag{5}$$

It remains to find all independent functions of u_{ik} which satisfy (c). One such function is the determinant

$$I = \begin{vmatrix} u_{11}, & u_{12}, & \ldots, & u_{1n} \\ u_{21}, & u_{22}, & \ldots, & u_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{n1}, & u_{n2}, & \ldots, & u_{nn} \end{vmatrix}. \tag{6}$$

To prove this it is only necessary to show that the infinitesimal transformation of I is zero. From (3) the infinitesimal transformations of u_{ik} are at once found to be

$$\frac{\delta u_{ik}}{\delta t} = \sum_{j=1}^{n} (\phi_{jk} u_{ij} - \phi_{ij} u_{jk}). \tag{7}$$

The substitution from (7) into (6) gives

$$\frac{\delta I}{\delta t} = \sum_{k=1}^{n} \begin{vmatrix} u_{11}, & u_{12}, & \ldots, & \sum_{j=1}^{n} (\phi_{jk} u_{1j} - \phi_{1j} u_{jk}), & \ldots, & u_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{l1}, & u_{l2}, & \ldots, & \sum_{j=1}^{n} (\phi_{jk} u_{l} - \phi_{lj} u_{jk}), & \ldots, & u_{ln} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{n1}, & u_{n2}, & \ldots, & \sum_{j=1}^{n} (\phi_{jk} u_{nj} - \phi_{nj} u_{jk}), & \ldots, & u_{nn} \end{vmatrix},$$

or $$\frac{\delta I}{\delta t} = \sum_{k=1}^{n} \phi_{kk} I - \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{j=1}^{n} \phi_{lj} u_{jk} U_{lk},$$

where U_{lk} is the algebraic minor of u_{lk} in I. When the terms in the last summation which add up to zero are dropped, we have

$$\frac{\delta I}{\delta t} = \sum_{k=1}^{n} \phi_{kk} I - \sum_{l=1}^{n} \phi_{ll} I = 0.$$

Therefore I is a seminvariant.

Other seminvariants are easily obtained from I. In fact the operator

$$D_u = \sum_{i=1}^{n} \frac{\partial}{\partial u_{ii}}$$

transforms any seminvariant into a seminvariant. To prove this we shall

need the finite transformations of u_{ik}. The finite transformations of p_{ik} and q_{ik}* immediately show the transformations of u_{ik} to be

$$|a|\bar{u}_{ik} = \sum_{l=1}^{n}\sum_{j=1}^{n} A_{jl}a_{lk}u_{jl} \quad (i, k = 1, 2, \ldots, n), \tag{8}$$

where A_{ji} represents the minor of u_{ji} in the determinant $|a|$ of the transformation (2). Suppose now that ϕ is any seminvariant involving u_{ik}, so that

$$\phi(u_{ik}) = \phi(\bar{u}_{ik}) = \bar{\phi}(\bar{u}_{ik}).$$

Since

$$\frac{\partial\phi}{\partial u_{ii}} = \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{A_{ij}a_{ik}}{|a|}\frac{\partial\bar{\phi}}{\partial\bar{u}_{jk}},$$

we have the relation $\sum_{i=1}^{n}\frac{\partial\phi}{\partial u_{ii}} = \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{A_{ij}a_{ik}}{|a|}\frac{\partial\bar{\phi}}{\partial\bar{u}_{jk}}$.

If j and k are held constant while i is varied, the above equation reduces to

$$\sum_{i=1}^{n}\frac{\partial\phi}{\partial u_{ii}} = \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{A_{ik}a_{ik}}{|a|}\frac{\partial\bar{\phi}}{\partial\bar{u}_{kk}}.$$

Again, if j is held constant while i is varied, the equation becomes

$$\sum_{i=1}^{n}\frac{\partial\phi}{\partial u_{ii}} = \sum_{k=1}^{n}\frac{\partial\bar{\phi}}{\partial\bar{u}_{kk}}.$$

Therefore $D_u\phi$ is again a seminvariant unless it reduces to a constant. Evidently the operation with D_u can be repeated. The application of D_u r times to a seminvariant ϕ will be indicated by

$$D_u^r\phi = \frac{1}{r!}\sum_{i=1}^{n}\left(\frac{\partial}{\partial u_{ii}}\right)^r\phi.$$

The factor $1/r!$ is introduced to avoid unnecessary numerical coefficients.

Successive operations by D_u on I evidently give seminvariants, each consisting of the sum of the principal minors of a given order of I. That the n seminvariants thus obtained are independent is seen by putting

$$u_{ik} = 0 \quad (i \neq k).$$

For, they then become the n elementary symmetric functions of u_{ii}.

If the variables in (*c*) are changed to u_{ik} those equations assume the simple form

$$\sum_{j=1}^{n}\left(u_{jr}\frac{\partial f}{\partial u_{js}} - u_{sj}\frac{\partial f}{\partial u_{rj}}\right) = 0 \quad (r, s = 1, 2, \ldots, n). \tag{9}$$

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* Wilczynski, *loc. cit.*, Chap. IV, equation (9).

This system contains n^2 equations and n^2 variables, but the fact that there are n independent solutions of it shows that there must be at least n relations between its equations. It is only necessary to show that n^2-n of these equations are independent to prove that there are no more seminvariants involving only p_{ik}, p'_{ik}, and q_{ik}. Let us consider the equations of (9) for which $r \neq s$. Since the coefficient of $\partial f/\partial u_{rs}$ in (9) is always $u_{rr}-u_{ss}$ it is evident that the determinant of the coefficients can be so arranged as to have all the terms of its principal diagonal of the form $u_{\alpha\alpha}-u_{\beta\beta}$. Moreover, no other element of the determinant contains an u with like subscripts. Therefore the determinant is not identically equal to zero and the n^2-n equations are independent.

We have now proved the following theorem :—

All the independent seminvariants involving only p_{ik}, p'_{ik}, and q_{ik} are given by

$$I^{(r)} = D_u^r I \quad (r = 0, 1, \ldots, n-1).$$

The system of partial differential equations which a seminvariant involving the next higher derivatives of p_{ik} and q_{ik} must satisfy is obtained in the same way as in the previous case. It contains four sets of n^2 equations each, and is a complete system. All equations in three of the sets are easily seen to be independent since they can be so arranged that each equation contains a partial derivative not appearing in those which precede. Since there are $5n^2$ variables in these three sets of equations there must be just $2n^2$ solutions. These are easily seen to be the independent quantities

$$u_{ik},\ v_{ik} = u'_{ik} + \sum_{j=1}^{n} (p_{ij}u_{jk} - p_{jk}u_{ij}) \quad (i, k = 1, 2, \ldots, n). \tag{10}$$

It is verified by direct substitution that v_{ik} is cogredient with u_{ik} under the transformation (2). Therefore if $\phi(u_{ik})$ is a seminvariant, $\phi(v_{ik})$ will also be a seminvariant. Since the transformations of u_{ik} are linear in u_{ik}, $u_{ik}+\lambda v_{ik}$ will also be cogredient with u_{ik} where λ is an arbitrary constant, and $\phi(u_{ik}+\lambda v_{ik})$ will be a seminvariant for all values of λ. Thus, if $\bar{u}_{ik}$ and $\bar{v}_{ik}$ denote the transforms of u_{ik} and v_{ik} respectively, there results the equation

$$\phi(u_{ik}+\lambda v_{ik}) = \bar{\phi}(\bar{u}_{ik}+\lambda \bar{v}_{ik}),$$

where $\bar{\phi}(\bar{u}_{ik}+\lambda\bar{v}_{ik})$ represents the same function of $\bar{u}_{ik}$ and $\bar{v}_{ik}$ as is $\phi(u_{ik}+\lambda v_{ik})$ of u_{ik} and v_{ik}. On expanding both sides of the above equation and equating like powers of λ, we have

$$\sum_{i=1}^{n}\sum_{k=1}^{n} v_{ik}\frac{\partial\phi}{\partial u_{ik}} = \sum_{i=1}^{n}\sum_{k=1}^{n} \bar{v}_{ik}\frac{\partial\bar{\phi}}{\partial\bar{u}_{ik}}.$$

Therefore $$D_{uv} = \sum_{i=1}^{n} \sum_{k=1}^{n} v_{ik} \frac{\partial}{\partial u_{ik}}$$

is an invariantive operator and transforms a seminvariant involving u_{ik} into one involving p''_{ik}, q'_{ik}. The result of operating s times with D_{uv} upon $I^{(r)}$ will be denoted by $I^{(rs)}$, *i.e.*,

$$D^s_{uv} I^{(r)} = \frac{1}{s!} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(v_{ij} \frac{\partial}{\partial u_{ij}} \right) I^{(r)} = I^{(rs)}.$$

It should be observed that $I^{(r1)}$ is the same as the derivative of $I^{(r)}$ with respect to x.

Let us call $U_{\alpha\beta}$ the algebraic minor of $u_{\alpha\beta}$ in I. It is easy to verify that $U_{\alpha\beta}$ is transformed cogrediently with $u_{\beta\alpha}$. In fact, the direct substitution from (7) gives

$$\frac{\delta U_{\alpha\beta}}{\delta t} = \sum_{j=1}^{n} (\phi_{j\alpha} U_{j\beta} - \phi_{\beta j} U_{\alpha j}).$$

Therefore the operator $$D_{vU} = \sum_{i=1}^{n} \sum_{j=1}^{n} U_{ji} \frac{\partial}{\partial v_{ij}}$$

must transform a seminvariant into a seminvariant. The result of operating once upon $I^{(rs)}$ with D_{vU} will be denoted by $I_1^{(rs)}$. We shall have no occasion to use this operator more than once.

The fourth set of the system of equations which the functions of u_{ik} and v_{ik} must satisfy, may be written in terms of the variables u_{ik} and v_{ik}. The equations then become

$$(11) \qquad R_{rs} f = \sum_{j=1}^{n} \left(u_{jr} \frac{\partial f}{\partial u_{js}} - u_{sj} \frac{\partial f}{\partial u_{rj}} + v_{jr} \frac{\partial f}{\partial v_{js}} - v_{sj} \frac{\partial f}{\partial v_{rj}} \right) = 0$$

$$(r, s = 1, 2, \ldots, n).$$

It is obvious that there is one relation $\sum_{r=1}^{n} R_{rr} f \equiv 0$ between them. If u_{ik} $(i \neq k)$ is put equal to zero it is seen from the matrix of the equations that this is the only relation. There are therefore just n^2+1 independent solutions. We have found $\frac{n^2+3n}{2}$ solutions of the form

$$I^{(rs)} \quad (r < n,\ r+s \leqslant n),$$

and $\frac{n^2-3n+2}{2}$ of the form $$I_1^{(rs)} \quad (s > 2,\ r+s \leqslant n).$$

It remains to prove these

$$\frac{n^2+3n}{2} + \frac{n^2-3n+2}{2} = n^2+1$$

seminvariants independent. For purpose of simplification we shall put u_{ik} $(i \neq k)$, and prove the seminvariants independent even in this special case.

Let us consider the functional matrix of these n^2+1 functions of the independent variables u_{ii}, v_{ik} $(i, k = 1, 2, \ldots, n)$. The only non-vanishing terms in the partial derivatives of $I^{(r0)} = I^{(r)}$ and $I^{(r1)}$ appear in $\frac{\partial(I^{(r,0)}, I^{(r,1)})}{\partial(u_{ii}, v_{ii})}$. But this determinant of order $2n$ is precisely the square of the determinant of order n which arises in proving $I^{(r)}$ independent. Therefore it is different from zero and it is only necessary to show the existence of a non-vanishing minor of order n^2-2n+1 in the columns obtained from derivatives with respect to v_{ik} $(i \neq k)$. It will be shown that there exists a determinant of order n^2-2n+1 in these columns which has the coefficient of $u_{nn}^{\alpha} v_{nn}^{\beta}$ different from zero. By u_{nn}^{α} is meant the highest power of u_{nn} in the determinant, and by v_{nn}^{β} the highest power of v_{nn} multiplying u_{nn}^{α}. Let us denote by $\mathfrak{J}^{(rs)}$ $(r+s < n,\ r < n-1)$, and $\mathfrak{J}_1^{(rs)}$ $(r+s < n,\ s > 2)$ the seminvariants in the case when there are $n-1$ equations in (A) which correspond to $I^{(rs)}$ and $I_1^{(rs)}$ respectively. Then, in the seminvariants $I^{(rs)}$ $(r+s < n,\ s > 1)$, u_{nn} always occurs linearly with $\mathfrak{J}^{(rs)}$ as a coefficient. Likewise in $I_1^{(rs)}$ $(r+s < n,\ s > 2)$ the factor $u_{nn} v_{nn}$ occurs linearly with $\mathfrak{J}_1^{(rs)}$ as a coefficient.

Let us now consider that part of the functional matrix made up of the partial derivatives with respect to v_{ik} $(i \neq k)$ of $I^{(rs)}$ $(r+s \leqslant n,\ s > 1)$ and $I_1^{(rs)}$ $(r+s \leqslant n,\ s > 2)$. Every term obtained by taking partial derivatives of $I^{(rs)}$ $(r+s < n,\ s > 1)$ with respect to v_{ik} $(i, k < n)$ will contain u_{nn}, while those with respect to v_{in} or v_{ni} will not contain u_{nn}. For $I^{(rs)}$ $(r+s = n,\ s > 1)$ neither set will contain u_{nn}. Again, for $I_1^{(rs)}$ $(r+s < n,\ s > 2)$, the first set will contain $u_{nn} v_{nn}$, while the second set will not, and for $I_1^{(rs)}$ $(r+s = n,\ s > 2)$ neither set will contain $u_{nn} v_{nn}$. Therefore the coefficient of $u_{nn}^{\alpha} v_{nn}^{\beta}$ will have for one factor some determinant of maximum order of the functional matrix of

$$\mathfrak{J}^{(rs)}\ (r+s < n,\ s > 1) \quad \text{and} \quad \mathfrak{J}_1^{(rs)}\ (r+s < n,\ s > 2)$$

with respect to the variables v_{ik} $(i, k < n)$. But such a determinant must be one factor of the corresponding determinant of maximum order of the functional matrix for the case of $n-1$ equations in (A). If we assume our seminvariants independent for $n-1$ equations, this determinant must be different from zero.

We shall now show that the remaining matrix which has one more column than it has rows contains a non-vanishing determinant of maxi-

mum order. Let us consider the determinant of the above matrix obtained by omitting the column arising from partial derivatives with respect to one variable, say $v_{n,\,n-1}$. In this determinant let us put all v_{ik} equal to zero, except $v_{21}, v_{32}, v_{43}, \ldots, v_{n-1,\,n-2}, v_{n,\,n-1}, v_{1n}$. It is then easily seen that the determinant consists of the product of one non-vanishing determinant of the first order and $n-2$ non-vanishing determinants of the second order.

We have now shown that our n^2+1 seminvariants are independent, provided that the corresponding seminvariants in the case of $n-1$ equations in (A) are independent. Wilczynski* has found the complete system of seminvariants in the case of two equations, and has shown that the seminvariants obtained from the above set by putting $n=2$ are independent. Moreover, it is easy to prove that the seminvariants are independent if $n=3$. Thus we are led to the conclusion that our seminvariants are independent. We now have the following theorem:—

All the independent seminvariants involving only $p_{ik}, p'_{ik}, p''_{ik}, q_{ik}, q'_{ik}$ *are given by* $I^{(rs)}$ $(r<n,\ r+s\leqslant n)$ *and* $I_1^{(rs)}$ $(s>2,\ r+s\leqslant n)$.

The fact that the column obtained from partial derivatives with respect to $v_{n,\,n-1}$ can be omitted in the matrix shows that none of the columns obtained from partial derivatives with respect to $v_{\alpha\beta}$ $(\alpha=\beta+1)$ need be used.

Let us continue our search for seminvariants by finding those containing the next higher derivatives of p_{ik} and q_{ik}. In this case there are $2n^2$ more variables and n^2 more independent equations than in the previous case. These equations form a complete system, so that there are n^2 more seminvariants. Of these n^2-n+1 can be taken to be the derivatives with respect to x of the seminvariants already found. The quantities w_{ik} formed from v_{ik} in the same way as v_{ik} were formed from u_{ik} must be cogredient with u_{ik} and v_{ik}. Therefore

$$J=\begin{vmatrix} w_{11}, & w_{12}, & \ldots, & w_{1n} \\ w_{21}, & w_{22}, & \ldots, & w_{2n} \\ \ldots & \ldots & & \ldots \\ w_{n1}, & w_{n2}, & \ldots, & w_{nn} \end{vmatrix} \tag{12}$$

is a seminvariant, and

$$\sum_{i=1}^{n}\frac{\partial}{\partial w_{ii}}=D_w \quad \text{and} \quad \sum_{i=1}^{n}\sum_{k=1}^{n} w_{ik}\frac{\partial}{\partial v_{ik}}=D_{vw}$$

* Wilczynski, *loc. cit.*, pp. 95–101.

are invariant operators. Thus operating on J $n-2$ times with D_w, we have $n-1$ independent seminvariants involving only w_{ik}.

We shall prefer to obtain the n^2-n+1 seminvariants mentioned above otherwise than by differentiation. If we operate once on the seminvariants containing v_{ik} with D_{vw} just n^2-n+1 seminvariants linear in w_{ik} are obtained. It remains to show that these seminvariants together with J and the $n-2$ seminvariants obtained by operating on J $n-2$ times with D_w, make up the n^2 additional independent seminvariants arising in this case.

Let us put $u_{ik} = 0$ $(i \neq k)$ and consider the functional matrix of the $2n^2+1$ seminvariants involving u_{ii}, v_{ik}, w_{ik}. That part in which derivatives of seminvariants containing only u_{ii} and v_{ik} with respect to those variables are taken, will be the same as in the previous case, and therefore will contain a determinant of maximum order which is different from zero. Again, that part obtained by taking partial derivatives of the n seminvariants linear in w_{ik} and not containing v_{ik} will have terms different from zero only for partial derivatives with respect to the n w_{ii}. Since

$$\frac{\partial}{\partial w_{ij}} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \frac{\partial}{\partial v_{ij}} \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} \frac{\partial}{\partial u_{ij}} I^{(r)} = \frac{\partial}{\partial v_{ij}} \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} \frac{\partial}{\partial u_{ij}} I^{(r)},$$

this determinant is the same as that for the functional matrix of the seminvariants containing v_{ik} linearly in the previous case. It is therefore not identically equal to zero.

It remains to prove that the functional matrix of the remaining seminvariants with respect to w_{ik} $(i \neq k)$ is different from zero. From the way the terms are obtained it is seen that the part containing partial derivatives of seminvariants linear in w_{ik} will be the same as that part containing partial derivatives with respect to v_{ik} $(i \neq k)$ in the previous case. Hence there is a non-vanishing determinant obtained by omitting columns arising from partial derivatives with respect to w_{21}, w_{32}, ..., $w_{n,\,n-1}$. It is now only necessary to show that in the functional matrix of

$$D_w^t J \quad (t = 0, 1, \ldots, n-2),$$

with respect to w_{ik}, the determinant made up of these columns does not vanish when all other terms are made zero by a proper choice of w_{ik}. As none of the previous factors into which the determinant of the general functional matrix has been broken up contains a w_{ik}, those factors will not be affected by this choice. Let us therefore put all w_{ik} $(i \neq k)$ equal to zero except w_{12}, w_{23}, ..., $w_{n-1,\,n}$. Then in the functional matrix of

$$L_w^t J \quad (t = 0, 1, \ldots, n-2),$$

with respect to w_{ik}, all terms are zero except those arising from

$$\frac{\partial (D^t_{vv} J)}{\partial (w_{21},\ w_{32},\ \ldots,\ w_{n,\ n-1})}.$$

This determinant of order $n-1$ consists of $\pm\, w_{n-1,\,n}\, w_{11} w_{22} \ldots w_{n-2,\,n-2}$ multiplied by the corresponding determinant for the case of $n-1$ equations in (A). If therefore this determinant is different from zero for one value of n, it is also for the next. But it is obviously different from zero for $n = 2$ and $n = 3$. Therefore the general functional matrix possesses a non-vanishing determinant of maximum order. We have now proved the following theorem :—

All seminvariants involving only p_{ik}, p'_{ik}, p''_{ik}, q_{ik}, q'_{ik}, *are functions of*

$$(13) \qquad \begin{cases} I^{(rs)}\ (r < n,\ r+s \leqslant n), \quad I_1^{(rs)}\ (s > 2,\ r+s \leqslant n), \\ D_{vw} I^{(rs)}\ (s > 0,\ r+s \leqslant n), \quad D^t_{vv} J\ (t = 0,\ 1,\ \ldots,\ n-2). \end{cases}$$

The seminvariants involving the next higher derivatives of p_{ik} and q_{ik} must satisfy a complete system of partial differential equations containing n^2 more equations, and $2n^2$ more variables than in the previous case. There are therefore n^2 more independent seminvariants. But the derivatives of the seminvariants (13) give this number of independent seminvariants. In fact a continuation of this process shows that all independent seminvariants involving higher derivatives of p_{ik} and q_{ik} can be found by differentiation of those already found. Therefore the following theorem has been proved :—

The complete system of seminvariants of (A) *consists of* (13), *and seminvariants obtained from* (13) *by successive differentiation.*

A PROBLEM OF ANALYSIS SITUS

By G. N. WATSON.

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1. The theorem, which is due to Jordan,* that a regular closed curve possesses an interior and an exterior, is well known and has far reaching consequences in the theory of functions of a complex variable as developed by mathematicians on the lines first indicated by Cauchy.

It will be recalled that, when a regular closed curve is given, if Q is a fixed point not on the curve, while P is a variable point of the curve, then, if θ be the angle between QP and the x-axis, Jordan's theorem is, effectively, that θ changes by 2π or by zero as P describes the curve, and that in the former case Q is called an interior point of the curve, while in the latter case Q is called an exterior point of the curve.

In this paper, I propose to investigate a theorem somewhat similar to Jordan's theorem, namely that, when a simple closed curve with a *continuously turning tangent*† is given, if ψ be the angle between the tangent at a variable point P of the curve and the x-axis, *then* ψ *changes by* 2π *as* P *describes the curve.*

This theorem can be employed‡ in proving Macdonald's theorem concerning a relation between the number of zeros of an analytic function $f(z)$, and the number of zeros of its derivate $f'(z)$ in a prescribed region of the plane of the complex variable z.

2. As the terminology of *Analysis Situs* is not fixed, and some of its

* *Cours d'Analyse*, t. I, pp. 96–103. The substance of a fairly simple proof (due to Ames, *American Journal*, Vol. XXVII, pp. 343–380) is given in my tract, "Complex Integration and Cauchy's Theorem" (*Camb. Math. Tracts*, No. 15).

† The precise meaning to be attributed to this phrase will be found below in § 3; the case in which the tangent has a finite number of abrupt changes of direction is discussed in §§ 8, 9.

‡ Whittaker and Watson, *Modern Analysis*, p. 121. Macdonald stated and proved his theorem in the *Proceedings*, Vol. XXIX, p. 576. His proof is of a different nature from that given in the *Modern Analysis*, and hence it does not raise the question discussed in this paper.

theorems are little known, it will be convenient to give a summary of the definitions and theorems* which are preliminary to the analysis employed in this paper.

Let $$x = x(t), \quad y = y(t),$$

where $x(t)$, $y(t)$ are real continuous one-valued functions of the real parameter t for all values of t such that $t_0 \leqslant t \leqslant T$. The functions are to be such that they do not assume the same pair of values† for *different* values of t in the range $t_0 < t < T$. Then the set of points (x, y) determined by the values of t in the range $t_0 \leqslant t \leqslant T$ is called a *simple curve* joining the end points (x_0, y_0) and (X, Y), where $x_0, \ldots$ have been written for $x(t_0), \ldots$. If the end points coincide, the curve is said to be *closed.*

A curve defined *either* by the equation $y = f(x)$, $(x_0 \leqslant x \leqslant x_1)$, *or else* by the equation $x = \phi(y)$, $(y_0 \leqslant y \leqslant y_1)$, where f (or ϕ) is a one-valued continuous function of its argument is called an *elementary curve.*

A simple curve is said to be *regular*‡ if it can be divided into a *finite* number of parts, each of which is an elementary curve.

If a simple closed curve is given, and if $T - t_0 = \omega$, then ω is called the *primitive period* associated with the pair of functions $x(t)$, $y(t)$; these functions are then defined for *all* real values of t by the equations

$$x(t+n\omega) = x(t), \quad y(t+n\omega) = y(t),$$

where n is any integer.

If $Q(x_1, y_1)$ is a point not on a given simple closed curve, it can be shewn that it is possible to define a continuous function of t, $\theta(t)$ say, such that

$$\cos\theta(t) = \{x(t)-x_1\}/r, \quad \sin\theta(t) = \{y(t)-y_1\}/r,$$

where $$r = +\sqrt{[\{x(t)-x_1\}^2 + \{y(t)-y_1\}^2]}.$$

It can then be shewn that, if the curve is *regular*,

$$\theta(t+\omega) - \theta(t) = 2n\pi,$$

where n is an integer depending only on (x_1, y_1); this integer is called the *order* of Q with respect to the regular closed curve.

Jordan's theorem obviously is that $n = 0$ or ± 1; if $n = 0$, Q is said to be a point exterior to the given curve, while if $n = \pm 1$, Q is said to be a point of the interior of the curve.

* The reader is referred to Ames' memoir, or to my tract, for a complete account of the theorems now quoted.

† This condition ensures that the curve has no double points.

‡ Ames uses this term in a more restricted sense.

Since the values of t form an ordered set,* the points of a simple curve can be regarded as ordered, the order of points on the curve being the order of the corresponding values of t. Such an ordered set of points is called an *oriented* curve; two oriented curves C_1, C_2 having a common arc σ are said to have the *same orientation* if the points of σ are in the same order, whether σ is regarded as belonging to C_1 or to C_2. If the points are not in the same order, C_1 and C_2 are said to have *opposite orientations*.

If an oriented curve C_1 is given by the equations

$$x = x(t), \quad y = y(t), \quad (t_0 \leqslant t \leqslant T),$$

and if we put $t = -t'$, $x(t) \equiv x'(t')$, $y(t) \equiv y'(t')$, then the point (x', y') traces out an oriented curve C_2 as t' increases from $-T$ to $-t_0$; the curves C_1, C_2 consist of the same set of points but have opposite orientations. It is consequently easily seen that when a regular closed curve is given it is possible to orient it (by suitable choice of parameter) in such a way that the order of interior points is $+1$ (not -1); such an oriented curve is said to be described *counter clockwise* as its parameter increases by the primitive period. It is invariably supposed that this choice of parameter is made.

The following theorem, due to Ames, is of importance in our investigation :—

AMES' THEOREM.†—Let R_1, R_2 be the sets of points formed by the interiors of two regular closed curves C_1, C_2, respectively; and let an arc σ_1 of C_1 coincide in position with an arc σ_2 of C_2. Then (i) if R_1, R_2 have no common point, the orientations of σ_1, σ_2 are opposite; but (ii) if every point of R_1 be a point of R_2, the orientations of σ_1, σ_2 are the same.

3. It is now possible to enunciate the theorem to be proved in this paper.

Let a simple closed curve Γ *be defined by the equations*

$$x = x(t), \quad y = y(t), \quad (t_0 \leqslant t \leqslant T),$$

and let the continuous functions $x(t)$, $y(t)$ *be defined for all real values of* t *outside the range* (t_0, T) *in the customary manner, so that they form a pair of periodic functions which has primitive period* $T - t_0 = \omega$.

* They may be ordered in ascending order of magnitude.

† *Loc. cit.*, p. 362.

Further, let the functions $x(t)$, $y(t)$, thus defined for all real values of t, have continuous differential coefficients with regard to t for all such values of t; it is to be assumed that these differential coefficients do not simultaneously vanish† for any value of t.*

Then (I) *the curve is regular,‡ so that Jordan's theorem is applicable to it :§ and* (II) *it is possible to define a function of t, say $\psi(t)$, by the equations*

$$\cos\psi(t) = \frac{\dot{x}}{+\sqrt{(\dot{x}^2+\dot{y}^2)}}, \qquad \sin\psi(t) = \frac{\dot{y}}{+\sqrt{(\dot{x}^2+\dot{y}^2)}},$$

which is continuous for all real values of t, and is such that

$$\psi(t+\omega)-\psi(t) = 2\pi.$$

4. The method of investigation which will be adopted is substantially "the method of subdividing regions into *suitable* regions" employed in a well known proof of the Heine-Borel theorem and kindred theorems.

We first observe that, since $\dot{x}$ and $\dot{y}$ never simultaneously vanish, and since $\dot{x}$ and $\dot{y}$ are continuous functions of t for all real values of t. there exists‖ a function $\psi(t)$, which may be regarded as a real, one-valued, continuous¶ function of t for all real values of t, and which satisfies the equations

$$\cos\psi(t) = \dot{x}/\sqrt{(\dot{x}^2+\dot{y}^2)}, \quad \sin\psi(t) = \dot{y}/\sqrt{(\dot{x}^2+\dot{y}^2)}.$$

Further, since x and y, *qua* functions of t, have period ω, it follows that $\dot{x}$ and $\dot{y}$ also have period ω, and so

$$\cos\psi(t+\omega) = \cos\psi(t), \quad \sin\psi(t+\omega) = \sin\psi(t);$$

* This is the analytical interpretation of the phrase "continuously turning tangent" employed in § 1; Ames finds it convenient to describe the curve as "smooth" at a point at which $x(t)$ and $y(t)$ have continuous differential coefficients. In future we shall denote differential coefficients with regard to t by dots.

† The removal of this restriction is discussed in § 8.

‡ This result is stated by Ames, who considered that the methods to be employed in a formal proof were so obvious that a proof might be left to the reader; the proof is given here only because it follows in a very few lines from part of the analysis employed in proving (II).

§ Since the results stated in § 2 are true for the curve under discussion, it will be tacitly assumed after the end of § 4 (except in § 10), that the requisite change of parameter is made (if necessary) to ensure that the curve Γ has the conventional orientation; this change will not affect the continuity of the differential coefficients of the coordinates with respect to the parameter; in fact, it will merely change their signs.

‖ Cf. *Modern Analysis*, pp. 538, 539.

¶ The function $\psi(t)$ may be rendered *definite* in any convenient manner, *e.g.*, by choosing $\psi(0)$ to satisfy the inequalities $-\pi < \psi(0) \leqslant \pi$.

and therefore $$\psi(t+\omega)-\psi(t) = 2n\pi,$$

where n is an integer.

Moreover n is constant, *i.e.*, independent of t. For, as t varies continuously, so do $\psi(t)$ and $\psi(t+\omega)$, while n can vary only *per saltus ;* and therefore n is constant.*

Let Γ' be any arc of Γ and let τ be the parameter of any point of Γ'.

Then Γ' belongs to one of the four following classes† according to the values of $\psi(\tau)$ on Γ' :

(A) For *one or more* values of τ, $|\cos\psi(\tau)| \geqslant \cos\frac{1}{10}\pi$, and, for *all* values of τ, $|\cos\psi(\tau)| > \sin\frac{1}{10}\pi$.

(B) For *one or more* values of τ, $|\cos\psi(\tau)| \leqslant \sin\frac{1}{10}\pi$, and, for *all* values of τ, $|\cos\psi(\tau)| < \cos\frac{1}{10}\pi$.

(C) For *all* values of τ, $\sin\frac{1}{10}\pi < |\cos\psi(\tau)| < \cos\frac{1}{10}\pi$.

(D) For *one or more* values of τ, $|\cos\psi(\tau)| \geqslant \cos\frac{1}{10}\pi$, and for *one or more* other values of τ, $|\cos\psi(\tau)| \leqslant \sin\frac{1}{10}\pi$.

Arcs of these types will be denoted by the symbols Γ_A, Γ_B, Γ_C, Γ_D respectively.

It is now easy to prove that Γ is a *regular* curve.

For $\psi(t)$ is a continuous function of t, and, since continuity involves *uniformity* of continuity (Heine's theorem), we can divide the range (t_0, T) into a *finite* number of (closed) subdivisions, such that if t, t' are *any* two values of the parameter which are both in any one subdivision, then $|\psi(t')-\psi(t)|$ is less than, say, $\frac{1}{10}\pi$.

Corresponding to this subdivision of the range (t_0, T), Γ is divided into a finite number of arcs, *which are all of the types* Γ_A, Γ_B, Γ_C [for, on an arc of the type Γ_D, $\psi(t)$ must change by at least as much as $\frac{3}{10}\pi > \frac{1}{10}\pi$].

It will now be shewn that each of the arcs Γ_A and Γ_C is an elementary curve of the type $y = f(x)$; an exactly similar proof (which is left to the reader) will then shew that each of the curves Γ_B and Γ_C is an elementary curve of the type $x = \phi(y)$: where f and ϕ denote continuous functions of their respective arguments.

Take any one of the arcs Γ_A or Γ_C, and let the parameters of its end points be t_1, t_2 $(0 < t_2-t_1 < \omega)$.

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* Cf. *Tract*, p. 6, footnote.

‡ The angle $\frac{1}{10}\pi$ is introduced in the following statements for the sake of definiteness. It could be replaced by any sufficiently small angle.

Then, when $t_1 \leqslant t \leqslant t_2$, we have

$$|\cos\psi(t)| > \sin\tfrac{1}{10}\pi,$$

and so $\dot{x}(t)$ is never zero and is therefore one-signed in the range (t_1, t_2); therefore $x(t)$ is monotonic (in the strict sense) in this range, and it is also continuous. Hence, writing $x(t_1) \equiv x_1$, $x(t_2) \equiv x_2$, we see that, in the range (x_1, x_2), t is a continuous function* of x; and so, since y is a continuous function of t, y is also a continuous function of x; indeed the differential coefficient of y with regard to x exists and is equal to $\dot{y}/\dot{x}$.

Consequently each of the arcs Γ_A and Γ_C is an elementary curve of the type $y = f(x)$, and, similarly, each of the arcs Γ_B and Γ_C is an elementary curve of the type $x = \phi(y)$. Since the number of arcs $\Gamma_A, \Gamma_B, \Gamma_C$ is finite, the curve Γ is regular; so theorem (I) is completely proved.

5. It is convenient to base the proof of theorem (II), namely that

$$\psi(t+\omega) - \psi(t) = 2\pi,$$

on the following lemma :—

LEMMA.—Let P_0 be a point (which is not an end point) of the elementary curve $y = f(x)$, which forms part of the given regular closed curve Γ; let P_0T and P_0U be the progressive and regressive tangents† at P_0 to Γ; also let P_0Q be drawn parallel to that one of the directions of the y-axis which makes all points of the line P_0Q which are sufficiently near to P_0 (except the end point P_0) lie *inside* Γ.

Then the principal values of the angles $T\hat{P}_0Q$, $Q\hat{P}_0U$ *are both* positive *and definitely* less than π.

[The corresponding form of the lemma for the elementary curve $x = \phi(y)$ merely involves the alteration that P_0Q is to be drawn parallel to one of the directions of the x-axis.]

Let P_0 have parameter t_0 and coordinates (x_0, y_0).

Then it can be shewn that it is possible to choose‡ a positive number r so small that no points of Γ except those on $y = f(x)$ lie inside or on the circle with centre P_0 and radius r.

With such a choice of r, let the point (x_0, y_0+r) be called B, while (x_0, y_0-r) is called B_1.

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* See, *e.g.*, Hardy, *A Course of Pure Mathematics*, § 109.

† So that UP_0T is a straight line inclined $\psi(t) + 2r\pi$ [not $\psi(t) + (2r+1)\pi$] to the x-axis.

‡ Ames' memoir, p. 356; or *Tract*, p. 10.

Let P be any point of Γ, with parameter t, and let the angles which BP and B_1P make with the x-axis be* $\theta(t)$ and $\theta_1(t)$; and let

$$\phi(t) = \theta(t) - \theta_1(t).$$

Now it can be proved that,† *if* $x(t_0+\delta) > x(t_0)$ *for sufficiently small positive values of* δ, then integers n_1, n_2 exist such that

$$\phi(t_0) = (2n_1+1)\pi, \quad \phi(t_0+\delta) > (2n_1+1)\pi,$$

$$\phi(t_0+\omega-\delta) < (2n_2+1)\pi, \quad \phi(t_0+\omega) = (2n_2+1)\pi,$$

and that $\phi(t)$ *is not an odd multiple of* π *when* $t_0+\delta < t < t_0+\omega-\delta$.

Hence $n_2 > n_1$, and so the order of B exceeds the order of B_1; and therefore B is *inside the curve and above* P_0; therefore Q is above P_0 when $\dot{x}(t_0)$ is positive; similarly Q is below P_0 when $\dot{x}(t_0)$ is negative.

The principal value of $T\hat{P}_0Q$ is therefore the principal value of $\frac{1}{2}\pi-\psi(t_0)$ or $\frac{3}{2}\pi-\psi(t_0)$, according as $\dot{x}(t_0) > 0$ or < 0.

Also, according as $\dot{x}(t_0) \gtrless 0$, we have

$$\begin{aligned}\sin T\hat{P}_0Q &= \sin\{\pi \mp \tfrac{1}{2}\pi - \psi(t_0)\} \\ &= \pm\cos\psi(t_0) \\ &= \pm\dot{x}(t_0) \div \sqrt{[\{\dot{x}(t_0)\}^2 + \{\dot{y}(t_0)\}^2]},\end{aligned}$$

and consequently $\sin T\hat{P}_0Q$ is essentially positive. Since the principal value of $T\hat{P}_0Q$ does not exceed π numerically, *it follows* that the principal value of $T\hat{P}_0Q$ is *positive and less than* π.

Similarly the principal value of $Q\hat{P}_0U$ is the principal value of

$$\{\psi(t_0)+\pi\} - \tfrac{1}{2}\pi \quad \text{or of} \quad \{\psi(t_0)+\pi\} - \tfrac{3}{2}\pi,$$

according to the sign of $\dot{x}(t_0)$, and hence the principal value of $Q\hat{P}_0U$ is also positive and less than π; and the lemma is proved.

[The proof for an elementary curve of the type $x = \phi(y)$ is left to the reader.]

6. We shall now investigate the total change in ψ as the point of contact of a tangent of a curve traverses the curve when the curve is of a special type and has a *discontinuously turning tangent.*

* The functional symbols θ and ϕ are used in this sense only in this section of the paper.

† See note ‡ on opposite page.

Suppose that the curve Γ of the type specified in § 3 is given, and that a rectangle R is given* which meets Γ in two points E, F only, the arc† EF of Γ which lies inside R being part (or the whole) of an arc of the type‡ Γ_A; it is supposed that, if t, t' be the parameters of *any* two points of the arc EF, then $|\psi(t)-\psi(t')| < \frac{1}{10}\pi$, that the sides of the rectangle are parallel to the axes, and that E, F lie on the sides m_1, m_2 of the rectangle parallel to the y-axis; let G be the end of m_2 which is inside Γ, and let H be the end of m_1 which is inside Γ.

Then Jordan's theorem is applicable to the curve $EFGH$, and by Ames' theorem (see § 2) the curves $EFGH$ and Γ have the same orientations.§

Let t_1 and t_2 be the parameters of E and F regarded as points of Γ.

Let t' be a parameter for the curve $EFGH$ and ω' the associated period; and let $\psi'(t')$ be the angle between the tangent and the x-axis. It is obviously legitimate to take $t' \equiv t$ when $t_1 \leqslant t' \leqslant t_2$, and also

$$\psi'(t') = \psi(t)$$

in this range.

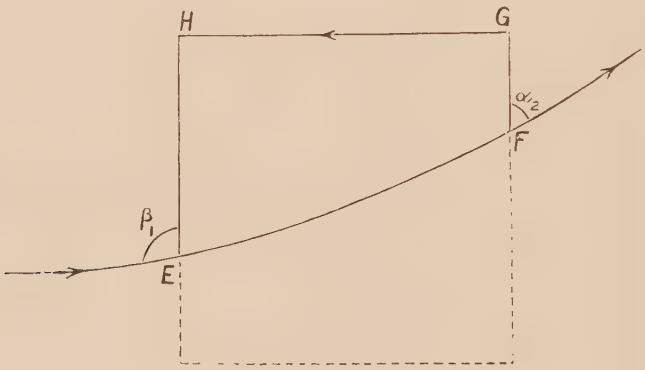

FIG. 1.

Now *define* the changes of ψ' at E, F, G, H to be the *principal values* of the respective angles. Let α_2, β_1 be the changes of ψ' at F and E; then α_2 and β_1 are positive and less than π, by § 5; also the changes of ψ' at G and H are $+\frac{1}{2}\pi$ (not $-\frac{1}{2}\pi$) by a proof similar to that of § 5.

---

* Methods of constructing such rectangles are given in § 7.

† It is supposed that this arc EF is given the same orientation as Γ.

‡ The investigation for an arc of the type Γ_B is similar to that for Γ_A, and is consequently left to the reader.

§ The proof that every point inside $EFGH$ is also inside Γ is left to the reader.

Now, as in § 4, $\psi'(t'+\omega')-\psi'(t')=2n'\pi$,

where n' is constant for values of t' in the range (t_1, t_2).

Now, if t_3', t_4' be the parameter of G and H, we have

$$\begin{aligned}\psi'(t'+\omega')-\psi'(t') = &\{\psi'(t'+\omega')-\psi'(t_4'+0)\}+\{\psi'(t_4'+0)-\psi'(t_4'-0)\}\\ &+\{\psi'(t_4'-0)-\psi'(t_3'+0)\}+\{\psi'(t_3'+0)-\psi'(t_3'-0)\}\\ &+\{\psi'(t_3'-0)-\psi'(t_2'+0)\}+\{\psi'(t_2'+0)-\psi'(t_2'-0)\}\\ &+\psi'(t_2'-0)-\psi'(t').\end{aligned}$$

Make $t'\to t_1+0$, and we get*

$$\psi'(t'+\omega')-\psi'(t')=\beta_1+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi+\alpha_2-\{\psi'(t_2-0)-\psi(t_1+0)\}.$$

The expression on the right lies between $\pi-\frac{1}{10}\pi$ and $3\pi+\frac{1}{10}\pi$; and so $2n'\pi=2\pi$, *i.e.*, $n'=1$.

Therefore $2\pi=\beta_1+\pi+\alpha_2+\psi(t_2)-\psi(t_1)$.

That is to say $\psi(t_2)-\psi(t_1)=2\pi-(\beta_1+\alpha_2+\pi)$.

6*a*. Next suppose that the rectangle R (with sides parallel to the arcs) meets Γ in two points E, F, only, as before, but that the arc EF of Γ which lies inside R is part (or the whole) of one of the arcs Γ_c. If E and F are on opposite sides of R, we get precisely the result of § 6. Accordingly, we suppose that E and F are on adjacent sides of R; there are two cases to be considered, according as one or three vertices of the rectangle inside Γ. We shall investigate the former case (Fig. 2) in detail and leave the latter (Fig. 3) to the reader.

Let G be the vertex of R which is inside Γ; then since an arc Γ_c may be represented by an equation of the form $y=f(x)$, *and also* by an equation of the form $x=\phi(y)$, it follows that if α_2 be the principal value of the change of ψ' at F while β_1 is the principal value of the change of ψ' at E, where ψ' is the angle between the tangent at a point of EFG and the x-axis, then α_2 and β_1 both lie between 0 and π. Hence, as in § 6, we get†

$$\psi(t_2)-\psi(t_1)=2\pi-(\beta_1+\alpha_2+\tfrac{1}{2}\pi),$$

where t_1, t_2 are the parameters of E and F on Γ.

---

* $\psi'(t')$ is continuous in each of the ranges $(t_2'+0, t_3'-0)$, $(t_3'+0, t_4'-0)$, $(t_4'+0, t_1'+\omega'-0)$, and is therefore constant in each of these ranges.

† Each corner of the rectangle inside Γ contributes $\frac{1}{2}\pi$ in the analysis of § 6; hence the difference between this result and that of § 6.

In like manner, if there were three vertices of the rectangle inside Γ, we should get

$$\psi(t_2)-\psi(t_1) = 2\pi-(\beta_1+\alpha_2+\tfrac{3}{2}\pi).$$

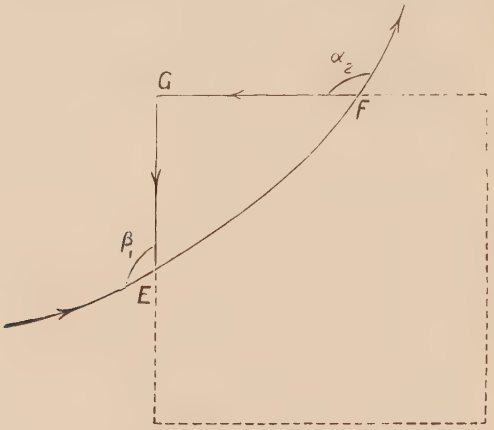

FIG. 2.

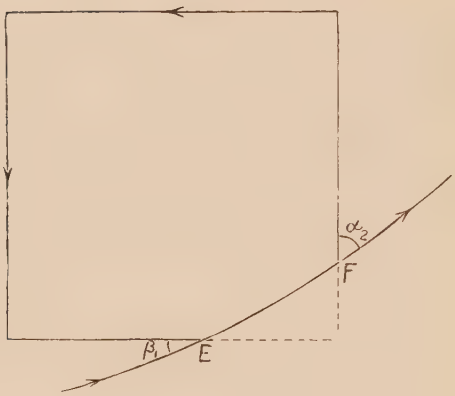

FIG. 3.

To sum up, we see that, if a rectangle R with sides parallel to the axes meets Γ in two points E and F only, so that the oriented arc EF of Γ is inside R, where all points of EF belong to not more than one of the arcs Γ_A, Γ_B, Γ_C, and E, F are on opposite sides of R (unless EF is part of an arc Γ_C, in which case they may be on adjacent sides), and if t_1, t_2 $(0 < t_2-t_1 < \omega)$ be the parameters of E and F, then

$$\psi(t_2)-\psi(t_1) = 2\pi-(\beta_1+\alpha_2+\tfrac{1}{2}r\pi).$$

In this equation r is the number of vertices of the rectangle which are

inside Γ, α_2 is the principal value of $\lambda(t_2)-\psi(t_2)$, and β_1 is the principal value of $\psi(t_1)-\lambda(t_1)+\pi$, where $\lambda(t)$ denotes a value of the angle between the side of the rectangle (drawn *towards the interior* of Γ) through the point with parameter t and the x-axis.

7. We shall now divide the interior of Γ into a number of smaller regions. By the arguments of § 4, we can divide Γ into a *finite* number of arcs, such that if t, t' be the parameters of any two points on the same arc, then $|\psi(t)-\psi(t')|<\frac{1}{20}\pi$. Let these arcs (in order) be called $\sigma_1, \sigma_2, \ldots, \sigma_s$.

Let P be any point on Γ, and let Q be any other point on Γ, such that P, Q are not on the same arc σ nor on consecutive arcs. Then the distance PQ has a positive lower bound, say 4δ. Cover the plane with a network of squares with sides parallel to the axes, the side of each square being of length δ.

[If a vertex of any square lies on Γ, shift the network slightly until this is no longer the case.]

The arcs σ are of the types (A), (B) and (C); not of type (D), also portions of Γ which are in the same square, or in two consecutive squares, must belong to the same arc σ or to consecutive arcs; also arcs σ of types (A) and (B) cannot be consecutive, for they must be separated by at least five arcs of type (C).

Now the arcs σ (taken in order) may be arranged in groups, each group consisting of one or other of three types:

(1) The first type consists of a number of consecutive arcs of types (A) and (C), each arc of type (C) being consecutive to at least one arc of type (A); the number of arcs in the group should be the largest possible. It therefore consists of arcs of types (A) and (C), those of the former type which are not consecutive being separated by not more than two arcs of type (C); and the group begins and ends with an arc of type (C).

(2) The second type consists of a number of consecutive arcs of types (B) and (C), arranged according to same laws of arrangement as were postulated for types (A) and (C) in (1).

(3) Each group of this type consists of a number of consecutive arcs of type (C) which do not adjoin an arc of type (A) or type (B).

Any group of arcs of types (1), (2), (3) respectively, will be called a curve of type (A′), (B′), (C′) respectively.

Then each arc of type (A′) or type (C′) is an elementary curve of

the form $y = f(x)$; and each arc of type (B′) or type (C′) is an elementary curve of the form $x = \phi(y)$.

Take any curve of type (A′), and let the lines of the network (parallel to the y-axis) which it crosses be $l_1, l_2, \ldots, l_p$; the points where it crosses l_1 and l_p must be on the extreme arcs of type (C), because on these extreme arcs $|\cos\psi(t)| \geqslant \cos(\frac{1}{10}\pi + \frac{1}{20}\pi + \frac{1}{20}\pi)$, and so the difference of the abscissæ of the ends of one of these arcs $\geqslant 4\delta\cos\frac{1}{5}\pi > \delta$; and so also the difference of the abscissæ of the ends of the other of the arcs of type (C) exceeds δ.

Between l_1 and l_2 the curve cannot cross more than one side of a square parallel to the x-axis; for the difference of the ordinates at the points where the curve crosses l_1 and l_2 is less than $\delta\tan\frac{1}{5}\pi < \delta$. Similar reasoning applies to the sides of squares between l_2 and l_3, l_3 and l_4, and so on.

Obliterate all* the sides of squares which are parallel to the axis of x, and which meet the curve between l_1 and l_p.

Carry out this process of obliteration for all the curves (A′), and a similar process of obliteration† of sides of squares parallel to the y-axis for all the curves (B′).

We then have the plane divided into a number of squares and rectangles, such that the curve Γ meets the perimeter of any square or rectangle either in no points or in two points (but not more).

Now obliterate the whole of the network outside Γ, and the surviving part of the network gives (i) a number of squares inside Γ, and (ii) a number of incomplete squares and rectangles of which the curvilinear boundaries make up Γ; each of these latter curves consists of an arc of Γ and part of a rectangle inside Γ, making up a curve with discontinuously turning tangent of the type considered in §§ 6, 6a.

Let ν be the number of squares, and μ the number of other regions, so that the number of points where surviving lines of the network meet Γ is μ; let $t_1, t_2, \ldots, t_\mu$ be the parameters of these points, in order, and let

$$t_{\mu+1} = t_1 + \omega.$$

Add up the equations of the type

$$\psi(t_2) - \psi(t_1) = 2\pi - (\beta_1 + \alpha_2 + \tfrac{1}{2}r_1\pi),$$

* The actual side of the square only is to be obliterated; not the whole of the line of the network on which it lies.

† This process is not carried out for the curves (C′), since they do not intersect the perimeter of any square in more than two points.

and we get

$$\psi(t_{\mu+1})-\psi(t_1) = 2\mu\pi - \sum_{m=1}^{\mu}(\alpha_m+\beta_m+\tfrac{1}{2}r_m\pi)$$

$$= 2(\mu+\nu)\pi - \left\{\sum_{m=1}^{\mu}(\alpha_m+\beta_m+\tfrac{1}{2}r_m\pi)+2\nu\pi\right\}.$$

Now α_m, β_m are positive angles less than π whose cosines are equal but opposite in sign; and so $\alpha_m+\beta_m=\pi$.

Also $4\nu+\Sigma r_m$ is the total number of *corners** inside Γ, while 2μ is the number of corners on Γ. Consequently

$$\psi(t_1+\omega)-\psi(t_1) = 2N\pi-\tfrac{1}{2}M\pi,$$

where N is the total number of regions into which the interior of Γ is divided, and M is the total number of corners inside and on Γ.

Now some lines of the network terminate on other lines of the network and not on the curve. Produce these lines until they meet the curve but no further.† Then the process of drawing each segment of these lines increases‡ N by 1, and M by 4, since two additional corners at each end of the segment are inserted.

Consequently $2N-\frac{1}{2}M$ is unaffected by drawing these lines. Now obliterate in turn each segment of the lines parallel to the x-axis; each obliteration diminishes N by 1 and M by 4, and at the end of these obliterations we are left with Γ and a number of lines parallel to the y-axis and terminated by Γ. Obliterate these in turn, and (as before) each obliteration diminishes N by 1 and M by 4, and so does not affect $2N-\frac{1}{2}M$. Therefore

$$2N\pi-\tfrac{1}{2}M\pi = 2N_0\pi-\tfrac{1}{2}M_0\pi,$$

where N_0 is the number of regions inside Γ and M_0 the number of corners when the whole network is obliterated.

But, by Jordan's theorem, $N_0=1$, and M_0 is obviously zero. Therefore

$$\psi(t_1+\omega)-\psi(t_1) = 2\pi,$$

which is the result stated in Theorem II.

8. We now consider what happens when $\dot{x}$ and $\dot{y}$ both vanish for a finite number of values of t in the range (t_0, T).

* A corner is an angle less than or equal to π formed by two intersecting arcs AP, BP with tangents at P.

† Cf. *Tract*, note at end of p. 11.

‡ This is a slight extension of a theorem due to Ames, *loc. cit.*, p. 356.

Let τ be such a value of t and let Q be the point with parameter τ. Then, if both $\dot{y}/\dot{x}$ and $\dot{x}/\dot{y}$ are discontinuous at τ, $\psi(t)$ is discontinuous; in this case we shall agree to define $\psi(\tau+0)$ by the convention that $\psi(\tau+0)-\psi(\tau-0)$ is to have its principal value (which is *not* equal to π).

If either $\dot{y}/\dot{x}$ or $\dot{x}/\dot{y}$ (or both) is continuous, and if neither $\dot{x}$ nor $\dot{y}$ changes sign at τ, $\psi(t)$ may be defined so as to be continuous at τ, and the proof of § 7 is valid.

If either $\dot{y}/\dot{x}$ or $\dot{x}/\dot{y}$ (or both) is continuous, and* $\dot{x}$ or $\dot{y}$ changes sign, $\psi(\tau+0)-\psi(\tau-0)$ must be an odd multiple of π.

In this case we define $\psi(\tau+0)$ to be $\psi(\tau-0)+\pi$ or $\psi(\tau-0)-\pi$, according as a line drawn through Q parallel to the y-axis† has all points on it (which are sufficiently near to Q) *outside* or *inside* the curve.

It seems hardly necessary to write out all possible cases; that shewn in Fig. 4, which is one of the most difficult, will be sufficient; Q is a cusp, and $\psi(\tau+0)-\psi(\tau-0)=+\pi$.

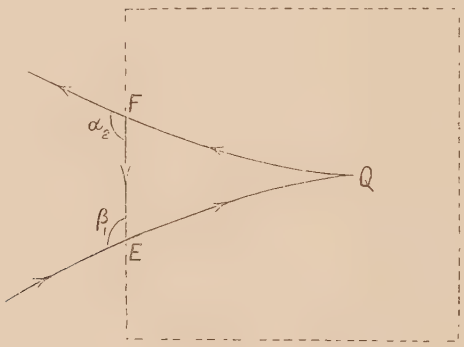

FIG. 4.

Choose the points τ to be end points of the arcs σ of § 7. Draw the network as before, with the additional proviso that none of the points Q are to be on the sides of the network. Obliterate temporarily the interiors of all the squares containing the points Q and let $4\delta_1$ be the lower bound of distances between surviving portions of consecutive arcs σ which have portions obliterated. Cut up the squares of the network into smaller squares whose sides are less than δ_1.

* In this case Q is a cusp.

† If $\psi(\tau-0)$ is an odd multiple of $\frac{1}{2}\pi$, we draw it parallel to the x-axis.

Consider a square containing one of the points Q; treating the part of the curve inside this square after the manner of § 6, we find that there is an integer n, such that

$$\begin{aligned} 2n_1\pi &= \psi'(t_1+\omega'+0)-\psi'(t_1+0) \\ &= \alpha_2+\beta_1+\psi(t_2)-\psi(t_1) \\ &= \alpha_2+\beta_1+\{\psi(\tau+0)-\psi(\tau-0)\}+\{\psi(t_2)-\psi(\tau+0)\} \\ &\qquad +\{\psi(\tau-0)-\psi(t_1)\}. \end{aligned}$$

Now α_2 and β_1 lie between $\frac{1}{2}\pi \pm \frac{1}{20}\pi$, and so $2n_1\pi$ lies between

$$2\pi+\tfrac{1}{10}\pi+\tfrac{1}{20}\pi+\tfrac{1}{20}\pi \quad \text{and} \quad 2\pi-\tfrac{1}{10}\pi-\tfrac{1}{20}\pi-\tfrac{1}{20}\pi,$$

and therefore $n_1 = 1$.

Consequently $$\psi(t_2)-\psi(t_1) = 2\pi-(\alpha_2+\beta_1),$$

an equation of the same form as those in §§ 6, 6a with $r = 0$.

The portions of the curve inside the smaller squares are now treated after the manner of § 7, and the required result follows.

9. The case in which $\dot{x}$ and $\dot{y}$ have a finite number of finite discontinuities is worth mention; at such a point (with parameter τ), where $\dot{x}$ or $\dot{y}$ is discontinuous, $\psi(t)$ is (in general) discontinuous, and the curve has an abrupt change of direction. To define $\psi(t)$ in such circumstances, we suppose that $\psi(t)$ varies continuously (except at the points τ) as t increases from t_0 to $t_0+\omega$, and that the change in $\psi(t)$ as t increases through the value τ is *numerically less than* π, so that $\psi(\tau+0)-\psi(\tau-0)$ has its principal value.* By the methods of § 8, it is now easy to shew that, as usual

$$\psi(t_0+\omega)-\psi(t_0) = 2\pi,$$

and it seems unnecessary to write out the details of a proof. For instance, if the cusp in Fig. 4 be replaced by an angle for which the discontinuity in ψ is positive, the arguments of the case worked out in § 8 hold almost *verbatim*.

A verbal statement in the case of a curve with a finite number of discontinuities in the direction of its tangent is as follows.

The sum of the changes in ψ along the continuously turning portions of the curve plus the sum of the exterior angles (taken numerically less than π) at the discontinuities is equal to 2π.

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* These considerations shew that the case under discussion is, in reality, no more general than the case worked out in § 8.

10. In the case when $\ddot{x}$, $\ddot{y}$ are continuous for all values of t, the result of Theorem II can be put into a definitely arithmetical form, for we have

$$\psi(t+\omega)-\psi(t)=\int_t^{t+\omega}\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\,dt,$$

and hence the result:

Let $x(t)$, $y(t)$ be real functions with continuous second differential coefficients, such that they do not assume the same pair of values for different values of t in the range $t_0<t<T$, such that $\dot{x}$ and $\dot{y}$ do not simultaneously vanish, so that x and y have period $\omega=T-t_0$. Then it is possible to find values of h and k, such that

$$(x-h)^2+(y-k)^2$$

does not vanish, and such that

$$\int_t^{t+\omega}\frac{(x-h)\,\dot{y}-(y-k)\,\dot{x}}{(x-h)^2+(y-k)^2}\,dt$$

is not zero; the value of this integral can then only be $+2\pi$ or -2π. And

$$\int_t^{t+\omega}\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\,dt$$

is equal to $+2\pi$ or -2π according as the former integral was equal to $+2\pi$ or -2π.

It does not seem to be possible to prove that the value of the last integral is $\pm 2\pi$ by any obvious method more direct than that employed in this paper.

FUNCTIONS OF POSITIVE TYPE AND RELATED TOPICS IN GENERAL ANALYSIS

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I. *Statement of the Problem.*

In a memoir "On the Foundations of the Theory of Linear Integral Equations,"* Prof. E. H. Moore has discussed the foundations of a theory of the general integral equation

$$\xi = \eta - zJ\kappa\eta, \tag{G}$$

including as instances the classical theory of integral equations and the theory of an infinite (or finite) number of linear equations in an infinite (or finite) number of variables. Later, in an address before the Fifth International Congress of Mathematicians at Cambridge, August 1912,† he has discussed the fundamental functional operation of the general theory on the foundation which he has called Σ_5, and has given other instances of the theory. In each of these papers, results are stated and used which are proved in an earlier paper by Prof. Moore.‡ We would refer also to a presentation of this theory by Bolza,§ which offers a ready approach to an understanding of the principles of the generalization.

In the system Σ_5 we have as basis

$$\left(\mathfrak{A}\,;\ \mathfrak{P}\,;\ \mathfrak{M}\,;\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*\,;\ J\right),$$

where $\mathfrak{A}$ is the class of all real or the class of all complex numbers, $\mathfrak{P}$ is

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* *Bulletin Amer. Math. Soc.*, Ser. 2, Vol. 18 (1912), pp. 334–362. In reference to this paper, it will be denoted by I.

† "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations," *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge, August 1912. In reference to this paper, it will be denoted by II.

‡ "Introduction to a Form of General Analysis," *New Haven Mathematical Colloquium*, 1906.

§ *Jahresbericht d. Deutschen Mathem.-Vereinigung*, Vol. 23 (1914), pp. 248–303.

a general range in the sense of the true general, $\mathfrak{M}$ is a class of functions on $\mathfrak{P}$ to $\mathfrak{A}$, $\mathfrak{K}$ is a class of functions on the composite range $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$ and is defined as the $*$-composite of the class $\mathfrak{M}$ with itself, J is a general functional operation on the class $\mathfrak{K}$ to $\mathfrak{A}$.* As Prof. Moore has shown (I, p. 361), we secure the general Fredholm theory of the equation (G) by postulating that the class $\mathfrak{M}$ has the properties $LCDD_0$, and that the functional operation J has the properties L and M. We denote the basis with these postulates on its elements as the system Σ_5^F. We secure the Hilbert-Schmidt theory for the complex-valued Hermitian kernel κ, *i.e.*, for a function $\kappa(st)$ such that $\kappa(st) = \bar{\kappa}(ts)$, by postulating that the class $\mathfrak{A}$ is the class of all complex numbers, that the class $\mathfrak{M}$ has the properties $LCDD_0R$, and that the functional operation J has the properties $LMHPP_0$. The basis with these postulates on its elements is the system Σ_5. Prof. Moore, in lectures at the University of Chicago, indicated the desirability of removal of postulation of the property P_0 from the functional operation, and had obtained Theorem I of this paper with its corollaries before the author took up this question. The Hilbert-Schmidt theory without postulation of the property P_0 on the functional operation is carried through in § IV below. In these lectures Prof. Moore showed also that the property H is a consequence of the properties L and P if the class $\mathfrak{A}$ is complex.

Among the instances Prof. Moore has cited, an important one is that suggested by the analogy of the sphere and the ellipsoid which led to the replacement of a unary operation J_p on a basis he denoted by Σ_4 by the binary operation J_{st} of the basis Σ_5.† This instance does not derive its importance from its application to equation (G) in the Fredholm theory, but, from the standpoint of the Hilbert-Schmidt theory and of the geometry of a function space, it leads to fundamentally new results. In this instance, given a system of type Σ_5, we define J as operative on a function κ no longer as merely $J\kappa$ but as $J^2_{(su)(vt)}\kappa(st)\omega(uv)$ and write (to simplify, leaving out arguments) with no ambiguity as to meaning

$$J_\omega \kappa = J^2_{(13)(42)}\kappa\omega.$$

If, now, ω is Hermitian, and further if ω as to the operation J is of positive type, in notation ω^P, that is in case for every function μ of the class $\mathfrak{M}$,

$$J^2_{(12)(34)}\bar{\mu}\omega\mu \geqslant 0,$$

* Elements of the class $\mathfrak{A}$ will be denoted by small Roman letters: a, b, c; of the classes $\mathfrak{M}$ and $\mathfrak{K}$ by small Greek letters: μ, ξ, η, ζ, ϕ, ψ and κ, λ, ρ, ω, respectively.

† I, pp. 349–350, and II, § 4 (*c*).

and further, if ω is definitely so, in notation ω^{P_0}, that is in case

$$J^2_{(12)(34)}\bar{\xi}\omega\xi = 0 \text{ implies } \xi = 0,$$

we have secured for the functional operation J_ω the properties $LMHPP_0$, and J_ω is operative as the functional operation for a new instance of type Σ_5.

Mercer* has discussed the positive Hermitian kernel and has shown its importance from the standpoint of expansibility in a uniformly convergent series of the characteristic functions. It is the purpose of this paper to discuss the positive Hermitian kernel in the general analysis as regards the instance cited above, that is the instance suggested by the analogy of the sphere and the ellipsoid, together with certain other problems in the general analysis connected with this problem. We set before us then, the problem of determining conditions on the function ω sufficient to secure the equivalence of the properties P_{J_ω} and P_J for every Hermitian function of the class $\mathfrak{K}$. In other words, we wish to determine conditions on ω sufficient to secure the relation

$$\kappa^H \ :\supset: \ J^4\bar{\xi}\omega\kappa\omega\xi \geqslant 0 \ (\xi) \qquad \cdot\sim\cdot \qquad J^2\bar{\xi}\kappa\xi \geqslant 0 \qquad (\xi)\dagger \qquad \text{(I)}$$

where in integration by the functional operation J adjacent variables are integrated out in pairs.

II. *Definition of Terms. The Systems* Σ_5', Σ_7', *and* Σ_7.

We first define the general systems we shall use in the discussion. Dropping the postulate P_0 on the functional operation J of the system Σ_5, we obtain what we shall call Σ_5'; viz.,

$$\Sigma_5' : \left(\mathfrak{A}\,;\ \mathfrak{P}\,;\ \mathfrak{M}^{\text{on } \mathfrak{P} \text{ to } \mathfrak{A}.LCDD_0R}\,;\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*\,;\ J^{LMHP \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}}\right).$$

We obtain the system Σ_7' by introducing into the system Σ_5' a new class $\mathfrak{N}$, with the properties LD_0R, and having the class $\mathfrak{M}$ as a sub-class. Further, we postulate that J is operative on the class $(\mathfrak{N}\mathfrak{N})_L$. We thus

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* *Phil. Trans.*, A, Vol. 209 (1909).

† To be read: "For every Hermitian (H) function κ (of the class $\mathfrak{K}$), it is true that $(:\supset:)\ J^4\bar{\xi}\omega\kappa\omega\xi \geqslant 0$ for every ξ of the class $\mathfrak{M}$ is equivalent to $(\cdot\sim\cdot)\ J^2\bar{\xi}\kappa\xi \geqslant 0$ for every ξ (of the class $\mathfrak{M}$)."

obtain

$$\Sigma_7' : \Big(\mathfrak{A}\,;\,\mathfrak{P}\,;\,\mathfrak{N}^{LD_0R \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}}\,;\,\mathfrak{M}^{LCDD_0R.B_0\mathfrak{N}}\,;\,\mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*\,;$$

$$J^{LMHP \text{ on } \left(\substack{(\mathfrak{N}\mathfrak{N})_L \\ \mathfrak{K}}\right) \text{ to } \mathfrak{A}}\Big).^*$$

As to notation, we shall denote functions of the class $\mathfrak{N}$ by ν, α, β, γ.

It is clear that any system of type Σ_5 or Σ_5' is also of type Σ_7', since the class $\mathfrak{N}$ may be taken as identical with the class $\mathfrak{M}$. Thus any theorem which is proved on the foundation Σ_7' is also true on the foundation Σ_5 or Σ_5', functions of the class $\mathfrak{N}$ being replaced by functions of the class $\mathfrak{M}$. The system Σ_7' derives its importance from the fact that, on it as foundation, we can prove certain theorems for a class with postulates less restrictive than those placed on the class $\mathfrak{M}$, without disturbing the postulates on the class $\mathfrak{M}$, and consequently the properties of the class $\mathfrak{K}$.

Before setting up the system Σ_7 we shall define certain properties of sequences and of classes of functions other than the elementary ones used in setting up the system Σ_5. The properties named have been investigated for the case in which the operation J is that of integration, but not in generalization. These will be given on the foundation Σ_7', and then the system Σ_7 will be obtained by postulating one of these properties for the class $\mathfrak{N}$ of Σ_7'. In writing the definitions of these properties as well as the theorems to follow, we shall use the notations of Peano as modified and extended by Moore. The properties here defined will also be written out in full,† and by comparison of the two forms the reader will come to a complete understanding of the few simple symbols we shall use and be able to interpret readily statements made throughout the paper. The more involved of the theorems will also be written out in full.

(1) $\{\alpha_n\}^{\text{complete for } \mathfrak{N}} : \alpha^{\mathfrak{N}} \cdot \supset \cdot J\bar{a}\alpha = \sum_n J\bar{a}\alpha_n J\bar{a}_n\alpha,$

* The property $B_0\mathfrak{N}$ is the property "is contained in the class $\mathfrak{N}$."

† (1) A sequence $\{\alpha_n\}$ of functions α_n (of the class $\mathfrak{N}$) is said to have the property "complete for $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ ($\alpha^{\mathfrak{N}}$) it is true that $(\cdot\supset\cdot)$

$$J\bar{a}\alpha = \sum_n J\bar{a}\alpha_n J\bar{a}_n\alpha.$$

(2) $\{a_n\}^{\text{general for } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: e \cdot\supset\cdot \exists (a_1, \ldots, a_{n_e})$

$$\exists J\overline{\left(\alpha - \sum_{n=1}^{n_e} a_n a_n\right)}\left(\alpha - \sum_{n=1}^{n_e} a_n a_n\right) < e,$$

(3) $\{a_n\}^{\text{closed as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: J\bar{\alpha}a_n = 0 \quad (n) \cdot\supset\cdot J\bar{\alpha}\alpha = 0.$

A function ω of the class $\mathfrak{K}$ may have any of the following properties with respect to functions of the class $\mathfrak{N}$,

(4) $\omega^{\text{positive as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} \cdot\supset\cdot J^2\bar{\alpha}\omega\alpha \geqslant 0,$

(5) $\omega^{\text{closed as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: J\bar{\alpha}\omega = 0 \cdot\supset\cdot J\bar{\alpha}\alpha = 0,$

(6) $\omega^{\text{ultra-closed as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: J(J\bar{\alpha}\omega)\overline{(J\bar{\alpha}\omega)} = 0 \cdot\supset\cdot J\bar{\alpha}\alpha = 0,$*

(7) $\omega^{\text{definite as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: J^2\bar{\alpha}\omega\alpha = 0 \cdot\supset\cdot J\bar{\alpha}\alpha = 0,$

(8) $\omega^{\text{general for } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: e \cdot\supset\cdot \exists \beta_e^{\mathfrak{N}} \exists J\overline{(\alpha - J\omega\beta_e)}(\alpha - J\omega\beta_e) < e.$

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(2) A sequence $\{a_n\}$ of functions a_n is said to have the property "general for $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that, given a number e (> 0), it is true that there exists $(\exists)$ a set $(a_1, \ldots, a_{n_e})$ of numbers, the number of elements depending on e (n_e), such that

(3) $J\overline{\left(\alpha - \sum_{n=1}^{n_e} a_n a_n\right)}\left(\alpha - \sum_{n=1}^{n_e} a_n a_n\right) < e.$

(3) A sequence $\{a_n\}$ of functions a_n is said to have the property "closed as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that the relation, $J\alpha a_n = 0$ for every n, implies the relation $J\bar{\alpha}\alpha = 0$.

(4) A function ω (of the class $\mathfrak{K}$) is said to have the property "positive as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that $J^2\bar{\alpha}\omega\alpha \geqslant 0$.

(5) A function ω is said to have the property "closed as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ the relation, $J\bar{\alpha}\omega = 0$, implies the relation $J\bar{\alpha}\alpha = 0$.

(6) A function ω is said to have the property "ultra-closed as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that the relation, $J(J\bar{\alpha}\omega)\overline{(J\bar{\alpha}\omega)} = 0$, implies the relation $J\bar{\alpha}\alpha = 0$.

* This property is equivalent, for a Hermitian function ω to the definite property for the function $J\omega\omega$. If the operation J has the definite (P_0) property, it reduces to ordinary closure of the function ω.

(7) A function ω is said to have the property "definite as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that the relation $J^2\bar{\alpha}\omega\alpha = 0$ implies the relation $J\bar{\alpha}\alpha = 0$.

(8) A function ω is said to have the property "general for $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that given a positive number e it is true that there exists a function β depending on e and of the class $\mathfrak{N}$ $(\beta_e^{\mathfrak{N}})$, such that $J\overline{(\alpha - J\omega\beta_e)}(\alpha - J\omega\beta_e) < e$.

The properties C_J and S, applicable to the class $\mathfrak{N}$, are defined as follows:

(9) $\mathfrak{N}^{C_J} : \{a_n^{\mathfrak{N}}\} \ni \underset{mn}{\mathrm{L}} J\overline{(a_m - a_n)}(a_m - a_n) = 0 \cdot \supset \cdot \exists\, a^{\mathfrak{N}} \ni \underset{n}{\mathrm{L}} J\overline{(a - a_n)}(a - a_n) = 0,$

(10) $\mathfrak{N}^{S} : \{a_n^{\mathfrak{N}}\}^{\text{U.O.}} \cdot (\{a_n\} \ni \sum_n a_n \bar{a}_n \text{ converges}) \cdot \supset \cdot \exists\, a^{\mathfrak{N}} \ni a_n = J\bar{a}_n a \ (n),$*

The property C_J is a closure property in that, if the class $\mathfrak{N}$ has the property C_J, it is closed as to convergence in the mean. We shall show that the properties C_J and S are equivalent as applied to the class $\mathfrak{N}$ of Σ_7'. These properties are, however, not possessed by the class of all continuous functions on the interval (0, 1) with respect to the operation of integration on that interval. They are possessed by the class of all functions integrable together with their squares on that interval with respect to the operation of integration in the sense of Lebesgue. This is the primary reason for the introduction of the class $\mathfrak{N}$ into the system Σ_5'. But, with this introduction, the definite property P_0 of the functional operation J does not obtain, and we are thus led to the removal of the postulate of this property to secure the systems we have named. The system Σ_7 on which, as foundation, each theorem we state may be proved, is obtained from the system Σ_7' by postulating that the class $\mathfrak{N}$ has the property C_J. Thus we have

$$\Sigma_7 : \Big(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{N}^{LD_0RC_J \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{M}^{LCDD_0R.B_0\mathfrak{N}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHP \text{ on } \binom{(\mathfrak{N}\mathfrak{N})_L}{\mathfrak{K}} \text{ to } \mathfrak{A}}\Big).$$

The system Σ_7 is introduced, as we have pointed out, to secure a class $\mathfrak{N}$ with the properties C_J and S. In instance IV, the class $\mathfrak{N}$ may be taken as the class of all functions integrable with their squares in the

(9) A class $\mathfrak{N}$ is said to have the property C_J, in case for every sequence $\{a_n\}$ of functions a_n of the class $\mathfrak{N}$, such that $\underset{mn}{\mathrm{L}} J\overline{(a_m - a_n)}(a_m - a_n) = 0$, it is true that there exists a function a of the class $\mathfrak{N}$, such that $\underset{n}{\mathrm{L}} J\overline{(a - a_n)}(a - a_n) = 0$.

(10) A class $\mathfrak{N}$ is said to have the property S, in case for every sequence $\{a_n\}$ of functions a_n, unitary and orthogonal [*i.e.* $J\bar{a}_i a_j = 0$ $(i \neq j)$, $1\,(i = j)$], and $(\cdot)$ sequence $\{a_n\}$ of numbers a_n such that $\sum_n a_n \bar{a}_n$ converges, it is true that there exists a function a in the class $\mathfrak{N}$ such that $a_n = J\bar{a}_n a$ for every n.

It will be noted that the number of dots on a sign of implication indicates the order of logical weight of the implication—the main implication having the greatest number of dots, and the other implications punctuated in order.

* It is understood that in case there is only a finite number N of functions in the set $\{a_n\}$, then there are exactly N numbers in the set $\{a_n\}$,

sense of Lebesgue on the interval (0, 1), while in instances II_n and III_2, the classes for $\mathfrak{N}$ may be taken as identical with the classes $\mathfrak{M}$ in the respective instances. On the foundation Σ_7 we shall prove the equivalence of the properties complete, general, and closed, for a unitary and orthogonal set of functions of $\mathfrak{N}$, the relations being taken with respect to the class $\mathfrak{N}$. On this foundation we shall also prove the equivalence, for a Hermitian kernel function ω, of the properties, general for $\mathfrak{N}$, ultra-closed as to $\mathfrak{N}$, and the existence of a unitary and orthogonal set of characteristic functions of ω which is complete for $\mathfrak{N}$, providing that there exists in the class $\mathfrak{N}$ a function α such that $J\bar{\alpha}\alpha \neq 0$.

III. *General Observations on the Problem with Σ_7' as Foundation.*

The problem as we have stated it in § I may be extended on the foundation Σ_7' to a problem in which the positive relations are taken with respect to the class $\mathfrak{N}$ instead of with respect to the class $\mathfrak{M}$. We remark, however, that by virtue of Theorems VI and IX the relations, positive as to $\mathfrak{N}$ and positive as to $\mathfrak{M}$, are equivalent. If we make this extension the problem becomes, to obtain conditions on ω, sufficient to secure the relation

$$_K{}^H :\supset : {}_K{}^{P_{J_\omega}} \text{ as to } \mathfrak{N} \cdot\sim\cdot {}_K{}^{P_J} \text{ as to } \mathfrak{N}. \tag{1}$$

Concerning this problem, the following relations are clear at once.

(*a*) For any Hermitian function ω of the class $\mathfrak{K}$,

$$_K{}^{HP_J} \text{ as to } \mathfrak{M} \cdot\supset\cdot {}_K{}^{P_{J_\omega}} \text{ as to } \mathfrak{N},$$

since $Ja\omega$ is a function of the class $\mathfrak{M}$ for every α,* and accordingly,

$$_K{}^H :\supset : J^2\bar{\xi}_K\xi \geqslant 0 \quad (\xi) \quad \cdot\supset\cdot \quad J^4\bar{\alpha}\omega_K\omega\alpha \geqslant 0 \quad (\alpha).$$

Thus our problem, as stated above, reduces to that of obtaining conditions on ω sufficient to secure the relation

$$_K{}^H :\supset : {}_K{}^{P_{J_\omega}} \text{ as to } \mathfrak{N} \cdot\supset\cdot {}_K{}^{P_J} \text{ as to } \mathfrak{N}. \tag{2}$$

We obtain such conditions in Theorem XIII, and equivalent conditions in later theorems.

(*b*) It is clear that, in order to secure relation (2), the function ω must

* *Vide* Theorem II below.

be closed as to $\mathfrak{M}$. For, suppose there exists in $\mathfrak{M}$ a function ξ such that $J\bar{\xi}\xi \neq 0$ and $J\bar{\xi}\omega = 0$. Choose $\kappa = -\xi\bar{\xi}$. We have, then, a function κ such that $J^4\bar{\alpha}\omega\kappa\omega\alpha = 0$ for every α, but there exists a function, namely ξ, such that

$$J^2\bar{\xi}\kappa\xi = -J^2\bar{\xi}\xi\bar{\xi}\xi = -(J\bar{\xi}\xi)^2 < 0,$$

and accordingly relation (2) does not hold.

(*c*) Finally, in case, for a given Hermitian function ω and a function ξ, the equation $J\bar{\alpha}\omega = \bar{\xi}$ is solvable for α, we have at once

$$\kappa^H :\supset: J^4\bar{\alpha}\omega\kappa\omega\alpha \geqslant 0 \quad \cdot\supset\cdot \quad J^2\bar{\xi}\kappa\xi \geqslant 0.$$

In case, then, ω is Hermitian and such that we can solve for α the equation $J\alpha\omega = \mu$ for every function μ in the class $\mathfrak{M}$, we have

$$\kappa^H :\supset: \kappa^{HP_{J_\omega} \text{ as to } \mathfrak{N}} \quad \cdot\supset\cdot \quad \kappa^{P_J \text{ as to } \mathfrak{M}}.$$

Before taking up a discussion of the conditions on ω described above, we shall obtain generalizations of existence theorems concerning characteristic functions and numbers, and for this the foundation Σ_7' will be sufficient. Then we shall take up the equation $J\alpha\omega = \xi$ in generalization.

IV. *The Hilbert-Schmidt Theory for a Complex-Valued Hermitian Kernel on the Foundation* Σ_7'.*

The first three theorems we shall prove are theorems of a general nature, made necessary on the foundation Σ_7', because of the introduction of the class $\mathfrak{N}$, and the fact that the functional operation is no longer restricted by postulation of the definite property P_0. They are of fundamental importance in consideration of relations on the foundation Σ_7'.

THEOREM I.— $\alpha :\supset: J\bar{\alpha}\alpha = 0 \quad \cdot\sim\cdot \quad J\bar{\alpha}\beta = 0 \quad (\beta)$.†

(A) That $J\bar{\alpha}\beta = 0 \quad (\beta) \quad \cdot\supset\cdot \quad J\bar{\alpha}\alpha = 0$ is obvious.

(B) $\qquad J\bar{\alpha}\alpha = 0 \quad \cdot\supset\cdot \quad J\bar{\alpha}\beta = 0 \quad (\beta)$.

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* References to treatments of this theory in the classical instance IV would be too numerous to mention. The treatment we shall give of a generalization of the theory will follow, in substance, that given by E. Schmidt, *Math. Ann.*, Vol. 63. Changes are necessary in the statements and proofs of the theorems, since the functional operation is not restricted by postulation of the definite property P_0, and the classes entering are not restricted so as to contain only real valued functions.

† "For any function α (of the class $\mathfrak{N}$) the relation $J\bar{\alpha}\alpha = 0$ is equivalent to the relation $J\bar{\alpha}\beta = 0$ for every function β (of the class $\mathfrak{N}$)."

(*a*) $\quad J\bar{\alpha}\alpha = 0 \;\cdot\; J\bar{\beta}\beta \neq 0 \quad \cdot\supset\cdot\; J\bar{\alpha}\beta = 0.$

For, let

$$\gamma = \alpha - \frac{\beta J\bar{\beta}\alpha}{J\bar{\beta}\beta}, \quad \text{and so} \quad \alpha = \gamma + \frac{\beta J\bar{\beta}\alpha}{J\bar{\beta}\beta}. \tag{3}$$

Then $J\bar{\beta}\gamma = 0$, and from (3),

$$0 = J\bar{\alpha}\alpha = J\bar{\gamma}\gamma + \frac{J\bar{\alpha}\beta J\bar{\beta}\alpha}{J\bar{\beta}\beta},$$

and, since J has the properties H and P, we have

$$J\bar{\alpha}\beta = 0.$$

(*b*) $J\bar{\alpha}\alpha = 0 \;.\; J\bar{\beta}\beta = 0 \quad \cdot\supset\cdot\; J\bar{\alpha}\beta = 0.$

For, let $\qquad \gamma = \alpha - \beta J\bar{\beta}\alpha,$

and we have $\qquad J\bar{\gamma}\gamma = -2J\bar{\alpha}\beta J\bar{\beta}\alpha.$

Since J has the properties H and P, it follows that

$$J\bar{\alpha}\beta = J\bar{\beta}\alpha = 0.$$

COROLLARY I (Inequality of Schwarz).—

$$\alpha \;\cdot\; \beta \;\cdot\supset\cdot\; J\bar{\alpha}\beta J\bar{\beta}\alpha \leqslant J\bar{\alpha}\alpha J\bar{\beta}\beta.$$

If $J\bar{\beta}\beta \neq 0$, the relation is obtained from the substitution (3) of the theorem since J has the property P.

If $J\bar{\beta}\beta = 0$, we have from the theorem $J\bar{\alpha}\beta = 0$ for every α, and accordingly

$$0 = J\bar{\alpha}\beta J\bar{\beta}\alpha = J\bar{\alpha}\alpha J\bar{\beta}\beta.$$

COROLLARY II (Orthogonalization of a set of functions).—

$$(\alpha_1, \ldots, \alpha_n) \;\cdot\supset\cdot\; \exists\, (\beta_1, \ldots, \beta_m) \;\ni J\bar{\beta}_i\beta = 0 \quad (i \neq j),$$

$$\beta_i = \sum_{j=1}^{n} a_{ij}\alpha_j \;\; (i = 1, \ldots, m), \qquad \alpha_i = \sum_{j=1}^{m} b_{ij}\beta_j \;\; (i = 1, \ldots, n).$$

THEOREM II.— $\quad \kappa \;\cdot\; \alpha \;\cdot\supset\cdot\; (J\alpha\kappa)^{B_0\mathfrak{M}} \;\cdot\; (J\kappa\alpha)^{B_0\mathfrak{M}}.$

The theorem follows readily from the definition of the class $\mathfrak{K}$.

THEOREM III.—

$$\kappa :\supset: J^2_{(13)(42)}\bar{\kappa}\kappa = 0 \quad \cdot\sim\cdot \quad J^2_{(13)(42)}\omega\kappa = 0 \ (\omega) \ \cdot\sim\cdot \ J^2\alpha\kappa\beta = 0 \ (\alpha\beta).$$

Proof.—(A) $J^2_{(13)(42)}\bar{\kappa}\kappa = 0 \quad \cdot\sim\cdot \quad J^2_{(13)(42)}\omega\kappa = 0 \ (\omega).$

For, in $J^2_{(13)(42)}$, we have a functional operation which has the properties $LMHP$ on $(\mathfrak{K}\mathfrak{K})_L$ to $\mathfrak{A}$.* Accordingly, we have this operation as the functional operation for a new instance of Σ'_7, and can use Theorem I.

(B) $J^2_{(13)(42)}\omega\kappa = 0 \ (\omega) \ \cdot\sim\cdot \ J^2\alpha\kappa\beta = 0 \ (\alpha, \beta).$

(1) $J^2_{(13)(42)}\omega\kappa = 0 \ (\omega) \ \cdot\supset\cdot \ J^2\alpha\kappa\beta = 0 \ (\alpha, \beta).$

Since ω may be taken as $\xi\cdot\eta$, where ξ and η are arbitrary functions of $\mathfrak{M}$, we have $J^2\xi\kappa\eta = 0$ for every ξ and η. Accordingly, for every ξ, $J\xi\kappa$ is a function of the class $\mathfrak{M}$, such that $J(J\xi\kappa)(\overline{J\xi\kappa}) = 0$, and by Theorem I, for every function β of the class $\mathfrak{N}$ we have $J^2\xi\kappa\beta = 0$. In an analogous way we show $J^2\alpha\kappa\beta = 0$ for every α and β.

(2) $J^2\alpha\kappa\beta = 0 \ (\alpha, \beta) \ \cdot\supset\cdot \ J^2_{(13)(42)}\omega\kappa = 0 \ (\omega).$

To prove this relation we note that any function ω of the class $\mathfrak{K}$ may be expressed as the limit of a sequence of functions of the class $(\mathfrak{M}\mathfrak{M})_L$, uniformly convergent relative to a scale function of that class, and the theorem follows at once.

COROLLARY.—

$$\kappa^H :\supset: J^2_{(23)(41)}\kappa\kappa = 0 \quad \cdot\sim\cdot \quad J^2_{(23)(41)}\kappa\omega = 0 \ (\omega) \quad \cdot\sim\cdot \quad J^2\alpha\kappa\beta = 0 \quad (\alpha, \beta).$$

THEOREM IV.—

$$\kappa^H :\supset: J^2_{(23)(41)}\kappa\kappa \neq 0 \quad \cdot\sim\cdot \quad \exists \ (z_0, \ \phi^- = 0) \ni \phi = z_0 J\kappa\phi.\dagger$$

---

* The proof of this statement is omitted. However, $J^2_{(13)(42)}$ is immediately obtainable as the functional operation in the instance Prof. Moore has called the $*$-composite of systems, *vide* II, § 4 (*b*). To prove that $J^2_{(13)(42)}$ has the property P, we use Corollary II, Theorem I.

† The notation $\phi^- = 0$ denotes the relation: ϕ is not identically zero. It negates the relation: $\phi(p) = 0$ for every p.

LEMMA.— $J^2_{(23)(41)\kappa\kappa} = 0 \quad \cdot\sim\cdot \quad J^2_{(23)(41)\kappa_r\kappa_r} = 0,$

where κ_r is the iterated kernel

$$J^{r-1}_{(23)(45)\ldots(2r-2,\,2r-1)\,\kappa\kappa\,\ldots\,\kappa}.$$

(A) That $J^2_{(23)(41)\kappa\kappa} = 0 \quad \cdot\supset\cdot \quad J^2_{(23)(41)\kappa_r\kappa_r} = 0,$

follows at once from the Corollary of Theorem III.

(B) $J^2_{(23)(41)\kappa_r\kappa_r} = 0 \quad \cdot\supset\cdot \quad J^2_{(23)(41)\kappa\kappa} = 0.$

(1) $r = 2^k$.

We have $J^2_{(23)(41)\kappa_r\kappa_r} = 0 \cdot\supset\cdot J^2\alpha\kappa_r\beta = 0 \quad (\alpha, \beta)$

$$\cdot\supset\cdot J(J\alpha\kappa_{\frac{r}{2}})\overline{(J\alpha\kappa_{\frac{r}{2}})} = 0 \quad (\alpha)$$

$$\cdot\supset\cdot J^2\alpha\kappa_{\frac{r}{2}}\beta = 0 \quad (\alpha, \beta)$$

$$\cdot\supset\cdot J^2_{(23)(41)\kappa_{\frac{r}{2}}\kappa_{\frac{r}{2}}} = 0.$$

Applying this result k times we have the desired relation.

(2) $2^k < r < 2^{k+1}$. Let $r = 2^{k+1} - s$.

We have $J^2_{(23)(41)\kappa_r\kappa_r} = 0 \cdot\supset\cdot J^2_{(23)(41)\kappa_{r+s}\kappa_{r+s}} = 0$

$$\cdot\supset\cdot J^2_{(23)(41)\kappa_{2^{k+1}}\kappa_{2^{k+1}}} = 0,$$

and, from (1), $\cdot\supset\cdot J^2_{(23)(41)\kappa\kappa} = 0.$

*Proof of Theorem IV.**—

(A) $\kappa^H :\supset: J^2_{(23)(41)\kappa\kappa} \neq 0 \quad \cdot\supset\cdot \quad \exists\,(z_0,\ \phi^- = 0) \ni \phi = z_0 J\kappa\phi.$

On the foundation Σ^F_5 which secures the Fredholm theory and includes Σ'_5 as an instance, we have

$$-\frac{1}{D(z)}\frac{d}{dz}D(z) = \sum_{n=0}^{\infty} z^n J_{(21)\kappa_{n+1}} = \sum_{n=0}^{\infty} a_{n+1}z^n. \qquad (4)$$

where $a_n = J_{(21)\kappa_n}$ and $D(z)$ is the Fredholm determinant

* We have diverged from the method of proof by Schmidt by introducing the Fredholm theory. Cf. Kneser, *Die Integralgleichungen*, Braunschweig (1911), pp. 234-6; also Heywood and Fréchet, *L'Equation de Fredholm*, Paris (1912), pp. 83-86.

for κ. Using the Schwarz inequality for the instance offered by $*$-composition of systems Σ_5',* we have

$$J^2_{(13)(42)}\,\bar{\kappa}_{n-1}\,\kappa_{n+1}\,J^2_{(13)(42)}\,\bar{\kappa}_{n+1}\,\kappa_{n-1} \leqslant J^2_{(13)(42)}\,\bar{\kappa}_{n-1}\,\kappa_{n-1}\,J^2_{(13)(42)}\,\bar{\kappa}_{n+1}\,\kappa_{n+1},$$

or $$(J_{(21)}\kappa_{2n})^2 \leqslant J\kappa_{2n-2}J\kappa_{2n+2},$$

or $$(a_{2n})^2 \leqslant a_{2n-2}a_{2n+2}.$$

Further, since $J^2_{(23)(41)}\,\kappa\kappa \neq 0$, we have, by the Lemma,

$$a_{2n} > 0 \quad (n).$$

Thus $$\frac{a_{2n+2}}{a_{2n}} \geqslant \frac{a_{2n}}{a_{2n-2}},$$

and so $$\frac{a_{2n+2}}{a_{2n}} \geqslant \frac{a_4}{a_2} \quad (n).$$

The absolute value of the ratio of terms of two successive odd indices in the last series of (4), is

$$|z^2|\,\frac{a_{2n+2}}{a_{2n}}.$$

Thus, if we choose $|z| > \sqrt{\dfrac{a_2}{a_4}}$, we see that the series (4) diverges. Accordingly, $D(z)$ has a root z_0, such that $|z_0| \leqslant \sqrt{\dfrac{a_2}{a_4}}$, and from the Fredholm theory on the foundation Σ_5^F, the relation given in (A) follows at once.

(B) $\kappa^H :\supset: \exists\,(z_0,\ \phi^- = 0)\ \mathfrak{z}\ \phi = z_0 J\kappa\phi\ \cdot\supset\cdot\ J^2_{(23)(41)}\,\kappa\kappa \neq 0.$

Suppose $$J^2_{(23)(41)}\,\kappa\kappa = 0.$$

Then, by the Corollary of Theorem III,

$$J^2_{(23)(45)}\,\kappa\kappa\phi = 0,$$

that is, $J\kappa\phi = 0$, and so $\phi = z_0 J\kappa\phi = 0$,

which is contrary to hypothesis. Thus

$$J^2_{(23)(41)}\,\kappa\kappa \neq 0.$$

THEOREM V. $\kappa^H \cdot \exists\,(z_0,\ \phi^- = 0)\ \mathfrak{z}\ \phi = z_0 J\kappa\phi\ \cdot\supset\cdot\ J\bar{\phi}\phi \neq 0.$

Suppose $$J\bar{\phi}\phi = 0.$$

* Cf. II, § 4 (*b*).

Then $$z_0\bar{z}_0 J(J\bar{\phi}\kappa)(J\kappa\phi) = 0,$$

and, by Theorem I, $$J^2_{(12)(34)}\,\bar{\phi}\kappa\kappa = 0.$$

Thus $J\bar{\phi}\kappa = 0$, and so $\phi = 0$, which is contrary to hypothesis. Our assumption is untenable and $J\bar{\phi}\phi \neq 0$.

Note.—This theorem is of importance in that it enables us to normalize a set of characteristic functions of a Hermitian kernel. That is, given any set of characteristic functions, we can obtain an equivalent set of which the functions are unitary and orthogonal.

THEOREM VI.—

$$\kappa^H \cdot \exists\,(z_0,\ \phi \ni J\bar{\phi}\phi \neq 0) \ni \phi = z_0 J\kappa\phi \cdot \supset \cdot z_0^R \cdot (\kappa^{P \text{ as to } \mathfrak{M}} \cdot \supset \cdot z_0 > 0).$$

Proof.— $$\phi = z_0 J\kappa\phi \quad \text{and so} \quad J\bar{\phi}\phi = z_0 J^2\bar{\phi}\kappa\phi,$$

$$\bar{\phi} = \bar{z}_0 J\bar{\phi}\kappa \quad \text{and so} \quad J\bar{\phi}\phi = \bar{z}_0 J^2\bar{\phi}\kappa\phi.$$

Thus $$z_0 J^2\bar{\phi}\kappa\phi = \bar{z}_0 J^2\bar{\phi}\kappa\phi \quad \text{and} \quad z_0 = \bar{z}_0.$$

Further, if κ is of positive type as to the class $\mathfrak{M}$, $J^2\bar{\phi}\kappa\phi > 0$, and $z_0 > 0$.

Definition.—A set of characteristic functions of a Hermitian kernel κ is said to be complete for κ in case every characteristic function of κ can be expressed linearly in terms of functions of this set. The proof of the existence of such a set may be obtained as a direct generalization of the proof by Schmidt.*

THEOREM VII.—

$$\kappa^H \cdot \{\phi_n\}^{\text{U.O. complete for } \kappa} \cdot \{z_n\}^{\text{characteristic numbers for } \kappa}$$

$$\cdot \sum_n \frac{\phi_n\bar{\phi}_n}{z_n} \text{ converges } (\mathfrak{P}\mathfrak{P};\ \mathfrak{K}) :\supset: \omega = \kappa - \sum_n \frac{\phi_n\bar{\phi}_n}{z_n} \cdot \supset \cdot J^2_{(23)(41)}\,\omega\omega = 0.$$

Proof.—Suppose $$J^2_{(23)(41)}\,\omega\omega \neq 0.$$

Using Theorems IV and V, we secure the existence of a number c and function ψ such that $J\bar{\psi}\psi \neq 0$ and $\psi = cJ\omega\psi$.

Then $$J\bar{\phi}_k\psi = cJ\bar{\phi}_k(J\omega\psi) = c\left[J(J\bar{\phi}_k\kappa)\psi - \frac{J\bar{\phi}_k\psi}{z_k}\right] = 0.$$

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* *Loc. cit.* pp. 444–6.

But $\psi = cJ\omega\psi$ and so $\psi = cJ\kappa\psi$.

Since the set $\{\phi_n\}$ is complete for κ,

$$\psi = \sum_{n=1}^{m} a_n \phi_n.$$

Thus $$J\bar{\psi}\psi = \sum_{n=1}^{m} a_n J\bar{\psi}\phi_n = 0,$$

which denies the property secured for ψ. Accordingly, we have

$$J^2_{(23)(41)}\,\omega\omega = 0.$$

THEOREM VIII.—

$$\kappa^H \cdot \{\phi_n\}^{\text{U.O. complete for } \kappa} \cdot \{z_n\}^{\text{characteristic numbers for } \kappa} \cdot \alpha \ni J\bar{\phi}_n\alpha = 0 \ (n) \cdot \supset \cdot J^2\beta\kappa\alpha = 0 \quad (\beta).$$

Proof.—We have, without postulation of the property P_0 on the operation J, the relations sufficient to prove that the set $\{\phi_n\}$ is a complete characteristic set for κ_4 with the corresponding set $\{z_n^4\}$ of characteristic numbers, and that

$$\sum_n \frac{\phi_n\bar{\phi}_n}{z_n^4} \text{ converges } (\mathfrak{PP}\,;\ \mathfrak{K}).*$$

Thus, by Theorem VII,

$$\omega = \kappa_4 - \sum_n \frac{\phi_n\bar{\phi}_n}{z_n^4} \cdot \supset \cdot J^2_{(23)(41)}\,\omega\omega = 0.$$

By the Corollary of Theorem III,

$$J^2\bar{\alpha}\kappa_4\alpha = \sum_n \frac{J\bar{\alpha}\phi_n J\bar{\phi}_n\alpha}{z_n^4} = 0,$$

that is, $$J(J\bar{\alpha}\kappa_2)(J\kappa_2\alpha) = 0,$$

and, by Theorem I, $$J^2\bar{\alpha}\kappa_2\alpha = 0,$$

that is, $$J(J\bar{\alpha}\kappa)(J\kappa\alpha) = 0,$$

and, by Theorem I, $$J^2\beta\kappa\alpha = 0 \quad (\beta).$$

* Cf., Schmidt, *loc. cit.*, pp. 447–451. The proofs of the relations we have stated may be obtained as direct generalizations of the proofs given by Schmidt. The relative uniform convergence is secured from the dominance of the class $\mathfrak{K}$ by the class $(\mathfrak{MM})$.

THEOREM IX.—

$$\kappa^H \cdot \{\phi_n\}^{\text{U.O. complete for } \kappa} \cdot \{z_n\}^{\text{characteristic numbers for } \kappa}$$

$$:\supset: (\alpha, \beta) \cdot \supset \cdot J^2\bar{\alpha}\kappa\beta = \sum_n \frac{J\bar{\alpha}\phi_n J\bar{\phi}_n\beta}{z_n}.$$

Proof.—Let $\xi = J\kappa\beta$ and $\eta = \xi - \sum_n \phi_n J\bar{\phi}_n\xi$. (5)

The series on the right converges uniformly, relative to a scale function of the class $\mathfrak{M}$.* Then

$$J\bar{\phi}_k\eta = 0 \quad (k),$$

and, by Theorem VIII,

$$J^2\bar{\gamma}\kappa\eta = J^2\bar{\eta}\kappa\gamma = 0 \quad (\gamma).$$

But $J\bar{\eta}\eta = J\bar{\eta}\xi = J^2\bar{\eta}\kappa\beta$ and so $J\bar{\eta}\eta = 0$. (6)

By Theorem I, $J\bar{\eta}\gamma = J\bar{\gamma}\eta = 0 \quad (\gamma)$,

and so $J\bar{\alpha}\xi = \sum_n J\bar{\alpha}\phi_n J^2\bar{\phi}_n\kappa\beta$,

and so $J^2\bar{\alpha}\kappa\beta = \sum_n \frac{J\bar{\alpha}\phi_n J\bar{\phi}_n\beta}{z_n}$.

Thus we have secured the Hilbert-Schmidt existence theorems and the Hilbert expansion theorem (Theorem IX) for a complex-valued Hermitian kernel on the foundation Σ_7' in which the functional operation J is not restricted by postulation of the property P_0. The condition $\kappa^- = 0$ in the classical theory is replaced by the condition $J^2_{(23)(41)}\kappa\kappa \neq 0$, and the theory is, in general, the same.

V. *The Non-Hermitian Kernel in General Analysis.*†

In the discussion of the equation $Ja\omega = \xi$, we shall use a generalization of results obtained by E. Schmidt, in which he has discussed the unsymmetric kernel in instance IV, the continuous case, in relation to characteristic functions and numbers, and in which, for a continuous function κ, he has proved the existence of a set of numbers $\{z_n\}$ and of

* The proof of this statement is contained in that of the theorem in II, § 5 (*l*), in which the property P_0 of the functional operation is not used.

† Cf. E. Schmidt, *loc. cit.*, pp. 459–466.

functions $\{\phi_n\}$ and of functions $\{\psi_n\}$, such that

$$\phi_n(t) = z_n \int_0^1 \psi_n(s)\,\kappa(s,t)\,ds \quad \text{and} \quad \psi_n(s) = z_n \int_0^1 \kappa(s,t)\,\phi_n(t)\,dt,$$

for every n. In the present discussion, we shall generalize these results not only by generalizing the range and the functional operation, but also by extending them in application to a kernel function on a range which is the composite of two ranges not necessarily the same. For this we shall consider two systems Σ' and Σ'' of type Σ_5'. In this way we shall secure, in case $\Sigma' = \Sigma'' = \Sigma^{\text{IV}}$, the results given in the classical instance above. However, by varying Σ' and Σ'', we obtain distinctly new results. For example, if we take $\Sigma' = \Sigma^{\text{IV}}$, $\Sigma'' = \Sigma^{\text{II}_u}$, our results are as follows: for a given set of continuous functions $(\xi_1, \ldots, \xi_n)$ on the interval (0, 1), not all zero, there exists a set of numbers $\{z_i\}$ and of continuous functions $\{\phi_i\}$, and of numbers $(x_{1i}, \ldots, x_{ni})$, such that

$$\phi_i = z_i \sum_{j=1}^{n} x_{ji}\,\xi_j \quad \text{and} \quad x_{ki} = z_i \int_0^1 \bar{\xi}_k(s)\,\phi_i(s)\,ds \quad (k, i).$$

We introduce, as we have said, two systems,

$$\Sigma' : \Big(\mathfrak{A};\ \mathfrak{P}';\ \mathfrak{M}'^{LCDD_0R \text{ on } \mathfrak{P}' \text{ to } \mathfrak{A}};\ \mathfrak{K}' = (\mathfrak{M}'\mathfrak{M}')_*;\ J'^{LMHP \text{ on } \mathfrak{K}' \text{ to } \mathfrak{A}}\Big),$$

$$\Sigma'' : \Big(\mathfrak{A};\ \mathfrak{P}'';\ \mathfrak{M}''^{LCDD_0R \text{ on } \mathfrak{P}'' \text{ to } \mathfrak{A}};\ \mathfrak{K}'' = (\mathfrak{M}''\mathfrak{M}'')_*;\ J''^{LMHP \text{ on } \mathfrak{K}'' \text{ to } \mathfrak{A}}\Big),$$

and form the system

$$\Sigma : \Big(\mathfrak{A};\ \mathfrak{P} = \mathfrak{P}'\mathfrak{P}'';\ \mathfrak{M} = (\mathfrak{M}'\mathfrak{M}'')_*;\ \mathfrak{K} = (\mathfrak{K}'\mathfrak{K}'')_*;\ J = J'J''\Big).$$

Since Σ' and Σ'' are of type Σ_5', it follows from the theory of $*$-composition of systems that Σ is also of type Σ_5'. Functions of the classes in Σ' and Σ'' will be given single and double accents respectively, while those in the classes of Σ will be unaccented.

THEOREM X.—

$$\hat{\xi} :\supset: J'^2_{(12)(34)}\, J''^2_{(23)(41)}\, \bar{\bar{\xi}}\hat{\xi}\bar{\hat{\xi}}\hat{\xi} \neq 0 \quad \cdot\supset\cdot \quad \exists\, (\{\phi_n'\}^{\text{U.O.}}\ \{\phi_n''\}^{\text{U.O.}}\ \{z_n > 0\})$$

$$\exists\ \phi_n' = z_n J''\hat{\xi}\phi_n'' \cdot\ \phi_n'' = z_n J'\,\bar{\bar{\xi}}\phi_n' \quad (n).$$

Proof.—It is clear that the function $J'\bar{\bar{\xi}}\hat{\xi}$ has the properties H and P with respect to the operation J'', and that

$$J''^2_{(23)(41)}\,(J'\bar{\bar{\xi}}\hat{\xi})(J'\bar{\bar{\xi}}\hat{\xi}) \neq 0.$$

Accordingly, by Theorems IV, V, and VI, we have a set of characteristic functions $\{\phi_n''\}$ for this function as kernel, which we may assume to be unitary and orthogonal with respect to J'', and a corresponding set of positive characteristic numbers $\{z_n^2\}$. Since the function ϕ_n'' is, for every n, a characteristic function for $J'\bar{\xi}\xi$ with characteristic number z_n^2, we have

$$\phi_n'' = z_n^2 J''(J'\bar{\xi}\xi)\,\phi_n'' \qquad (n),$$

and if we let $$\phi_n' = z_n J''\xi\phi_n'' \ (z_n > 0) \quad (n),$$

we have the desired relation except for the unitary, orthogonal character of the set $\{\phi_n'\}$.

To prove this, we have

$$\begin{aligned} J'\bar{\phi}_n'\phi_m' &= z_n z_m J'(J''\bar{\phi}_n''\bar{\xi})(J''\xi\phi_m'') \\ &= z_n z_m J''^2\,\bar{\phi}_n''(J'\bar{\xi}\xi)\,\phi_m'' \\ &= \frac{z_n}{z_m} J''\bar{\phi}_n''\phi_m'', \end{aligned}$$

and since the set $\{\phi_n''\}$ is unitary and orthogonal, so also is the set $\{\phi_n'\}$.

THEOREM XI.—Under the hypotheses and notation of Theorem X, we have

$\{\phi_n''\}$ U.O. complete for $J'\bar{\xi}\xi$. $\{z_n^2\}$ characteristic numbers for $J'\bar{\xi}\xi$

$\cdot\sim\cdot$ $\{\phi_n'\}$ U.O. complete set for $J''\xi\bar{\xi}$. $\{z_n^2\}$ characteristic numbers for $J''\xi\bar{\xi}$.

Proof.—Let $$\phi'' = z_0^2 J''(J'\bar{\xi}\xi)\,\phi''.$$

We wish first to show that $\phi' = z_0 J''\xi\phi''$ is a characteristic function of $J''\xi\bar{\xi}$ with characteristic number z_0^2. We have

$$\phi' = z_0^3 J''\xi[J''(J'\bar{\xi}\xi)\,\phi''] = z_0^2 J''\xi(J'\bar{\xi}\phi') = z_0^2 J'(J''\xi\bar{\xi})\,\phi'.$$

Thus we have proved that, with z_0^2 and ϕ'' assumed as characteristic number and function, respectively, of $J'\bar{\xi}\xi$, the function ϕ' defined as $z_0 J''\xi\phi''$ is a characteristic function of $J''\xi\bar{\xi}$ with characteristic number z_0^2. Similarly, with z_0^2 and ϕ' assumed as characteristic number and function

of $J''\hat{\xi}\bar{\xi}$, we can show that ϕ'', defined as $z_0 J'\bar{\xi}\phi'$, is a characteristic function of $J'\bar{\xi}\xi$ with characteristic number z_0^2.

We wish further to show that, if we have given a unitary and orthogonal set $\{\phi_n''\}$, complete for $J'\bar{\xi}\xi$ and obtain the set $\{\phi_n'\}$ by the relation $\phi_n' = z_n J''\hat{\xi}\phi_n''$ for every n, then the set $\{\phi_n'\}$ is complete for $J''\hat{\xi}\bar{\xi}$. To do this, we must show that any characteristic function ϕ' of $J''\hat{\xi}\bar{\xi}$ is a linear combination of functions of the set $\{\phi_n'\}$. We have

$$\phi' = z_0^2 J'(J''\hat{\xi}\bar{\xi})\phi'.$$

But we have shown that ϕ'' defined as $z_0 J'\bar{\xi}\phi'$ is a characteristic function of $J'\bar{\xi}\xi$ with characteristic number z_0^2, and since the set $\{\phi_n''\}$ is complete for $J'\bar{\xi}\xi$, we have

$$\phi'' = \sum_{n=1}^{m} c_n \phi_n''.$$

Thus
$$J''\hat{\xi}\phi'' = \sum_{n=1}^{m} c_n J''\hat{\xi}\phi_n'' = \sum_{n=1}^{m} c_n \frac{\phi_n'}{z_n}.$$

But
$$J''\hat{\xi}\phi'' = J''\hat{\xi}(z_0 J'\bar{\xi}\phi') = z_0 J'(J''\hat{\xi}\bar{\xi})\phi' = \frac{1}{z_0}\phi'.$$

Thus
$$\phi' = z_0 \sum_{n=1}^{m} \frac{c_n}{z_n}\phi_n'.$$

That the set $\{\phi_n''\}$ is complete for $J'\bar{\xi}\xi$ if the set $\{\phi_n'\}$ is complete for $J''\hat{\xi}\bar{\xi}$, as well as the desired relation concerning characteristic numbers, is proved in an analogous manner.

The unitary and orthogonal properties of the equivalence are proved as is the corresponding property in Theorem X.

VI. *The Equation $Ja\omega = \hat{\xi}$ in Generalization.*

As stated in § III$_c$, the problem of determining under what conditions on the function ω we shall have, for every Hermitian function κ, the equivalence of the properties P_{J_ω} and P_J, suggests at once the problem of the solution for a of the equation $Ja\omega = \hat{\xi}$. The necessary and sufficient condition for the solution of this equation, in instance IV, has been discussed by Picard.* While not so stated by Picard, his discussion really involves the problem which, in generalization, becomes: to decompose a

* *Rendiconti di Palermo*, Vol. 29 (1910).

function ξ into two functions η and ζ (that is, to determine η and ζ so that $\xi = \eta + \zeta$), where η is in the space ω (that is, there exists a function a such that $\eta = Ja\omega$) and ζ is orthogonal to ω (that is, $J\bar{\zeta}\omega = 0$). In case ω is closed as to $\mathfrak{R}$, that is, $J\bar{\zeta}\omega = 0$ implies $J\bar{\zeta}\zeta = 0$, we have secured the existence of a function η which differs from ξ only by one of the exceptional elements, viz., ζ such that $J\bar{\zeta}\zeta = 0$. In case J is restricted by postulation of the property P_0, the function a is a solution of the equation since we have $Ja\omega = \xi$.

If, now, as in § V, we use as foundation the $*$-composite of two systems Σ' and Σ'', the problem reduces to a problem well known in the geometry of a function space, by a suitable choice of the two systems. For, if we choose $\Sigma' = \Sigma'_5$ and $\Sigma'' = \Sigma^{\mathrm{II}}{}_{\eta}$, the problem becomes: given a set of continuous functions $(\xi'_1, \ldots, \xi'_n)$; to decompose a continuous function ξ' into the functions η' and ζ' where η' is in the space $(\xi'_1, \ldots, \xi'_n)_L$ $\left(\text{that is, there exists a sequence of numbers } (a_1, \ldots, a_n), \text{ such that } \eta' = \sum_{i=1}^{n} a_i \xi'_i\right)$ and ζ' is orthogonal to that space (that is, $J'\bar{\zeta}'\xi'_k = 0$ for every k).

As foundation, we use two systems of type Σ'_5 and Σ'_7, respectively,

$$\Sigma' : \left(\mathfrak{A};\ \mathfrak{P}';\ \mathfrak{M}'^{LCDD_0R \text{ on } \mathfrak{P}' \text{ to } \mathfrak{A}};\ \mathfrak{K}' = (\mathfrak{M}'\mathfrak{N}')_*;\ J'^{LMHP \text{ on } \mathfrak{K}' \text{ to } \mathfrak{A}}\right)$$

$$\Sigma'' : \left(\mathfrak{A};\ \mathfrak{P}'';\ \mathfrak{N}''^{LD_0R \text{ on } \mathfrak{P}'' \text{ to } \mathfrak{A}};\ \mathfrak{M}''^{LCDD_0R.B_0\mathfrak{N}''};\ \mathfrak{K}'' = (\mathfrak{M}''\mathfrak{M}'')_*;\right.$$
$$\left. J''^{LMHP \text{ on } \left(\begin{smallmatrix}(\mathfrak{N}''\mathfrak{N}'')_L \\ \mathfrak{K}''\end{smallmatrix}\right) \text{ to } \mathfrak{A}}\right),$$

and form the system

$$\Sigma : \left(\mathfrak{A};\ \mathfrak{P} = \mathfrak{P}'\mathfrak{P}'';\ \mathfrak{M} = (\mathfrak{M}'\mathfrak{M}'')_*;\ \mathfrak{K} = (\mathfrak{K}'\mathfrak{K}'');\ J = J'J''\right).$$

As in § V, we use single and double accents to denote functions of classes of Σ' and Σ'', respectively, while functions of classes in Σ are unaccented. The system Σ is of type Σ'_5. To obtain the theorem we wish to prove, we shall restrict the class $\mathfrak{N}''$ to classes having the property S, defined in § II, above. As we shall prove that, if $\mathfrak{N}''$ has the property C_J, it has also the property S,* this restriction might be imposed by restricting Σ'' to be of type Σ_7.

* Cor., Theorem XVI.

THEOREM XII.—

$$\mathfrak{N}''^{S} \therefore \supset \therefore \xi \ni J'^{2}_{(12)(34)} J''^{2}_{(23)(41)} \bar{\xi}\xi\bar{\xi}\xi \neq 0 \quad \cdot \xi' :\supset:$$

$$\exists (\eta', \zeta') \ni \xi' = \eta' + \zeta' \quad \cdot \exists \alpha'' \ni \eta' = J''\xi\alpha'' \quad \cdot J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0$$

$$\cdot \sim \cdot \quad \sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi' \text{ converges,}^*$$

where $\{\phi_n'\}$ is a unitary, orthogonal, complete set of characteristic functions of $J''\xi\bar{\xi}$ and $\{z_n^2\}$ is the set of corresponding characteristic numbers for $J''\xi\bar{\xi}$, as discussed in Theorems X and XI.

LEMMA.— $J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0 \cdot \supset \cdot J'\bar{\phi}_n'\zeta' = 0 \quad (n).$

For, by Theorem I, $J''J'\bar{\zeta}'\xi\phi_n'' = 0$ and so $J'\bar{\phi}_n'\zeta' = 0$, for every n.

Proof of the Theorem.—

(A) $\xi' = \eta' + \zeta' \cdot \eta' = J''\xi\alpha'' \cdot J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0 \cdot \supset \cdot$

$$\sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi' \text{ converges.}$$

For, $J'\bar{\phi}_n'\xi' = J'\bar{\phi}_n'\eta' + J'\bar{\phi}_n'\zeta' = J'\bar{\phi}_n'\eta' = J'\bar{\phi}_n' J''\xi\alpha''$

$$= \frac{J''\bar{\phi}_n''\alpha''}{z_n}.$$

Thus $J''\bar{\phi}_n''\alpha'' = z_n J'\bar{\phi}_n'\xi' \quad (n),$

and, since by Bessel's inequality,†

$$\sum_n J''\bar{\alpha}''\phi_n'' J''\bar{\phi}_n''\alpha'' \text{ converges,}$$

we have $$\sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi' \text{ converges.}$$

---

* If the class $\mathfrak{N}''$ has the property S, then for every function ξ, such that

$$J'^{2}_{(12)(34)} J''^{2}_{(23)(41)} \bar{\xi}\xi\bar{\xi}\xi \neq 0,$$

and for every function ξ' it is true that the relation: there exist two functions η' and ζ' such that $\xi' = \eta' + \zeta'$ and a function α'', such that $\eta' = J''\xi\alpha''$ and $J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0$, is equivalent to the relation: $\sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi'$ converges.

† By Bessel's inequality, we shall understand the relation,

$$\{\alpha_n''\}^{\text{U.O.}}\, \alpha'' \cdot \supset \cdot \sum_n J''\bar{\alpha}''\alpha_n'' J''\alpha_n''\alpha'' \leqslant J''\bar{\alpha}''\alpha''.$$

The removal of postulation of the definite property from the functional operation does not affect the validity of the proof as given in II, § 5 (f).

(B) $\sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi'$ converges $\cdot\supset\cdot\ \exists\,(\eta', \zeta')\ni \xi' = \eta' + \zeta'$

$$\cdot\exists\,\alpha''\ni\eta' = J''\xi\alpha''\cdot J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0.$$

Let $$x_n = J'\bar{\phi}_n'\xi',$$

and so $$\sum_n \bar{x}_n x_n z_n^2 \text{ converges.}$$

By the property S of the class $\mathfrak{N}''$, we secure the existence of a function α_1'', such that that $x_n z_n = J''\bar{\phi}_n''\alpha_1''$ for every n. Let $\xi_1' = J''\xi\alpha_1''$, and

$$x_n' = J'\bar{\phi}_n'\xi_1' \quad (n).$$

Then, $x_n' = J'\bar{\phi}_n'(J''\xi\alpha_1'') = J''(J'\bar{\phi}_n'\xi)\alpha_1'' = \dfrac{J''\bar{\phi}_n''\alpha_1''}{z_n} = x_n\ (n).$

Thus $$J'\bar{\phi}_n'\xi' = J'\bar{\phi}_n'\xi_1'.$$

Choose $$\eta' = \xi_1', \quad \zeta' = \xi' - \eta', \quad \alpha'' = \alpha_1'',$$

and we have $$J'\bar{\phi}_n'\zeta' = 0 \quad (n),$$

and, by Theorem VIII,

$$J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0.$$

In case $J'^2_{(12)(34)} J''^2_{(23)(41)}\bar{\xi}\xi\bar{\xi}\xi = 0$, the decomposition suggested in the theorem is always possible, since $J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0$ by the Corollary of Theorem III, for every function ζ' of the class $\mathfrak{M}'$. Accordingly, if we choose $\eta' = 0$, $\zeta' = \xi'$, and $\alpha'' = 0$, we have the desired decomposition.

Remarks on the conditions of Theorem XII.

(A) It is obvious that, in case there are only a finite number of characteristic numbers in the set $\{z_n^2\}$, the condition $\sum_n z_n^2 J'\bar{\alpha}_n'\phi_n' J'\bar{\phi}_n'\alpha'$ converges, is always satisfied. This is the case, for example, in the instance already cited in which Σ'' is Σ^{II_n}. In this instance we can choose

$$\mathfrak{N}'' = \mathfrak{M}'' = \mathfrak{M}^{\mathrm{II}_n},$$

and since J'' has the property P_0, the condition

$$J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0$$

reduces to $$J'\bar{\xi}\zeta' = 0.$$

Thus we obtain the theorem already cited; viz., given a system Σ' of type

Σ_5' and a set of functions $(\xi_1', \ldots, \xi_n')$ and a function ξ', these functions all being of the class $\mathfrak{M}'$, we can always decompose ξ' into a function of the space $(\xi_1', \ldots, \xi_n')_L$ and a function orthogonal to that space. In this instance, the condition

$$J'^2_{(12)(34)} J''^2_{(23)(41)} \bar{\xi}\xi\bar{\xi}\xi = 0,$$

which leads to the trivial decomposition noted above, reduces to

$$J' \bar{\xi}_i' \xi_i' = 0$$

for every i.

(B) In case there are an infinite number of functions, unitary and orthogonal, in the classes $\mathfrak{M}'$ and $\mathfrak{M}''$, we can always set up a function ξ and a corresponding function ξ', so that the condition of the theorem is not satisfied. Let $\{\phi_n'\}$ and $\{\phi_n''\}$ be two unitary and orthogonal sets of functions of the classes $\mathfrak{M}'$ and $\mathfrak{M}''$, respectively, each set containing an infinite number of functions. Since $\mathfrak{M}'$ and $\mathfrak{M}''$ each have the properties D and D_0, we secure the existence of functions ϕ_0' and ϕ_0'', and of sequences $\{a_n'\}$ and $\{a_n''\}$ of numbers greater than unity, such that

$$|\phi_n'| \leqslant a_n' \phi_0' \quad \text{and} \quad |\phi_n''| \leqslant a_n'' \phi_0'',$$

and accordingly

$$|\phi_n' \phi_n''| \leqslant a_n \phi_0' \phi_0'',$$

where a_n is equal to $a_n' a_n''$, and the relations hold for every n. Let

$$\xi = \sum_{n=1}^{\infty} \frac{\phi_n' \bar{\phi}_n''}{z_n} \quad \text{and} \quad \xi' = \sum_{n=1}^{\infty} \frac{\phi_n'}{z_n},$$

where z_n is equal to $n^2 a_n$ for every n. Since the classes $\mathfrak{M}$ and $\mathfrak{M}'$ are closed, the functions ξ and ξ' belong to these classes, respectively. But $\sum_n z_n^2 J' \bar{\xi}' \phi_n' J' \bar{\phi}_n' \xi'$ does not converge, since each term in the summation is unity.

(C) If we choose $\mathfrak{P}' = \mathfrak{P}''$, $\mathfrak{M}' = \mathfrak{M}''$, and $J' = J''$, and drop accents throughout to obtain one system Σ of type Σ_7', the decomposition becomes: given a function ω, to decompose a given function ξ into two functions η and ζ, such that $\eta = J\omega\alpha$ and $J(\overline{J\bar{\omega}}\zeta)(J\bar{\omega}\zeta) = 0$. With special reference to the instance IV, if we take $\mathfrak{N}''$ as the class of all functions integrable with their squares on the interval (0, 1) in the sense of Lebesgue and in all other respects Σ' and Σ'' as identical with Σ^{IV}, we obtain the theorem of Picard cited above. We show later that the class $\mathfrak{N}''$ so chosen has the property S.*

* *Vide* § XI (A) together with Corollary to Theorem XVI, and footnote § XI (C).

(D) Finally, to return to the problem suggested by the analogy of the sphere and the ellipsoid as reduced in § III$_a$, we make the reduction to a single system as in (C) and assume that the class $\mathfrak{N}$ $(=\mathfrak{N}'')$ has the property S. We see then, that, if a Hermitian function ω of the class $\mathfrak{K}$, definite as to the class $\mathfrak{M}$, and a function ξ of the class $\mathfrak{M}$, are so related that the decomposition of the theorem is possible, we have

$$J^4\bar{\alpha}\omega\kappa\omega\alpha \geqslant 0 \quad (\alpha) \quad \cdot\supset\cdot \quad J^2\bar{\xi}\kappa\xi \geqslant 0.$$

For, by the theorem, there exist functions η, ζ, and α, such that

$$\xi = \eta + \zeta, \quad \eta = J\omega\alpha, \quad \text{and} \quad J(J\bar{\zeta}\omega)(J\omega\zeta) = 0.$$

Since ω is definite as to $\mathfrak{M}$, this last relation is equivalent to

$$J\bar{\zeta}\zeta = 0.$$

Using Theorem I, we have

$$J^2\bar{\xi}\kappa\xi = J^2\bar{\eta}\kappa\eta = J^4\bar{\alpha}\omega\kappa\omega\alpha \geqslant 0.$$

VII. *Conditions on ω sufficient to secure the Equivalence of the Relations $\kappa^{HP_{J_\omega}}$ and κ^{HP_J}, on the Foundation Σ_7'.*

In this discussion we shall use the notion of a set of functions, complete for $\mathfrak{N}$, as we have already defined this relation, viz.,

$$\{\phi_n\}^{\text{complete for } \mathfrak{N}} : \alpha^{\mathfrak{N}} \cdot\supset\cdot J\bar{\alpha}\alpha = \sum_n J\bar{\alpha}\phi_n J\bar{\phi}_n\alpha.$$

Using this definition on the functions $\alpha+\beta$, $\alpha-\beta$, $\alpha+i\beta$, $\alpha-i\beta$, in turn, and combining results, we have immediately,

$$\{\phi_n\}^{\text{complete for } \mathfrak{N}} :\supset: (\alpha, \beta)^{\mathfrak{N}} \cdot\supset\cdot J\bar{\alpha}\beta = \sum_n J\bar{\alpha}\phi_n J\bar{\phi}_n\beta.$$

THEOREM XIII.—

$$\omega^{H\,\ni\,\exists}\,[\text{characteristic functions of } \omega]^{\text{U.O. complete for } \mathfrak{N}}$$
$$:\supset: \kappa^{HP_{J_\omega} \text{ as to } \mathfrak{N}} \cdot\supset\cdot \kappa^{P_J \text{ as to } \mathfrak{N}}.$$

Proof.—Denote the set of characteristic functions by $(\phi_1, \ldots, \phi_n, \ldots)$.*

* The set of functions may be either infinite or finite as, for example, in instance IV and instance II$_n$, respectively. The summations occurring later in the paper, unless otherwise indicated, are to be taken so as to include each one of these functions.

(A) Given a Hermitian function κ and a function a of the classes $\mathfrak{K}$ and $\mathfrak{N}$, respectively, let

$$a_n = J\bar{a}\phi_n \quad \text{and} \quad \xi_n = J\kappa\phi_n \quad (n),$$

and

$$k_{ij} = J\bar{\xi}_i\phi_j = J^2\bar{\phi}_i\kappa\phi_j \qquad (ij).$$

By Bessel's inequality, we secure at once the convergence of $\sum_n \bar{a}_n a_n$. Also, since

$$\sum_n (\xi_n\bar{\xi}_n)_R = (J\kappa\kappa)_R,^*$$

and the class $\mathfrak{K}$ is dominated by the class $(\mathfrak{M}\mathfrak{M})$, the series $\sum_n (\xi_n\bar{\xi}_n)_R$ converges to a function which is dominated by a nowhere negative function μ^2, of the class $(\mathfrak{M}\mathfrak{M})_R$, and accordingly

$$\sum_n |\, a_n\xi_n \,| \quad \text{converges} \quad (\mathfrak{P};\ \mu).^\dagger$$

Also, since

$$\sum_i (\sum_j k_{ij}\bar{k}_{ij}) = \sum_i J\bar{\xi}_i\xi_i = \sum_i J(J\kappa\phi_i\, J\bar{\phi}_i\kappa) \leqslant J_{(14)}J_{(23)}\kappa\kappa,^\ddagger$$

we secure the convergence of

$$\sum_i (\sum_j k_{ij}\bar{k}_{ij}) = \sum_{ij} k_{ij}\bar{k}_{ij}.$$

(B) Since the set $\{\phi_n\}$ is complete for $\mathfrak{N}$, we have

$$J\bar{a}\kappa = \sum_n J\bar{a}\phi_n\, J\bar{\phi}_n\kappa = \sum_n a_n\bar{\xi}_n.$$

Since, from (A), the convergence of this series is relatively uniform,

$$J^2\bar{a}\kappa a = \sum_i a_i J\bar{\xi}_i a = \sum_i a_i \sum_j J\bar{\xi}_i\phi_j J\bar{\phi}_j a = \sum_i a_i \sum_j k_{ij}\bar{a}_j.$$

Using the relations of convergence established in (A), we see that the order of summation of this last series is immaterial, and write

$$J^2\bar{a}\kappa a = \sum_{ij} a_i k_{ij}\bar{a}_j.$$

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* The suffix R on a function of two or more variables indicates that the arguments of the function are to be set equal.

† II, § 2 (h_3).

‡ This relation is stated in II, § 5 (k), and may be proved without postulation of the property P_0 on the operation J.

(C) Given any function β of the class $\mathfrak{M}$, we have

$$J^4\bar{\beta}\omega\kappa\omega\beta = \sum_{ij} \frac{J\bar{\beta}\phi_i}{z_i} J^2\bar{\varphi}_i\kappa\phi_j \frac{J\bar{\phi}_j\beta}{z_j}.$$

The proof of this statement involves relations similar to those used in (B), and, accordingly, we shall omit the discussion of questions of convergence, merely indicating the steps to be taken.

$$J^2\bar{\beta}\omega\kappa = \Sigma J(J\bar{\beta}\omega)\,\phi_i J\bar{\phi}_i\kappa \quad (\mathfrak{P};\ \mathfrak{M}),$$

$$J^4\bar{\beta}\omega\kappa\omega\beta = \sum_i J(J\bar{\beta}\omega)\,\phi_i J\bar{\phi}_i(J^2_{(23)(45)}\kappa\omega\beta) = \sum_i \frac{J\bar{\beta}\phi_i}{z_i} J\bar{\phi}_i \sum_j J\kappa\phi_j J\bar{\phi}_j (J\omega\beta)$$

$$= \sum_i \frac{J\bar{\beta}\phi_i}{z_i} \Sigma J^2\bar{\phi}_i\kappa\phi_j \frac{J\bar{\phi}_j\beta}{z_j} = \sum_{ij} \frac{J\bar{\beta}\phi_i}{z_i} J^2\phi_i\kappa\phi_j \frac{J\bar{\phi}_j\beta}{z_j}.$$

(D) For a given n we can solve for β the set of equations

$$\frac{J\bar{\beta}\phi_i}{z_i} = \begin{matrix} b_i\ (i = 1, \ldots, n) \\ 0\ \ (i > n) \end{matrix},$$

the solution being $\beta = \sum_{j-1}^{n} z_j\bar{b}_j\phi_j$.

Since, by hypothesis, for every β we have

$$J^4\bar{\beta}\omega\kappa\omega\beta \geqslant 0,$$

we see that $(b_1, \ldots, b_n)\ \cdot\supset\cdot\ \sum_{ij}^{1-n} b_i k_{ij} b_j \geqslant 0.$

But, since

$$\Big(\{a_i\} \ni \sum_{i=1}^{\infty} a_i\bar{a}_i \text{ converges}\Big) \cdot \Big(\{\{k_{ij}\}\} \ni \sum_{ij}^{1-\infty} k_{ij}\bar{k}_{ij} \text{ converges}\Big)$$

$$\cdot\supset\cdot \lim_n \sum_{ij}^{1-n} a_i k_{ij}\bar{a} = \sum_i^{1-\infty} a_i k_{ij}\bar{a}_j,$$

we have $\{a_i\} \ni \sum_i a_i\bar{a}_i$ converges $\cdot\supset\cdot\ \sum_{ij} a_i k_{ij}\bar{a}_j \geqslant 0.$

Using the results obtained in (B), we have the desired relation; viz.,

$$\alpha\ \cdot\supset\cdot\ J^2\bar{\alpha}\kappa\alpha \geqslant 0.$$

*Remarks on the Conditions of Theorem XIII. Equivalence, for a Hermitian Kernel Function ω, of the Properties: General for $\mathfrak{N}$; Existence of a Unitary, Orthogonal Set of Characteristic Functions of ω, which is either Complete for $\mathfrak{N}$ or General for $\mathfrak{N}$.**

Having, then, in view of § III$_a$, this condition on ω, sufficient to secure for every Hermitian function κ the equivalence of the properties, P_{J_ω} as to $\mathfrak{N}$ and P_J as to $\mathfrak{N}$, we discuss the meaning of the condition, that is, for what functions of the class $\mathfrak{K}$ can we establish the existence of a unitary and orthogonal characteristic set of functions, complete for $\mathfrak{N}$? In this discussion we use the definitions of § II.

THEOREM XIV.—

$$\Sigma_7' \cdot \mathfrak{N} \ni \exists\, a^{\mathfrak{N}} \ni J\bar{a}a \neq 0 \ .\supset.\ \omega^H \ :\supset:\ \omega^{\text{general for } \mathfrak{N}}$$

$$\cdot\sim\cdot\ \exists\,[\text{characteristic functions of } \omega]^{\text{U.O. general for } \mathfrak{N}}.$$

(A) $\omega^{H \text{ general for } \mathfrak{N}}\ \cdot\supset\cdot$

$$\exists\,[\text{characteristic functions of } \omega]^{\text{U.O. general for } \mathfrak{N}}.\dagger$$

By hypothesis,

$$a\ :\supset:\ e_1\ \cdot\supset\cdot\ \exists\,\beta_{e_1} \ni J\,\overline{(a-J\omega\beta_{e_1})}(a-J\omega\beta_{e_1}) < e_1.$$

We see immediately that, by virtue of our hypothesis on the class $\mathfrak{N}$, there must exist in that class a function β, such that

$$J^2\bar{a}\omega\beta \neq 0,$$

and accordingly $$J^2_{(23)(41)}\,\omega\omega \neq 0.$$

Thus we secure the existence of a set $\{\phi_n\}$ of characteristic functions of ω, which may be taken as unitary and orthogonal. Using relations (5) and (6) of the proof of Theorem IX, we have

$$J\omega\beta_{e_1} = \eta + \sum_n \phi_n J^2\phi_n\omega\beta_{e_1} = \eta + \sum_n \phi_n \frac{J\bar{\phi}_n\beta_{e_1}}{z_n}, \qquad (7)$$

and $$J\bar{\eta}\eta = 0.$$

---

* For a discussion of relations in this connection obtained in instance IV, see Hans Hahn, "Bericht über die Theorie der linearen Integralgleichungen," *Jahresbericht d. Deutschen Mathem.-Vereinigung*, Vol. 20 (1911), pp. 108, 109.

† Cf. Hilbert, *Göttinger Nachrichten*, 1904, p. 78, and Theorem XV, below.

Using these relations and Theorem 1, we have

$$\alpha :\supset: e_1 \cdot\supset\cdot \exists\, \beta_{e_1} \,\mathfrak{z}\, J\left(\overline{\alpha - \sum_n \phi_n \frac{J\bar{\phi}_n \beta_{e_1}}{z_n}}\right)\left(\alpha - \sum_n \phi_n \frac{J\bar{\phi}_n \beta_{e_1}}{z_n}\right) < e_1.$$

Given a positive number e, choose $e_1 = \frac{1}{2}e$, and n_e such that

$$\left| \dot{\sum_{n=n_{e_1}}} \frac{J\bar{\alpha}\phi_n J\bar{\phi}_n \beta_{e_1}}{z_n} + \frac{J\bar{\beta}_{e_1}\phi_n J\bar{\phi}_n \alpha}{z_n} - \frac{J\bar{\beta}_{e_1}\phi_n J\bar{\phi}_n \beta_{e_1}}{z_n^2} \right| < e_1.*$$

Choose $\beta_e = \beta_{e_1}$, $a_n = \dfrac{J\bar{\phi}_n \beta_e}{z_n}$ $(n = 1, \ldots, n_{e_1})$,

and we have the desired relation; viz.,

$$\alpha :\supset: e \cdot\supset\cdot \exists\, (a_1, \ldots, a_{n_e})\, \mathfrak{z}\, J\left(\overline{\alpha - \sum_{n=1}^{n_e} a_n \phi_n}\right)\left(\alpha - \sum_{n=1}^{n_e} a_n \phi_n\right) < e.$$

(B) $\omega^H \cdot \exists$ [characteristic functions of ω] $^{\text{U.O. general for } \mathfrak{N}}$ $\cdot\supset\cdot$ $\omega^{\text{general for } \mathfrak{N}}$.

Denote the set of characteristic functions by $\{\phi_n\}$. Then, by hypothesis,

$$\alpha :\supset: e \cdot\supset\cdot \exists\, (a_1, \ldots, a_{n_e})\, \mathfrak{z}\, J\left(\overline{\alpha - \sum_{n=1}^{n_e} a_n \phi_n}\right)\left(\alpha - \sum_{n=1}^{n_e} a_n \phi_n\right) < e.$$

Thus, by choosing $\beta_e = \sum_{n-1}^{n_e} a_n z_n \phi_n$, we show that

$$\alpha :\supset: e \cdot\supset\cdot \exists\, \beta_e \,\mathfrak{z}\, J(\overline{\alpha - J\omega\beta_e})(\alpha - J\omega\beta_e) < e.$$

THEOREM XV.—

$$\{\gamma_n\}^{\text{U.O.}} :\supset: \{\gamma_n\}^{\text{general for } \mathfrak{N}} \cdot\sim\cdot \{\gamma_n\}^{\text{complete for } \mathfrak{N}}.$$

(A) $$\{\gamma_n\}^{\text{U.O. general for } \mathfrak{N}} \cdot\supset\cdot \{\gamma_n\}^{\text{complete for } \mathfrak{N}}.\dagger$$

---

* $\dot{\sum_{n=n_{e_1}}}$ is to be interpreted as $\sum_{n=n_{e_1}}^{N}$ or $\sum_{n=n_{e_1}}^{\infty}$ according as there are a finite number N of functions in the set $\{\phi_n\}$ or an infinite number.

† This relation is, in instance IV, due to Hilbert, *Göttinger Nachrichten*, 1906, pp. 443-5.

By hypothesis,

$$a :\supset: e \cdot\supset\cdot \exists (a_1, \ldots, a_{n_e}) \ni J\overline{\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)}\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right) < e. \quad (8)$$

We wish to prove

$$a \cdot\supset\cdot Jaa = \dot{\sum_{n=1}} J\bar{a}\gamma_n J\bar{\gamma}_n a. \quad (9)$$

Suppose this relation does not hold. Then, by Bessel's inequality, we must have, for some function a,

$$J\bar{a}a > \dot{\sum_{n=1}} J\bar{a}\gamma_n J\bar{\gamma}_n a.$$

Let $$e = J\bar{a}a - \dot{\sum_{n=1}} J\bar{a}\gamma_n J\bar{\gamma}_n a > 0,$$

and, by relation (8), determine n_e and $(a_1, \ldots, a_{n_e})$. We have, however,

$$J\overline{\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)}\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)$$

$$= e + \dot{\sum_{n=n_e+1}} J\bar{a}\gamma_n J\bar{\gamma}_n a + \sum_{n=1}^{n_e} (J\bar{a}\gamma_n - \bar{a}_n)(J\bar{\gamma}_n a - a_n) \geqslant e.$$

Thus our assumption is untenable, and relation (9) holds.

(B) $$\{\gamma_n\}^{\text{U.O. complete for } \mathfrak{N}} \cdot\supset\cdot \{\gamma_n\}^{\text{general for } \mathfrak{N}}.$$

Under the hypothesis,

$$a \cdot\supset\cdot J\bar{a}a = \dot{\sum_{n=1}} J\bar{a}\gamma_n J\bar{\gamma}_n a,$$

we wish to prove

$$a :\supset: e \cdot\supset\cdot \exists (a_1, \ldots, a_{n_e}) \ni J\overline{\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)}\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right) < e.$$

Let $$a_n = J\bar{\gamma}_n a \quad (n).$$

Then

$$J\overline{\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)}\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right) = J\bar{a}a - \sum_{n=1}^{n_e} J\bar{a}\gamma_n J\bar{\gamma}_n a,$$

and using our hypothesis, by a proper choice of n_e, we have

$$J\left(\overline{\alpha - \sum_{n=1}^{n_e} a_n \gamma_n}\right)\left(\alpha - \sum_{n=1}^{n_e} a_n \gamma_n\right) < e.$$

Thus, on the foundation Σ_7', if ω is general for $\mathfrak{N}$, the relations $\kappa^{HP_{J_\omega} \text{ as to } \mathfrak{N}}$ and $\kappa^{HP_J \text{ as to } \mathfrak{N}}$, are equivalent for every κ.

We obtain in the next section, other conditions on a unitary and orthogonal set of functions, equivalent to that of complete for $\mathfrak{N}$, but these are to be given on the foundation Σ_7.

VIII. *The Equivalence, on the Foundation Σ_7, of the Properties closed as to $\mathfrak{N}$ and General for $\mathfrak{N}$, for a Unitary and Orthogonal Set of Functions of $\mathfrak{N}$.* On the Foundation Σ_7, the Equivalence, for a Hermitian Function ω, of the following Properties:* (A) *General for $\mathfrak{N}$*; (B) *Ultra-Closed as to $\mathfrak{N}$*; *the Existence of a Unitary and Orthogonal Set of Characteristic Functions of ω which is* (C) *General for $\mathfrak{N}$*, (D) *Complete for $\mathfrak{N}$*, (E) *Closed as to $\mathfrak{N}$.*

In this section, we continue the discussion of the content of the conditions of Theorem XIII, and obtain properties of a unitary and orthogonal set of functions of $\mathfrak{N}$, equivalent to the property, complete for $\mathfrak{N}$. To secure these equivalences, we shall use the system Σ_7, which we have defined as follows:

$$\Big(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{N}^{LD_0R.C_J \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{M}^{LCDD_0R.B_0\mathfrak{N}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHP \text{ on } \left(\begin{smallmatrix}(\mathfrak{N}\mathfrak{N})_L \\ \mathfrak{K}\end{smallmatrix}\right) \text{ to } \mathfrak{A}}\Big).$$

The closure property C_J for $\mathfrak{N}$ has been defined as follows:

$$\{a_n^{\mathfrak{N}}\} \ni \lim_{mn} J(\overline{a_m - a_n})(a_m - a_n) = 0 \ \cdot\supset\cdot\ \exists\, a^{\mathfrak{N}} \ni \lim_{n} J(\overline{a - a_n})(a - a_n) = 0.$$

Notationally, following Fischer, we write for the last relation

$$\lim_{n} a_n \sim a.$$

* These relations, in instance IV, have been obtained by Fischer, *Comptes Rendus*, Vol 144 (1907), pp. 1148–51. Theorems XVI and XVII are there proved for this instance. Cf. also Hans Hahn, *loc. cit.*

THEOREM XVI.—

$$(\{a_n\} \ni \sum_n a_n\bar{a}_n \text{ converges}) \cdot \{a_n\}^{\text{U.O.}} \cdot \supset \cdot \exists\, \alpha^{\mathfrak{N}}$$
$$\ni \alpha \sim \mathop{\mathrm{L}}_{n} \sum_{i=1}^{n} a_i a_i \cdot a_n = J\bar{a}_n \alpha \quad (n).$$

(A) We have

$$\mathop{\mathrm{L}}_{mn,\, m<n} J\left(\overline{\sum_{i=1}^{n} a_i a_i - \sum_{i=1}^{m} a_i a_i}\right)\left(\sum_{i=1}^{n} a_i a_i - \sum_{i=1}^{m} a_i a_i\right) = \mathop{\mathrm{L}}_{mn,\, m<n} \sum_{i=m}^{n} a_i\bar{a}_i = 0,$$

Thus, since $\mathfrak{N}$ has the property C_J, we secure the existence of a function α in the class $\mathfrak{N}$, such that

$$\alpha \sim \mathop{\mathrm{L}}_{n} \sum_{i-1}^{n} a_i a_i.$$

(B) By the Schwarz inequality,

$$(n,\, k) \cdot \supset \cdot J\left(\overline{\alpha - \sum_{i=1}^{n} a_i a_i}\right) a_k J\bar{a}_k \left(\alpha - \sum_{i=1}^{n} a_i a_i\right)$$
$$\leqslant J\left(\overline{\alpha - \sum_{i=1}^{n} a_i a_i}\right)\left(\alpha - \sum_{i=1}^{n} a_i a_i\right).$$

Thus $$\mathop{\mathrm{L}}_{n} J\left(\overline{\alpha - \sum_{i=1}^{n} a_i a_i}\right) a_k = 0,$$

for every k. But

$$n \geqslant k \cdot \supset \cdot J\left(\overline{\alpha - \sum_{i=1}^{n} a_i a_i}\right) a_k = J\bar{a} a_k - \bar{a}_k.$$

Thus $$a_k = J\bar{a}_k a.$$

COROLLARY.—The class $\mathfrak{N}$ of the system Σ_7 has the property S defined in § II.

THEOREM XVII.—$\{a_n\}^{\text{U.O.}} :\supset: \{a_n\}^{\text{closed}} \cdot \sim \cdot \{a_n\}^{\text{general}}$.

(A) $$\{a_n\}^{\text{U.O. closed}} \cdot \supset \cdot \{a_n\}^{\text{general}}.$$

Given a function α of the class $\mathfrak{N}$, let $a_n = J\bar{a}_n \alpha$ for every n. By Theorem XVI, we secure the existence of a function β in the class $\mathfrak{N}$, such that

$$\mathop{\mathrm{L}}_{n} J\left(\overline{\beta - \sum_{i=1}^{n} a_i a_i}\right)\left(\beta - \sum_{i=1}^{n} a_i a_i\right) = 0 \quad \text{and} \quad a_n = J\bar{a}_n \beta \ (n). \quad (10)$$

Thus, for every n, $\quad J(\overline{\alpha-\beta})\,a_n = 0,$

and, since the set $\{a_n\}$ is closed,

$$J(\overline{\alpha-\beta})(\alpha-\beta) = 0.$$

From relations (10), we have

$$\lim_n \left(J\bar{\beta}\beta - \sum_{i=1}^{n} a_i\bar{a}_i\right) = 0. \tag{11}$$

From the inequality of Schwarz and the first of relations (10), we have

$$\lim_n J\overline{\left(\beta - \sum_{i=1}^{n} a_i a_i\right)}\left(\alpha - \sum_{i=1}^{n} a_i a_i\right) = 0.$$

Combining these relations, we have

$$\lim_n \left(J\bar{\alpha}\alpha - \sum_{i=1}^{n} a_i\bar{a}_i\right) = J(\overline{\alpha-\beta})(\alpha-\beta) = 0.$$

Thus, $\quad \lim_n J\overline{\left(\alpha - \sum_{i=1}^{n} a_i a_i\right)}\left(\alpha - \sum_{i=1}^{n} a_i a_i\right) = 0,$

and

$$\alpha \mathrel{:\supset:} e \cdot\supset\cdot \exists\, n_e \,\mathfrak{z}\, J\overline{\left(\alpha - \sum_{i=1}^{n_e} a_i a_i\right)}\left(\alpha - \sum_{i=1}^{n_e} a_i a_i\right) < e.$$

(B) $$\{a_n\}^{\text{U.O. general}} \cdot\supset\cdot \{a_n\}^{\text{closed}}.$$

Suppose $\{a_n\}$ is not closed, and let β be a function such that $J\bar{\beta}\beta > 0$ and $J\bar{\beta}a_n = 0$ for every n. Then

$$J\overline{\left(\beta - \sum_{i=1}^{n} a_i a_i\right)}\left(\beta - \sum_{i=1}^{n} a_i a_i\right) = J\bar{\beta}\beta + \sum_{i=1}^{n} a_i\bar{a}_i \geqslant J\bar{\beta}\beta > 0,$$

for every sequence $\{a_i\}$ and number n. Thus the set $\{a_n\}$ is not general. Our assumption is untenable and the set $\{a_n\}$ is closed.

THEOREM XVIII.—

$$\Sigma_\gamma \cdot \mathfrak{N}\ \mathfrak{z}\ \exists\ \alpha^{\mathfrak{N}}\ \mathfrak{z}\ J\bar{\alpha}\alpha \neq 0 \mathrel{\therefore\supset\therefore} \omega^H \mathrel{:\supset:} \omega^{\text{ultra-closed as to } \mathfrak{N}}$$

$$\cdot\sim\cdot \exists\, [\text{characteristic functions of } \omega]^{\text{U.O. closed as to } \mathfrak{N}}.$$

If ω is Hermitian and ultra-closed as to $\mathfrak{N}$, the condition we have imposed on the class $\mathfrak{N}$ is sufficient to secure the re-

lation $J^2_{(23)(41)}\,\omega\omega \neq 0$, and so, by Theorem IV, the existence of a set of characteristic functions, which may be taken as unitary and orthogonal and complete for ω. Using Theorems I and VIII, we have

$$\omega^H \cdot \{\phi_n\}^{\text{U.O. complete for } \omega} \cdot \alpha \;:\supset:$$

$$J(J\bar{\alpha}\omega)(J\omega\alpha) = 0 \;\cdot\sim\cdot\; J\bar{\phi}_n\alpha = 0 \quad (n),$$

and the desired equivalence follows at once.

To summarize, then, the equivalent relations we have obtained in Theorems XIV, XV, XVII, and XVIII, we state the following theorem.

THEOREM XIX.—*On the foundation Σ_7, if the class $\mathfrak{N}$ be such that in it there exists a function α such that $J\bar{\alpha}\alpha$ is not zero, the following properties are equivalent, for any Hermitian function ω of the class $\mathfrak{K}$: general; ultra-closed; the existence of a unitary and orthogonal set of characteristic functions of ω which is either general, or closed, or complete. Each relation is taken with respect to the class $\mathfrak{N}$.*

We recall that, in order to secure for the operation J_ω the properties of the functional operation of the system Σ_5, the postulates HPP_0 were placed on the function ω. On the other hand, on the foundation Σ_7, the postulation of any one of the equivalent properties of Theorem XIX on a Hermitian function ω is sufficient to secure for every Hermitian function κ the equivalence of the properties, positive as to J and positive as to J_ω. Thus, neither of the properties P or P_0 on the function ω is necessary to secure this equivalence. However, in connection with the instance suggested by the analogy of the sphere and the ellipsoid, the following theorem is of importance.

THEOREM XX.—*On the foundation Σ_7, if a Hermitian function ω is definite as to the class $\mathfrak{N}$, then, for every Hermitian function κ, the properties, positive as to J and positive as to J_ω, are equivalent.*

To prove this theorem, we note first that, if the class $\mathfrak{N}$ is such that for every function α of that class, $J\bar{\alpha}\alpha = 0$, we have the desired equivalence at once. If $\mathfrak{N}$ is such that there exists in the class a function α such that $J\bar{\alpha}\alpha \neq 0$, the hypothesis on ω is sufficient to secure the existence of a unitary and orthogonal set of characteristic functions of ω, which is closed as to $\mathfrak{N}$.

Since a Hermitian function ω which is definite is of either positive or

negative type, we see that, in Theorem XX, we have again the close analogy of the sphere and the ellipsoid in evidence, although the relation here involves Σ_7 with the class $\mathfrak{N}$ and the postulate C_J on that class.

IX. *Instances of Systems* Σ_7.*

The theorems we have proved have been on foundations none of which are more extensive than the system Σ_7 and, accordingly, we give instances of this system on which as foundation each theorem holds.

(A) $\mathfrak{A} \equiv$ [all complex numbers]; $\mathfrak{P} \equiv \mathfrak{P}^{IV} \equiv (0-1)$;

$\mathfrak{N} \equiv$ [all functions of the form $\alpha+i\beta$, where α and β are real, single-valued functions on the range $\mathfrak{P}$, integrable together with their squares in the sense of Lebesgue],

$\mathfrak{M} \equiv \mathfrak{M}^{IV} \equiv$ [all continuous functions of the class $\mathfrak{N}$],

$\mathfrak{K} \equiv \mathfrak{K}^{IV} \equiv (\mathfrak{M}\mathfrak{M})_* =$ [all single-valued, continuous functions on the range $\mathfrak{P}\mathfrak{P}$],

$$J_{st}\kappa(st) \equiv J_s\kappa(ss) \equiv \int_0^1 \kappa(ss)\,ds,$$

$$J\nu_1\nu_2 \equiv \int_0^1 (a_1+i\beta_1)(a_2+i\beta_2)\,ds$$

$$\equiv \int_0^1 a_1a_2\,ds + i\left[\int_0^1 (a_1\beta_2+\beta_1a_2)\,ds\right] - \int_0^1 \beta_1\beta_2\,ds;$$

$$J\sum_i a_i\hat{\nu}_i\tilde{\nu}_i = \sum_i a_iJ\hat{\nu}_i\tilde{\nu}_i.$$

The only property of the elements necessary for us to discuss is the property C_J for the class $\mathfrak{N}$. Fischer† has shown that the sub-class of $\mathfrak{N}$ consisting of all its real valued functions has this property. To extend to the whole class, we wish to prove

$$\{\nu_m^{\mathfrak{N}}\} \ni \lim_{m,n} \int_0^1 \overline{(\nu_m-\nu_n)}(\nu_m-\nu_n)\,ds = 0 \cdot \supset \cdot$$

$$\exists\, \nu^{\mathfrak{N}} \ni \lim_n \int_0^1 \overline{(\nu-\nu_n)}(\nu-\nu_n)\,ds = 0.$$

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* Except as regards the classes $\mathfrak{N}$, these instances have been cited by Prof. Moore. Cf. II, § 1 and § 4 (*a*).

† *Loc. cit.*, pp. 1023–4.

Let $$\nu_n = \alpha_n + i\beta_n,$$

where α_n and β_n are real valued functions, for every n, and the hypothesis reduces to

$$\underset{mn}{\mathrm{L}} \int_0^1 [(\alpha_m - \alpha_n)^2 + (\beta_m - \beta_n)^2]\, ds = 0,$$

or $$\underset{mn}{\mathrm{L}} \int_0^1 (\alpha_m - \alpha_n)^2\, ds = \underset{mn}{\mathrm{L}} \int_0^1 (\beta_m - \beta_n)^2\, ds = 0.$$

Thus, from the results of Fischer, there exist real valued functions α and β in the class $\mathfrak{N}$, such that

$$\underset{n}{\mathrm{L}} \int_0^1 (\alpha - \alpha_n)^2\, ds = 0 \quad \text{and} \quad \underset{n}{\mathrm{L}} \int_0^1 (\beta - \beta_n)^2\, ds = 0.$$

Accordingly,

$$\underset{n}{\mathrm{L}} \int_0^1 [\overline{(\alpha + i\beta) - (\alpha_n + i\beta_n)}][(\alpha + i\beta) - (\alpha_n + i\beta_n)]\, ds = 0.$$

Choose $\nu = \alpha + i\beta$, and we have the desired relation.

(B) $\mathfrak{A} \equiv$ [all complex numbers], $\mathfrak{P} \equiv \mathfrak{P}^{\mathrm{II}_n} \equiv (0, 1, \ldots, n)$,

$\mathfrak{N} \equiv \mathfrak{M} \equiv \mathfrak{M}^{\mathrm{II}_n} \equiv$ [all sequences of n elements from the class $\mathfrak{A}$],

$\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_* = (\mathfrak{M}^{\mathrm{II}_n}\mathfrak{M}^{\mathrm{II}_n})_* =$ [all double sequences of n^2 elements of $\mathfrak{A}$],

$$J_{st}\kappa(st) \equiv J_s\kappa(ss) \equiv \sum_{s=1}^{n} \kappa(ss) = \sum_{i=1}^{n} k_{ii}.$$

That the class $\mathfrak{N}$ of this instance has the property C_J is at once obtainable, since we have to consider only limits of a finite sum.

(C) $\mathfrak{A} \equiv$ [all complex numbers], $\mathfrak{P} \equiv \mathfrak{P}^{\mathrm{III}} \equiv (0, 1, 2, \ldots)$,

$\mathfrak{N} \equiv \mathfrak{M} \equiv \mathfrak{M}^{\mathrm{III}_2} \equiv$ [all infinite sequences $\{a_n\}$ of numbers a_n of $\mathfrak{A}$, such that $\sum_{n=1}^{\infty} a_n \bar{a}_n$ converges],

$\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_* \equiv (\mathfrak{M}^{\mathrm{III}_2}\mathfrak{M}^{\mathrm{III}_2})_* =$ [all infinite double sequences $\{\{k_{ij}\}\}$ of elements k_{ij} of $\mathfrak{A}$, such that there exists a function $\{a_i\}$ of the class $\mathfrak{M}$, such that $|k_{ij}| \leqslant |a_i a_j|$ for every i and j],

$$J_{st}\kappa(st) \equiv J_s\kappa(ss) \equiv \sum_{s=1}^{\infty} \kappa(ss) = \sum_{i=1}^{\infty} k_i\,.$$

Again, the only property of the elements we need to consider is the property C_J of the class $\mathfrak{R}$. The proof that $\mathfrak{R}$ has this property has been given by E. Schmidt.* The notion of "strong convergence" which he has used is, in this instance, equivalent to what we have called "convergence in the mean."

X. *The Equivalence of the Properties C_J and S for a Class $\mathfrak{R}$.*

Since we are to discuss properties of a single class $\mathfrak{R}$, we take as foundation a system simpler than those systems we have considered heretofore; viz.,

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{R}^{LD_0R \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ J^{LMHP \text{ on } (\mathfrak{R}\mathfrak{R})_L \text{ to } \mathfrak{A}}).$$

On this system as foundation, we have

THEOREM XXI.—$\mathfrak{R} \ni \exists \{a_n^{\mathfrak{R}}\}^{\text{U.O. complete for } \mathfrak{R}} :\supset: \mathfrak{R}^{C_J} \cdot\sim\cdot \mathfrak{R}^{S}$.

(A) $$\mathfrak{R}^{C_J} \cdot\supset\cdot \mathfrak{R}^{S}.$$

This relation is proved as in Theorem XVI, and is given in the Corollary to that theorem. It is clear that the proof there given is valid on the foundation given above.

(B) $$\mathfrak{R} \ni \exists \{a_n^{\mathfrak{R}}\}^{\text{U.O. complete for } \mathfrak{R}} :\supset: \mathfrak{R}^{S} \cdot\supset\cdot \mathfrak{R}^{C_J}.$$

To prove this relation, consider a sequence $\{\beta_n\}$ of functions of $\mathfrak{R}$, such that

$$\lim_{m,n} J(\overline{\beta_m - \beta_n})(\beta_m - \beta_n) = 0.$$

We have, then,

$$\lim_{m,n} \sum_i J(\overline{\beta_m - \beta_n})\, a_i\, J\bar{a}_i\, (\beta_m - \beta_n) = 0.$$

Let $$J\bar{a}_i \beta_n = a_{in} \quad (i, n).$$

Then $\sum_i a_{in}\bar{a}_{in}$ converges for every n, and

$$\lim_{m,n} \sum_i \overline{(a_{im} - a_{in})}(a_{im} - a_{in}) = 0.$$

Since, as we have shown in § IX (B) and § IX (C), the classes $\mathfrak{M}^{II_\alpha}$ and $\mathfrak{M}^{III_2}$ have the properties C_J, we secure the existence of a sequence $\{a_i\}$ which is either finite or infinite depending

* E. Schmidt, *Rendiconti di Palermo*, Vol. 25 (1908), pp. 58–60.

on the number of functions in the sequence $\{a_i\}$, and such that $\Sigma\, a_i\bar{a}_i$ converges, and furthermore

$$\lim_n \sum_i (\overline{a_i - a_{in}})(a_i - a_{in}) = 0.$$

Since, by hypothesis, the class $\mathfrak{N}$ has the property S, we secure the existence of a function β of the class $\mathfrak{N}$, such that $a_i = J\bar{a}_i\beta$ for every i. Then

$$\lim_n \sum_i J(\beta - \beta_n)a_i\, J\bar{a}_i(\beta - \beta_n) = 0,$$

or, since the set $\{a_i\}$ is complete for $\mathfrak{N}$,

$$\lim_n J(\beta - \beta_n)(\beta - \beta_n) = 0,$$

and so we have

$$\{\beta_n^{\mathfrak{N}}\} \ni \lim_{m,n} J\overline{(\beta_m - \beta_n)}(\beta_m - \beta_n) = 0 \cdot \supset \cdot \exists\, \beta^{\mathfrak{N}} \ni \lim_n J\overline{(\beta - \beta_n)}(\beta - \beta_n) = 0.$$

Thus, the class $\mathfrak{N}$ has the property C_J as was to be proved.

XI. *Summary.*

Taking, then, as a guide for our discussion, the positive kernel in the instance of the general theory suggested by the analogy of the sphere and the ellipsoid, we have been led to a discussion of conditions on a Hermitian function ω sufficient to secure, for every Hermitian kernel function κ, the equivalence of the properties, positive as to J and positive as to J_ω. We have considered a system in which the functional operation is not restricted by postulation of the definite property P_0, and have shown that the Hilbert-Schmidt theory as to theorems of existence of characteristic functions and numbers, together with expansion theorems, may be obtained on this system as foundation, the expansion theorems being stated with respect to functions of a class $\mathfrak{N}$ on which are placed postulates less restrictive than those placed on the class $\mathfrak{M}$ of the system Σ_5. We have generalized the theory of the non-symmetric kernel in the extended sense of a function on the composite of two ranges not necessarily the same. Using these results and with the same extensions, we have generalized a theorem by Picard on conditions for a solution of an integral equation of the first kind.

We have next considered conditions on a Hermitian function ω sufficient to secure the equivalence stated in our original problem. We

found as such a condition that there exists a unitary and orthogonal set of characteristic functions of ω which is complete for the class $\mathfrak{N}$, and showed that this is equivalent to the condition that ω be general for $\mathfrak{N}$. On the foundation Σ_7, in which the class $\mathfrak{N}$ is restricted by postulation of the property of closure as to convergence in the mean with respect to the functional operation J, we showed the equivalence of either of these conditions on ω to either of the conditions: ω, ultra-closed as to $\mathfrak{N}$, and the existence of a unitary and orthogonal set of characteristic functions of ω which is closed as to $\mathfrak{N}$.

To connect with the instance of the general theory suggested by the analogy of the sphere and the ellipsoid, we have shown that the condition that ω be Hermitian and definite as to $\mathfrak{N}$ is, on the foundation Σ_7, a sufficient condition to secure the equivalence, for every Hermitian function κ, of the properties, positive as to J and positive as to J_ω.

NOTE ON MR. WREN'S PAPER ON "SOME APPLICATIONS OF THE TWO-THREE BIRATIONAL SPACE TRANSFORMATION"*

By H. F. BAKER.

[Received February 23, 1916.—Read March 9th, 1916.]

AT the time when I saw the last draft of Mr. Wren's interesting paper, I had the curiosity to try how far the formulæ given by me in these *Proceedings*, June 25th, 1910, p. 187, would be applicable to prove Mr. Grace's theorem of the six transversals of a line, which is considered in Section IV of Mr. Wren's paper (p. 160). So far as the truth of the six-line theorem is concerned, it is difficult to add to Mr. Grace's and Mr. Wren's proof anything of geometrical interest; but the algebra in my preceding paper might be justified by its establishment of the double six theorem, which is harder to prove. For the present note there is no such justification, and yet the work seems to possess some interest because of the difficulty it exhibits in handling the involved determinants, and the reduction of the theorem to the vanishing of a skew determinant of odd order. A further development would, as it appears to me, be of algebraical interest.

1. Consider six lines intersecting (ξ), namely,

$$(A_1), \ldots, (A_5) \quad \text{and} \quad (A_6) = \omega_1\mu_1(A_1) + \ldots + \omega_5\mu_5(A_5).$$

The first five of these are met in fours by five other lines, all of which meet a line (ξ') given by

$$(x\xi') = \frac{C_{11}}{2\omega_1}(xA_1) + \ldots + \frac{C_{55}}{2\omega_5}(xA_5) - (xA_0)(\omega_1 c_{01} + \ldots + \omega_5 c_{05}),$$

where (A_0) denotes a general arbitrary line, c_{ij} denotes (A_iA_j), and C_{ij} are

* *Proc. London Math. Soc.*, Ser. 2, Vol. 15, 1916, p. 144.

the minors in

$$C = \begin{vmatrix} c_{00} & c_{01} & \dots & c_{05} \\ c_{10} & c_{11} & \dots & c_{15} \\ \dots & \dots & \dots & \dots \\ c_{50} & c_{51} & \dots & c_{55} \end{vmatrix},$$

while ω_i^2 and $\omega_i\omega_j$ are the minors in the determinant C_{00}, which vanishes in virtue of the fact that $(A_1)\dots(A_5)$ are all met by another line.

Now consider the five lines

$$(A_1)\dots(A_4)(A_6),$$

and form the determinant D for these having the same form as C, except that (A_5) is replaced by (A_6). Let the minors of the determinant D_{00} be now called ϵ_i^2, $\epsilon_i\epsilon_j$. Then we easily find

$$\epsilon_1 = \omega_1\omega_5(\mu_5-\mu_1), \quad \epsilon_2 = \omega_2\omega_5(\mu_5-\mu_2), \quad \dots, \quad \epsilon_4 = \omega_4\omega_5(\mu_5-\mu_4), \quad \epsilon_6 = \omega_5,$$

and the equation corresponding to

$$(\xi) = \omega_1(A_1)+\dots+\omega_5(A_5),$$

is

$$\mu_5\omega_5(\xi) = \epsilon_1(A_1)+\dots+\epsilon_4(A_4)+\epsilon_6(A_6).$$

Also we have $\epsilon_1 c_{01}+\dots+\epsilon_4 c_{04}+\epsilon_6 c_{06} = \mu_5\omega_5(\omega_1 c_{01}+\dots+\omega_5 c_{05})$,

and $D_{66} = C_{55}$, $D_{ii} = \mu_5^2\omega_5^2 C_{ii} - 2\mu_5\mu_i\omega_5\omega_i C_{i5} + \mu_i^2\omega_i^2 C_{55}$ $(i = 1, 2, 3, 4)$,

as we easily calculate.

The line which replaces (ξ'), say (ξ_5'), being given by

$$(x\xi_5') = \frac{D_{11}}{2\epsilon_1}(xA_1)+\dots+\frac{D_{44}}{2\epsilon_4}(xA_4)+\frac{D_{66}}{2\epsilon_6}(xA_6)-(xA_0)(\epsilon_1 c_{01}+\dots+\epsilon_4 c_{04}+\epsilon_6 c_{06}),$$

is thus given by

$$(x\xi_5') = \mu_5\omega_5(x\xi')+\tfrac{1}{2}\sum_{i=1}^{4}\left[\frac{D_{ii}}{(\mu_5-\mu_i)\omega_5\omega_i}+\frac{\mu_i\omega_i C_{55}}{\omega_5}-\frac{\mu_5\omega_5 C_{ii}}{\omega_i}\right](xA_i),$$

which, utilising the formula

$$c_{ij}' = -C\,\frac{\omega_i^2 C_{ij}+\omega_j^2 C_{ii}-2\omega_i\omega_j C_{ij}}{2\omega_i\omega},$$

given in *Proc. London Math. Soc.*, June 25th, 1910, p. 190, leads to

$$-\frac{C}{\mu_5}(x\xi_5') = -\tfrac{1}{2}C\omega_5(x\xi')+\frac{c_{51}'}{\mu_5-\mu_1}\mu_1(xA_1)+\frac{c_{52}'}{\mu_5-\mu_2}\mu_2(xA_2)+\dots$$
$$+\frac{c_{54}'}{\mu_5-\mu_4}\mu_4(xA_4).$$

Now consider the skew symmetrical determinant of which the elements are

$$\gamma'_{ij} = \frac{c'_{ij}}{\mu_i - \mu_j} \quad (i, j = 1, 2, \ldots, 5).$$

Then, by taking in turn every five of the lines $(A_1) \ldots (A_5)$, (A_6), we shall be led to six lines* expressible by

$$\begin{array}{llllllll} & & & & & & & \\ -\tfrac{1}{2}C\omega_1(x\xi') + & & & & \gamma'_{12}\mu_2(xA_2) + \ldots + & \gamma'_{15}\mu_5(xA_5) & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\tfrac{1}{2}C\omega_5(x\xi') + \gamma'_{51}\mu_1(xA_1) + & \gamma'_{52}\mu_2(xA_2) + \ldots + & \gamma'_{54}\mu_4(xA_4). & & & & & \end{array}$$

The skew symmetrical determinant whose elements are γ'_{ij} vanishes, and its minors form a symmetrical system, which we can therefore represent by ψ_i^2, $\psi_i\psi_j$. In fact, ψ_i will be rational in the elements γ'_{ij}, being a Pfaffian such as $\gamma'_{12}\gamma'_{34} - \gamma'_{13}\gamma'_{24} + \gamma'_{14}\gamma'_{23}$. Consider then the element (t), a straight line or linear complex, given by

$$(xt) = \frac{\psi_1}{\mu_1}(xA'_1) + \ldots + \frac{\psi_5}{\mu_5}(xA'_5).$$

The condition that this should be a line, namely $(tt) = 0$, is, in fact,

$$\Sigma\Sigma \frac{c'_{rs}\psi_r\psi_s}{\mu_r\mu_s} = 0,$$

or

$$\Sigma\Sigma \left(\frac{1}{\mu_s} - \frac{1}{\mu_r}\right) \gamma'_{rs}\psi_r\psi_s = 0,$$

or

$$-\sum_{r=1}^{5} \frac{\psi_r}{\mu_r} \sum_{s=1}^{5} \psi_s\gamma'_{sr} = 0,$$

which is identically satisfied in virtue of the relations

$$\sum_s \psi_s\gamma'_{sr} = 0.$$

This line meets (ξ'), in virtue of the relations $(\xi' A'_r) = 0$, and as $(A'_r A_s) = 0$, $(A'_r A_r) = C$ (p. 190 of the paper referred to), it meets each of the other five lines referred to. Thus, for example, the condition it should meet the line given by

$$-\tfrac{1}{2}C\omega_1(x\xi') + \gamma'_{12}\mu_2(xA_2) + \ldots + \gamma'_{15}\mu_5(xA_5),$$

is

$$C(\gamma'_{12}\psi_2 + \ldots + \gamma'_{15}\psi_5) = 0,$$

which is identically satisfied. We have thus shewn that, *if the line* (ξ) *be*

* *June 24th.*—These are the lines d of Mr. Wren's diagram, *loc. cit.*, p. 165, his line e being that here called (t).

intersected by the six lines (A_1), ..., (A_5), *and*

$$\mu_1\omega_1(A_1)+\ldots+\mu_5\omega_5(A_5),$$

the six lines derivable from the sets of five of these by the theorem of the double six, are themselves all met by the line

$$\frac{\psi_1}{\mu_1}(A'_1)+\ldots+\frac{\psi_5}{\mu_5}(A'_5),$$

where the ψ_1, ..., ψ_5 *are defined by the fact that* ψ_i^2 and $\psi_i\psi_j$ *are the minors of the skew symmetric determinant of five rows and columns whose elements are*

$$\frac{c'_{ij}}{\mu_i-\mu_j}.$$

2. This is the theorem proved by Mr. Wren, due to Mr. Grace. Mr. Wren's argument uses certain intermediary lines,* fifteen in number, which are necessary to the symmetry of his figure. The existence of these is suggested by the use above made of the skew symmetric determinant. For it is clear that if (A'_1), ..., (A'_n) be any *odd* number of lines, and $c'_{ij}=(A'_iA'_j)$, and if ψ_i^2, $\psi_i\psi_j$ be the minors of the vanishing skew symmetric determinant whose element is γ'_{ij}, defined by

$$\gamma'_{ij}=\frac{c'_{ij}}{\mu_i-\mu_j}\quad(i,j=1,\ldots,n),$$

wherein μ_1, ..., μ_n are arbitrary, then the element (t) defined by

$$(xt)=\frac{\psi_1}{\mu_1}(xA'_1)+\ldots+\frac{\psi_n}{\mu_n}(xA'_n),$$

is a line, the proof being as before. And this line obviously meets every line (if any) which meets all of (A'_1), ..., (A'_n).

Thus, for $n=3$, we may take $\psi_1=\gamma'_{23}$, $\psi_2=\gamma'_{31}$, $\psi_3=\gamma'_{12}$, and so obtain ten lines such as

$$\frac{\gamma'_{23}}{\mu_1}(A'_1)+\frac{\gamma'_{31}}{\mu_2}(A'_2)+\frac{\gamma'_{12}}{\mu_3}(A'_3).$$

In the case when (A'_1), &c., are as in the preceding work, this line meets (ξ'), (A_4), (A_5), and also meets (ξ'_1), (ξ'_2), (ξ'_3), as we see at once by remarking from the preceding formulæ that $(A'_r\xi'_r)=0$, and that

$$(A'_s\xi'_r)=-\mu_r\mu_s\gamma'_{rs}.$$

There are also 5 lines of which, for example, one meets every one of

$$(A_1),\ (A_6),\ (\xi'_2),\ (\xi'_3),\ (\xi'_4),\ (\xi'_5).$$

* *June 24th.*—Denoted by him by *e*, *loc. cit.*, p. 165.

When the theorem of the double six is established the whole configuration arises very naturally by consideration of the quadratic complexes having sixteen given lines, as suggested by Mr. Grace, *Trans. Camb. Phil. Soc.*, Vol. XVI, 1898, p. 164, § 19.

3. The work in § 1 deals with the six given lines having a common transversal in an unsymmetrical way. We may, however, build a symmetrical result, which includes the preceding, and furnishes a basis for an analytical treatment of the case when we begin with seven lines having a common transversal.

For this it is convenient to use an algebraic theorem which may be thus stated. Let λ_{rs} be any constants, and L be the determinant $|\lambda_{rs}|$, of n rows and columns. Let a_{ij} be the terms of a symmetric determinant $A = |a_{ij}|$ of n rows and columns, and let (r, s) denote the quadratic or lineolinear form defined by

$$(r, s) = \sum_i \sum_j a_{ij}\lambda_{ri}\lambda_{sj}.$$

Let Δ denote the determinant whose elements are (r, s), and Δ_{ij} the minors of this determinant. Further, let A_{rs} and L_{rs} be the minors in the determinants $|a_{rs}|$ and $|\lambda_{rs}|$. Then we have

$$\Delta = AL^2,$$

and Δ_{ii} is the quadratic form built with the minors A_{rs} as coefficients and the minors $(L_{i1}, \ldots, L_{in})$ as variables, while Δ_{ij} is the lineolinear form built with the same coefficients and the two rows of variables $(L_{i1}, \ldots, L_{in})$, $(L_{j1}, \ldots, L_{jn})$.

Now beginning as before with a line (ξ) which is a transversal of each of the five lines

$$(\bar{A}_i) = \lambda_{i1}(A_1) + \ldots + \lambda_{i5}(A_5) \quad (i = 1, \ldots, 5),$$

denoting the five lines obtained from these precisely as $(A_1'), \ldots, (A_5')$ were obtained from $(A_1), \ldots, (A_5)$ by $(\bar{A}_r')$, and the line thence obtained by the theorem of the double six, by $(\bar{\xi}')$, we find, for arbitrary (x),

$$(x\bar{A}_r') = L_{r1}(xA_1') + \ldots + L_{r5}(xA_5') - Q_r(x\xi),$$

$$(x\bar{\xi}') = L(x\xi') + Q_1(x\bar{A}_1) + \ldots + Q_5(x\bar{A}_5),$$

where

$$Q_r = \frac{1}{C}\Sigma\Sigma\, c_{ij}' L_{ri} L_{rj} / (\omega_1 L_{r1} + \ldots + \omega_5 L_{r5}).$$

If we have six lines meeting (ξ), namely $(\bar{A}_1), \ldots, (\bar{A}_5)$, and another

$(\bar{A}_6)$ determinable from (A_1), ..., (A_5) by coefficients λ_{61}, ..., λ_{65}, it becomes necessary for symmetry to replace the last equation by one of the form

$$(x\bar{\xi}'_r) = L_r(x\xi') + Q_{r1}(x\bar{A}_1) + \ldots + Q_{r6}(x\bar{A}_6),$$

wherein $Q_{rr} = 0$, there being six lines $(\bar{\xi}'_r)$. That these are all met by another line now presents itself as a consequence of the fact that the skew symmetric determinant of seven rows and columns

$$\begin{vmatrix} 0 & Q_{12} & \ldots & Q_{16} & L_1 \\ Q_{21} & 0 & \ldots & Q_{26} & L_2 \\ \ldots & \ldots & \ldots & \ldots & \\ -L_1 & -L_2 & \ldots & -L_6 & 0 \end{vmatrix}$$

vanishes identically. This work can easily be seen to contain that of § 1.

SOME SINGULARITIES OF SURFACES AND THEIR DIFFERENTIAL GEOMETRY

By H. W. TURNBULL.

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I.

Flex Points on a Surface.

1. When a surface is considered from the standpoint of algebraic geometry, the singularities which are most prominent are those at which the surface ceases to be approximately flat (*e.g.* nodes), and the reciprocal singularities. But from the differential geometry of a surface the peculiarities which arise involve umbilics, points of inflexion, and the parabolic line. Where the indicatrix is circular or parabolic the arrangement of the principal lines is peculiar. M. Darboux* has shewn what happens at umbilics and how the lines of curvature are grouped: in this paper we consider points at which the asymptotic lines take exceptional configurations.

Adopting the method of two parameters which is due to Gauss†, let the coordinates (x, y, z) of a point on the surface be analytic functions of variables p, q. Let

$$x_1 \equiv \frac{\partial x}{\partial p}, \qquad x_2 \equiv \frac{\partial x}{\partial q}, \qquad x_{11} \equiv \frac{\partial^2 x}{\partial p^2}, \qquad \&c.$$

The element of arc ds is given by

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2, \tag{1}$$

the direction cosines of the normal are X, Y, Z, while the asymptotic

* Darboux, *Théorie des Surfaces*, Vol. 4, p. 455.

† Forsyth, *Differential Geometry*, Ch. II.

lines are denoted by

$$L\,dp^2+2M\,dp\,dq+N\,dq^2=0\,; \tag{2}$$

E, F, G, L, M, N being the six fundamental magnitudes of the surface.

2. If in equation (2) the coefficients vanish simultaneously for some pair of values of p and q, then more than two asymptotic lines go through the point on the surface. Thus, if

$$L=M=N=0,$$

when $p=p_0$, $q=q_0$, we shall call (p_0, q_0) a *flex point*. It is an extension of a point of inflexion on a curve to the case of a surface.* We remark that the point lies on the parabolic line given by

$$LN-M^2=0.$$

Now $$Xx_1+Yy_1+Zz_1=0,$$

obviously (Cf. Forsyth, *l.c.*, p. 36). Hence by differentiating

$$\Sigma Xx_{11}-\Sigma X_1x_1=0,$$

generally. But $$\Sigma Xx_{11}\equiv L=0$$

at the flex. Therefore $$X_1x_1+Y_1y_1+Z_1z_1=0.$$

Similarly $$X_1x_2+Y_1y_2+Z_1z_2=0,$$

since $$\Sigma Xx_2=0.$$

Therefore $$X:Y:Z::X_1:Y_1:Z_1,$$

and similarly $$::X_2:Y_2:Z_2.$$

Thus the neighbouring normals are all parallel to the normal at the flex. This is not true of ordinary points on the parabolic line.

Again, let (x, y, z) be the coordinates of the flex, and (x', y', z') those of a near point on the surface, where

$$x'=x+x_1\delta p+x_2\delta q+\tfrac{1}{2}x_{11}\delta p^2+\ldots+\frac{1}{3!}(x_{111}\delta p^3+\ldots)+\ldots,$$

and the distance of (x', y', z') from the tangent plane at (x, y, z) is

$$X(x'-x)+Y(y'-y)+Z(z'-z).$$

Substituting for x', y', z', and remembering that ΣXx_1, ΣXx_2, ΣXx_{11}, &c.,

* In another sense the parabolic line corresponds in three dimensions to a point of inflexion in two dimensions.

vanish, this becomes

$$\tfrac{1}{6}\{\Lambda\delta p^3+3\Lambda'\delta p^2\delta q+3\Lambda''\delta p\,\delta q^2+\Lambda'''\delta q^3\},$$

neglecting fourth order terms, where

$$\Lambda=\Sigma Xx_{111},\quad \Lambda'=\Sigma Xx_{112},\quad \&c.\quad \text{(Cf. Forsyth, p. 59).}$$

It follows that the normal curvature ρ for any direction $(\delta p,\ \delta q)$ at this point is zero, and that ρ changes sign as a point on the normal section passes through the flex. Every plane through the normal at a flex cuts the surface in a curve with a point of inflexion at this point.

3. The extension to higher orders is clear. If all the quantities Λ, Λ', Λ'', Λ''' vanish we get a sort of point of undulation, and the surface does not cross the tangent plane at the point; for the distance of a neighbouring point from the tangent plane at $(x,\ y,\ z)$ is

$$\frac{1}{4!}(M\delta p^4+\ldots).$$

If all the coefficients vanish at $(p_0,\ q_0)$ till we reach the n-th set, then the distance of a neighbouring point from the tangent plane at $(p_0,\ q_0)$ is

$$\frac{1}{n!}(\ldots\mathfrak{X}\delta p,\ \delta q)^n+\ldots,$$

and any tangent line has $(n-1)$-th order contact with the surface at $(p_0,\ q_0)$. At ordinary points $n=2$.

4. Going back to the case when $n=3$, we see that we have a cubic form for the normal curvature. For three values of $\delta p/\delta q$ it vanishes. Thus at the flex there are three directions in which the normal section has a point of undulation. For a real surface there are two types of flexes corresponding to the character of the cubic equation: there may be three real or only one real root.

Imagine a point P to describe a little ring round the flex $(p_0,\ q_0)$. As P crosses the direction which makes the cubic vanish, the value of the perpendicular from P to the tangent plane at $(p_0,\ q_0)$ changes sign. For simplicity consider this plane horizontal.

(*a*) If three roots of the cubic are real, the three corresponding directions will divide the surface into six regions lying alternately above and below the tangent plane at the flex (Fig. 1). Such a flex is at the origin of the surface

$$z=xy(x+y).$$

The two directions given by the Hessian of the cubic are imaginary.

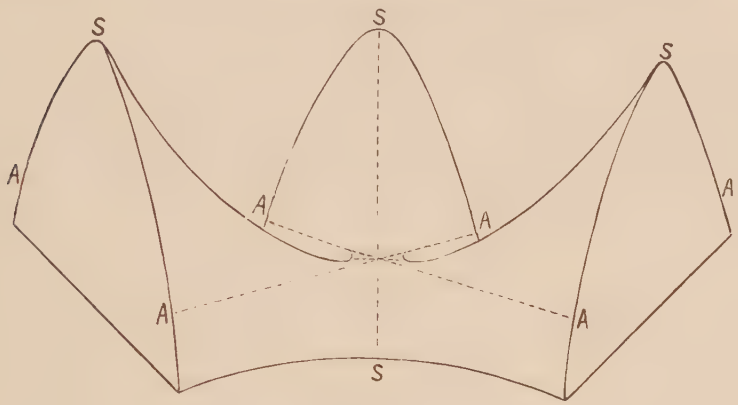

FIG. 1.

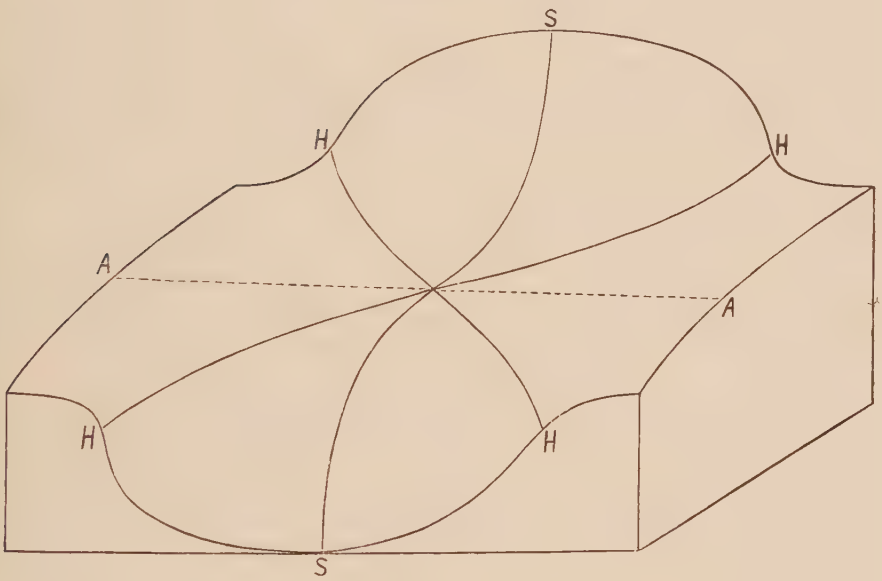

FIG. 2.

SS line of greatest slope, *HH* Hessian line, *AA* asymptotic lines given by the cubic equation (one line invisible in Fig. 1).*

* In mountainous regions these two types of flex points are not uncommon. Both occur in the English Lake District. Esk Hause is a well known place of the kind shewn in Fig. 1, the flex point being about midway between the two saddles and the tangent plane at the flex sloping downwards in a north-easterly direction. In the same district the depression containing Bleaberry Tarn between High Stile and Red Pike, near Buttermere, illustrates the second species of flex.

They are tangents to the parabolic line $LN-M^2=0$, which has a double point at a flex (we shall prove this later in § 12).

(*b*) If one root is real, the Hessian has two real roots, and the parabolic curve has a real double point, the two branches dividing the surface into four regions alternately synclastic and anticlastic. The one real direction AA given by the cubic crosses the anticlastic region (Fig. 2). The simplest example of this shape is at the origin of

$$z=x^3+y^3.$$

The lines of greatest slope are lines of curvature so that the point is a kind of umbilic.

Special cases occur when two or three roots of the cubic are equal. They can be illustrated by modifying Fig. 1 suitably.

In the general case when all the magnitudes L, M, N, and derivatives of higher orders vanish to the n-th order and those of the $(n+1)$-th do not all vanish, the surface is divided into $2(n+1)$ regions at the point, alternately above and below the tangent plane if the roots are real and unequal. The number is lessened by two for every pair of complex roots.

The Configuration of Asymptotic Lines near a Flex Point.

5. It is most useful now to consider the origin as flex point and the z-axis as normal. Then the equation of the surface takes the form

$$z=\frac{1}{3!}(ax^3+3bx^2y+3b'xy^2+a'y^3)+\frac{1}{4!}(ax^4+\ldots)+\ldots. \qquad (1)$$

Calling $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, ... by the symbols p, q, r, s, t as usual, it follows that

$$\frac{L}{r}=\frac{M}{s}=\frac{N}{t}$$

holds for every point of the surface, and it is obvious that equation (1) is the most general form subject to the condition that the origin is a simple flex; that is to say that L, M, N all vanish, but not all of the third derivates vanish when $x=y=z=0$.

The equation of asymptotic lines is

$$r\,dx^2+2s\,dx\,dy+t\,dy^2=0, \qquad (2)$$

or
$$\{ax+by+\tfrac{1}{2}(ax^2+\ldots)\}+2\{bx+b'y+\tfrac{1}{2}(\beta x^2+\ldots)\}\frac{dy}{dx}$$
$$+\{b'x+a'y+\ldots\}\left(\frac{dy}{dx}\right)^2=0.$$

We transform to new variables* u, p from x, y, where

$$x=\frac{du}{dp}=u_p, \qquad y=pu_p-u, \qquad p=\frac{dy}{dx}.$$

If x, y are small it follows that u is small, but p may be finite. Neglecting second order terms in x and y, and thus neglecting second order terms in u and du/dp, the differential equation becomes

$$(a+bp+2bp+2b'p^2+b'p^2+a'p^3)\frac{du}{dp}-(bu+2b'pu+a'p^2u)$$
$$+\text{second order terms}=0,$$

and now taking
$$u=\epsilon u',$$

where ϵ is an infinitesimal constant of the same order as x and y, the equation becomes

$$(a+3bp+3b'p^2+a'p^3)\frac{du'}{dp}-(b+2b'p+a'p^2)u'+\epsilon\Phi=0, \qquad (3)$$

where Φ vanishes if $\epsilon=0$.

When $\epsilon=0$, the solution is

$$u'=A(a'p^3+3b'p^2+3bp+a)^{\frac{1}{3}}, \qquad (4)$$

which is called the principal solution of equation (3); A being the arbitrary constant.

Writing U for the cubic form $a'p^3+\ldots$, it follows that neglecting ϵ^2 the asymptotic lines are given by

$$\left.\begin{aligned} u&=\epsilon u'=\epsilon A U^{\frac{1}{3}} \\ x&=\frac{du}{dp}=\epsilon A\frac{a'p^2+2b'p+b}{U^{\frac{2}{3}}} \\ y&=p\frac{du}{dp}-u=-\epsilon A\frac{b'p^2+2bp+a}{U^{\frac{2}{3}}} \end{aligned}\right\}. \qquad (I)$$

As long as we take two values p_0 and p_1 of p not separated by a root

* Cf. Darboux, *Théorie des Surfaces*, Vol. 4, "Note on Umbilics"; and Forsyth, p. 94, &c.

of the cubic $U = 0$, we can use the equation giving u' as a function of p in the form of a power series in ϵ.

6. Equations (I) determine the traces of the asymptotic lines projected upon the tangent plane $z = 0$. Having drawn one we may draw any number of the curves by taking different values of A.

Let α, β, γ be the roots of $U = 0$.

Then $y - \alpha x = 0$ is an asymptote of all the curves represented by equations (I). For, as p tends to α or β or γ, u tends to zero, but x and y tend to infinity. In fact, when p is near α,

$$x = K(p-\alpha)^{-\frac{3}{2}},$$

$$y - \alpha x = K_1(p-\alpha)^{\frac{1}{2}} = Lx^{-\frac{1}{3}}, \text{ say,}$$

where K, K_1, L remain finite and non-zero when $p = \alpha$. Hence the curve is cuspidal near the asymptote, for the curve is approximately

$$y - \alpha x = Lx^{-\frac{1}{3}},$$

near $y - \alpha x = 0$.

(a) Let $\alpha > \beta > \gamma$, all the roots being real. Referring to equa-

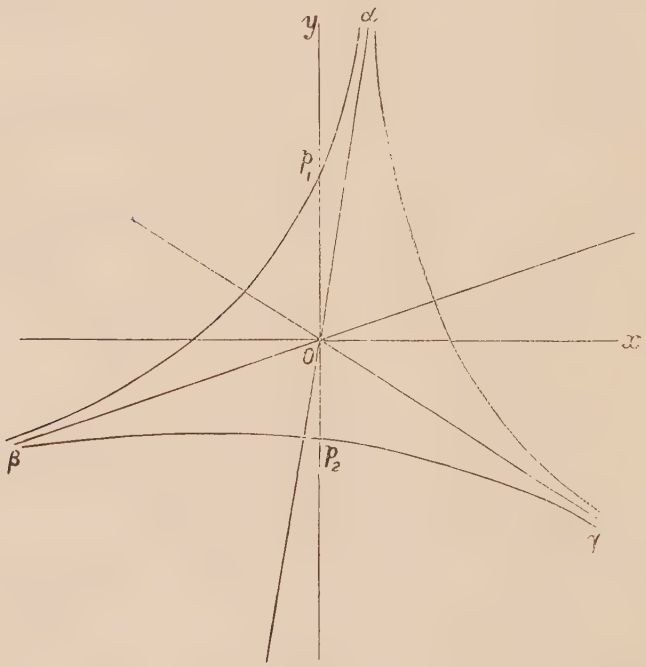

FIG. 3.

tions (I) the curve is

$$\frac{x}{\epsilon A(a'p^2+2b'p+b)} = \frac{y}{-\epsilon A(b'p^2+2bp+a)} = \frac{1}{(a'p^3+3b'p^2+3bp+a)^{\frac{2}{3}}},$$

and has a cusp at each asymptote as we have just seen. It is a sextic curve with three cusps. Moreover, $x = 0$ if $dU/dp = 0$; thus $x = 0$ between values α and β of p and between β and γ, say at p_1 and p_2 respectively.

Thus we can draw the curve. It is described clockwise or anticlockwise according to the sign of A. The family of asymptotic lines will therefore take up a form as in Fig. 4.

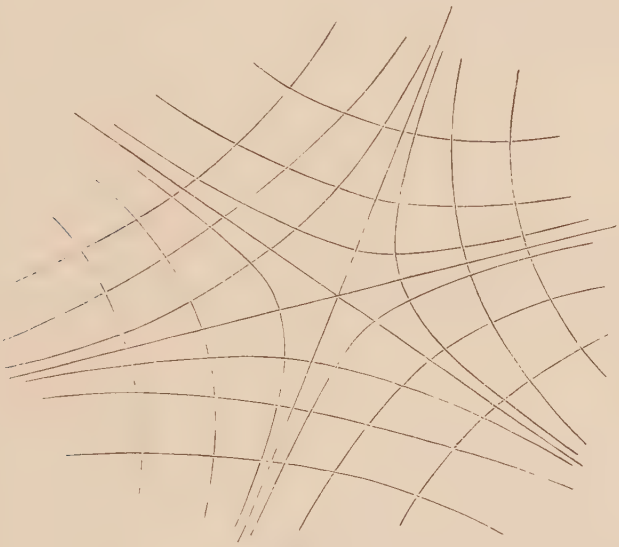

FIG. 4.

When $A = 0$, we get the three asymptotes, and they are the true asymptotic lines through the flex. For $A = \pm\lambda$, we get curves which are reflexions of each other in the origin. The configuration resembles that of lines of curvature near an umbilic. A reference to Fig. 1 shews how the lines actually lie on the surface.

(*b*) Let one root only be real. Then the equations of the curve being

$$\frac{x}{U_1} = \frac{y}{-U_2} = \frac{3\epsilon A}{U^{\frac{2}{3}}},$$

where U_1, U_2 are the first derivates of the cubic U, we may change U to

its canonical form $a_0^3 - b_0^3 t^3$, and choosing suitable axes ξ, η at the same origin, we have

$$\frac{\xi}{a_0^3} = \frac{\eta}{b_0^3 t^2} = \frac{\lambda}{(a_0^3 - b_0^3 t^3)^{\frac{2}{3}}},$$

where λ is a constant which need not be calculated.

This curve is easily traced, having cusps on the axes of ξ and η and at infinity on $a_0 - b_0 t = 0$, *i.e.*, $\frac{\xi}{a_0} = \frac{\eta}{b_0}$.

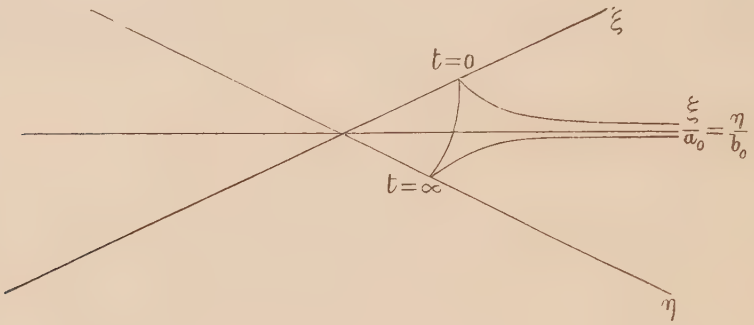

FIG. 5.

The network of lines can now be drawn for different values of λ. They take up the annexed form.

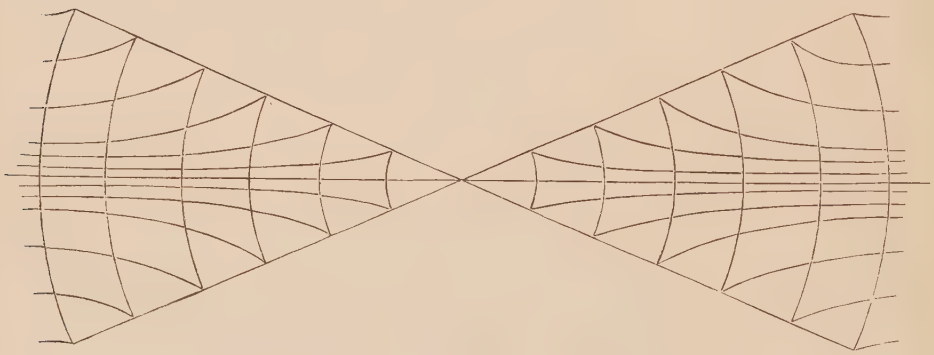

FIG. 6.

Through the flex one *real* asymptotic line passes, and two branches of a cusp locus. This locus is given by the Hessian of the cubic with one real root. It divides the surface into synclastic and anticlastic regions. So it is the parabolic line ($LN - M^2 = 0$) on the surface. A reference to Fig. 2 shews how the lines lie on the surface.

(*c*) If two roots of the cubic are equal, we can take the canonical form

$$\frac{\xi}{t^2} = \frac{\eta}{t} = \frac{\lambda}{\{t^2(t+1)\}^{\frac{2}{3}}},$$

and the surface near the flex is like a skew surface of the type

$$z = x^2 y.$$

The configuration is an obvious modification of Fig. 4.

(*d*) If three roots are equal, the surface is approximately cylindrical at the flex. The surface can be written as

$$z = x^3 + \text{higher order terms},$$

and the asymptotic lines reduce to the generators of the cylinder.

The Flex of any Order.

7. Let $$f = a_0 x^n + \binom{n}{1} a_1 x^{n-1} y + .. + a_n y^n$$

be a polynomial in x, y of order n. Let ϕ denote a sum of terms of higher order in x and y. Then at a flex of the general type the equation of the curve can be written

$$z = f + \phi. \tag{1}$$

Let f_{ij} denote $\frac{\partial^{i+j} f}{\partial x^i \partial y^j}$ divided by the numerical factor common to every term. Then the equation for asymptotic lines

$$r\,dx^2 + 2s\,dx\,dy + t\,dy^2 = 0,$$

becomes $$f_{20}dx^2 + 2f_{11}\,dx\,dy + f_{02}dy^2 = 0, \tag{2}$$

when we omit, as before, higher orders of x and y.

The solution of (2) gives the network of curves which are the asymptotic lines near the flex.

To solve this, let

$$\left.\begin{aligned} y &= px - u, \\ x &= \frac{du}{dp}, \\ p &= \frac{dy}{dx}, \\ y &= vx. \end{aligned}\right\} \tag{3}$$

so that

and also let

When we substitute v for y/x in $f, f_{01}, \ldots$, let the forms be denoted by $g, g_{01}, \ldots$, so that

$$\begin{aligned} g &= a_0+na_1v+\ldots+a_nv^n, \\ g_{10} &= a_0+(n-1)a_1v+\ldots+a_{n-1}v^{n-1}, \\ g_{11} &= a_1+\ldots \qquad\qquad +a_{n-1}v^{n-2}, \\ &\&c. \end{aligned}$$

Substituting for y, the equation (2) becomes a function of x, u and p, homogeneous in x and u of degree $n-2$, and since

$$g_{20}+2g_{11}v+g_{02}v^2 \equiv g\,;$$

therefore the coefficient of x^{n-2} in the resulting equation is the form g with v replaced by p.

Theoretically we may then write the equation as

$$g(p)\{(x-a_1u)(x-a_2u)\ldots(x-a_{n-2}u)\} = 0, \tag{4}$$

where $a_1, a_2, \ldots$ are definite functions of p.

If all the a's are distinct, there are $n-2$ solutions of (4) of type

$$\left.\begin{aligned} u &= c_r e^{\int^p a_r\,dp} \\ x &= ua_r \\ y &= u(a_r p-1) \\ y-px &= -u \end{aligned}\right\}, \tag{I}$$

c_r being an arbitrary constant.

8. Now let $\theta_1, \theta_2, \ldots, \theta_n$ be the values of v for which g vanishes, and let θ denote any one of these n quantities. Then in the first place

$$y-\theta x = 0$$

is an asymptote to the family of curves denoted by (I). See § 6.

If now we assume all the lines

$$y-\theta x = 0$$

real and distinct, we can determine the general shape of these curves and obtain the configuration of asymptotic lines. The question, however, is difficult, supposing pairs of values θ to be complex or coincident. Assume

then that the roots are real and distinct in the equation

$$g \equiv a_0 + n a_1 v + \ldots + a_n v^n = 0.$$

Consider the equation

$$p^2 g_{02} + 2p g_{11} + g_{20} = 0,$$

as that of a curve in the (p, v) plane. The curve will be of order n with two real and distinct asymptotes parallel with $p = 0$, and $n-2$ real and distinct asymptotes parallel with

$$v = 0. \qquad (\alpha)$$

The conditions for a double point are

$$\left.\begin{aligned} p g_{02} + g_{11} &= 0 \\ p g_{11} + g_{20} &= 0 \\ p^2 g_{03} + 2p g_{12} + g_{21} &= 0 \end{aligned}\right\}. \qquad (\alpha)$$

Hence

$$g_{20} g_{02} - g_{11}^2 = 0,$$

and also

$$g_{11}^2 g_{03} - 2 g_{11} g_{12} g_{02} + g_{21} g_{02}^2 = 0.$$

This latter is the same as

$$g_{02} g_{20} g_{03} - 2 g_{11} g_{12} g_{02} + g_{21} g_{02}^2 = 0,$$

or

$$g_{02} \left\{ \frac{d}{dv} (g_{20} g_{02} - g_{11}^2) \right\} = 0.$$

If $g_{02} = 0$, then by equation (α) either g_{11} and g_{20} vanish or p is infinite. But g is a general binary form and the three second polars g_{02}, g_{11}, g_{20} cannot vanish together. Also the Hessian $g_{02} g_{20} - g_{11}^2$ and its derivate cannot vanish together for the general form g. Thus no finite double point exists. (β)

9. Again, the line $p = v$ cuts the curve in n different points $\theta_1, \theta_2, \ldots, \theta_n$ given by $g = 0$, and the asymptotes cut this line at points between the first and last of these n points, since the roots of g_{02} lie between those of g_{01}, and therefore within the extremes of g. (γ)

Again, the line

$$p = v + a,$$

cuts the curve where

$$(v+a)^2 g_{02} + 2(v+a) g_{11} + g_{20} = 0,$$

or

$$g + 2a g_{01} + a^2 g_{02} = 0,$$

and all the roots of this equation in v are real. For, if $b_1, b_2, \ldots, b_{n-1}$ are

the roots of g_{01}, the series of ascending values

$$-\infty,\ b_1,\ b_2,\ \ldots,\ b_{n-1},\ \infty,$$

of v make the left member of the equation alternately $+$ and $-$, since both g and g_{02} change sign as v passes through a root of g_{01}.

$\therefore$ Hence lines parallel with $p = v$ cut the curve in n real and distinct points. (δ)

Moreover, each asymptote, $v =$ constant, cuts the curve in one finite point. (ϵ)

10. These considerations (α) ... (ϵ) enable the shape of the curve to be traced. The lines $p = v + a$ must be cut by two even branches exterior to $n-4$ odd branches as in the figure. For no two branches may cross

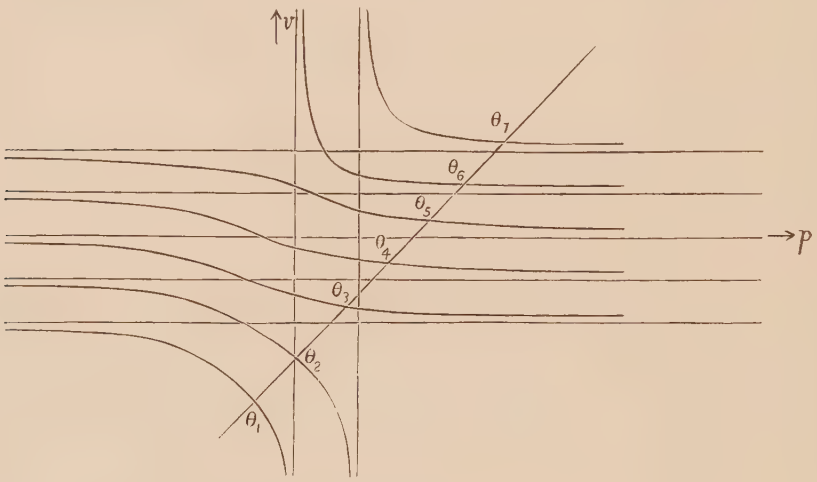

FIG. 7.

(β), nor can one branch bend back through 180°, and so allow a tangent parallel with $p = v$ (δ).

If n is odd all the branches are parts of one branch described continuously, say from θ_1 to θ_1. If n is even they belong to two branches, θ_1 to θ_1 and θ_2 to θ_2 (insert another branch horizontally in Fig. 7, leaving four, the outside branches, unaltered but increasing the odd number of interior branches by unity).

11. We can now proceed to the asymptotic curves, remembering that $p = dy/dx$, $v = y/x$, and that $y = \theta x$ is an asymptote. Let a point

describe the (p, v) curve from θ_1 to θ_1 and a corresponding point the asymptotic curve.

If n is odd the order in which the points θ are reached in the (p, v) curve is

$$\theta_1, \theta_3, \theta_5, \ldots, \theta_n, \theta_2, \theta_4, \ldots, \theta_{n-1}.$$

If n is even the order is

$$\left.\begin{matrix} \theta_1, \theta_3, \theta_5, \ldots \\ \theta_2, \theta_4, \theta_6, \ldots, \theta_n \end{matrix}\right\},$$

in two separate branches.

The general appearance of the figure shewing the two curves answering to equal and opposite values of c is the same. In each case the branch of the (p, v) curve from θ_1 to θ_3 (or any similar interval) gives a simple link from one asymptote to another (θ_1 to θ_3) in the (x, y) curve. It passes through an infinity of p (dy/dx), and cuts each line $v =$ constant (*i.e.*, radius through O) once only in the link. The opposite values of c enable us to trace one curve starting at one extremity of the θ_1 radius and the other curve starting at the other extremity. But in the case when n is

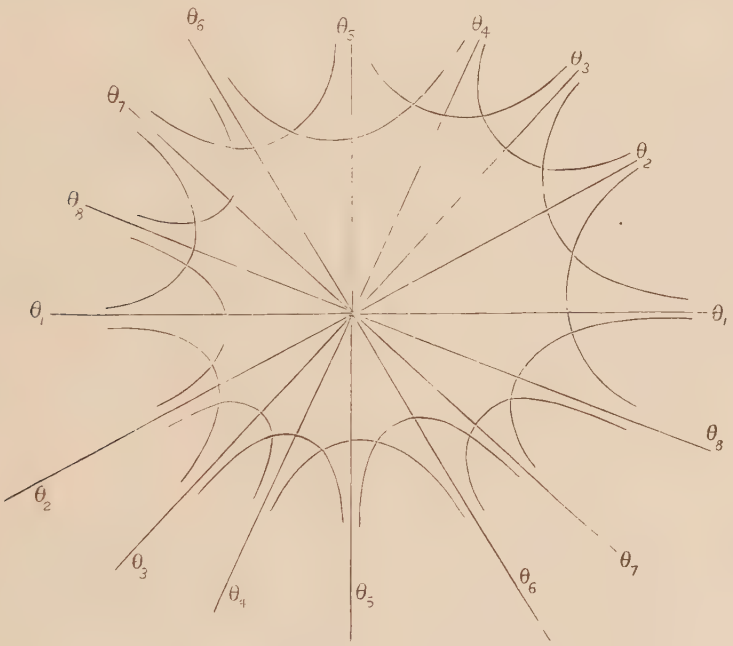

FIG. 8.

even, we must make a start at both θ_1 and θ_2 in order to draw the complete curves.

The Distribution of Conjugate Lines at a Flex.

12. At a flex the conjugate relation is not symmetrical. Consider a first order flex.

Let the surface have for its equation (in coordinates z, x_1, x_2)

$$z = \frac{1}{3!}\, a_x^3 + \text{higher orders},$$

a_x^3 being the symbolic form for $a_0x_1^3+3a_1x_1^2x_2+3a_2x_1x_2^2+a_3x_2^3$.

We can talk of the direction (x), meaning the direction of the point $(x_1, x_2, 0)$ from the origin.

Then the direction (y) is conjugate to (x), if

$$a_x^2 a_y = 0,$$

this being the direction about which the tangent plane rotates as the point of contact moves from the origin to a near point in the direction (x).

Hence two directions (x) have a conjugate in common. If (y) is either Hessian line the two values of (x) coincide for

$$H = (ab)^2 a_y b_y = 0.$$

Writing the cubic in the form X^3+Y^3 and H as XY and the Jacobian covariant as X^3-Y^3, it is easy to see that one root of H gives a conjugate to the other root; one root of X^3+Y^3 gives a conjugate to one root of X^3-Y^3 and also to itself.

The Hessian directions at the flex are tangents to the parabolic line.

We mentioned this in § 4*a*. It is easily proved by forming the expression $V^2(LN-M^2)$ of § 2.

To the first set of small quantities

$$V^2(LN-M^2) = \begin{vmatrix} a_0x_1+a_1x_2, & a_1x_1+a_2x_2 \\ a_1x_1+a_2x_2, & a_2x_1+a_3x_2 \end{vmatrix} \text{ which is } H,$$

and

$$V^2 = EG-F^2 = 1.$$

Hence, for $H(=(ab)^2a_xb_x) > 0$, the surface is synclastic with imaginary asymptotic lines; for $H < 0$, it is anticlastic with real asymptotic lines; for $H=0$ on the boundary, we have the two branches of the parabolic line at the flex.

Similarly the second order flex illustrates the binary quartic and its concomitants.

Here three directions have a common conjugate.

II.

The Relations between the Parabolic Line and the Asymptotic Curves in the Neighbourhood of Nodes and Tropes.

1. At an ordinary point of the parabolic line we may write the equation of the surface as

$$z = \frac{k}{2}x^2 + \frac{1}{6}(a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3) + \frac{1}{4!}(a_0x^4+\ldots), \text{ \&c.} \quad (1)$$

If we calculate p, q, r, s, t, we find that

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial x^2} = r = k+(a_0x+a_1y)+\tfrac{1}{2}(a_0x^2+2a_1xy+a_2y^2)+\ldots \\ \frac{\partial^2 z}{\partial x\,\partial y} = s = \quad (a_1x+a_2y)+\tfrac{1}{2}(a_1x^2+2a_2xy+a_3y^2)+\ldots \\ \frac{\partial^2 z}{\partial y^2} = t = \quad (a_2x+a_3y)+\tfrac{1}{2}(a_2x^2+2a_3xy+a_4y^2)+\ldots \end{aligned}\right\}.$$

Thus $$rt-s^2 = k(a_2x+a_3y)+\text{higher powers of } x, y. \quad (2)$$

This puts into prominence the fact that the asymptotic line ($z=0=x$) and the parabolic line ($z=0$, $a_2x+a_3y=0$) are not in the same direction in general. In fact, the parabolic line is not an envelope of the asymptotic curves, but a cusp locus. At the isolated points where $a_3=0$, the two curves appear to touch, in which case the asymptotic curve has not a cusp, but two ordinary branches in contact. M. Giraudet was the first to point this out (*Mem. Sci. d. Lille*, 1861).

If, however, both a_2, a_3 vanish, the origin is a double point on the parabolic curve, whose tangents are given by the terms of the second order in $rt-s^2$,

$$\left(a_2-\frac{2a_1^2}{k}\right)x^2+2a_3xy+a_4y^2. \quad (3)$$

Hence a flex ($k=0$) is not the only point at which the parabolic curve has a double point. The other case is obtained when the surface is expressible as

$$z = \frac{k}{2}x^2+\frac{x^2}{3!}(a_0x+3a_1y)+\frac{1}{4!}(a_0x^4+\ldots), \text{ \&c.} \quad (4)$$

The shape of the surface at such a double point of the parabolic line is here shewn, looked at from the outside and the inside.

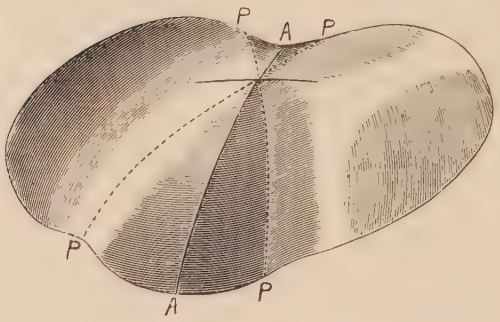

FIG. 9.

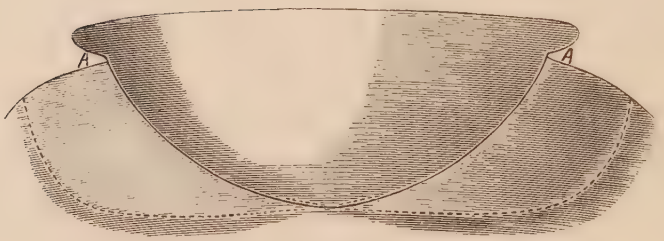

FIG. 10.

Parabolic line dotted, asymptotic line AA.

[Only the immediate neighbourhood of the double point is to be considered.]

2. *In the one case where the parabolic curve touches the asymptotic curve, the torsion of the former is zero.*

In fact we have $a_3 = 0$, and

$$\left.\begin{aligned} 0 = rt - s^2 &= ka_2x + k(a_2x^2 + 2a_3xy + a_4y^2) - 2(a_1x + a_2y)^2 + \dots \\ z &= \frac{k}{2}x^2 + \tfrac{1}{6}(a_0x^3 + \dots)\dots \end{aligned}\right\}$$

for the equations of the parabolic curve.

If y is the parameter of the curve, we have

$$\left.\begin{aligned} x &= p_0y^2 + p_1y^3 + \dots \\ z &= q_0y^4 + \dots \end{aligned}\right\} \qquad p_0 = \frac{a_2}{k} - \frac{a_4}{2a_2},$$

as possible expansions of x, z in terms of y.

Hence $\frac{dz}{dy}$, $\frac{d^2z}{dy^2}$, $\frac{d^3z}{dy^3}$ all vanish, so that the torsion is zero, since the curvature is not zero in general ($p_0 \neq 0$).

If then the parabolic curve is a true envelope of the asymptotic lines for a finite distance l, this portion l must be *a plane curve.* Such a curve l is a trope curve, lying in a singular tangent plane or trope. It is obvious that all the tropes (usually conics*) of a surface form branches of the parabolic curve because one of the principal radii of curvature of the surface along a trope is infinite. Thus the tropes are the only true envelopes of the asymptotic lines. As these singularities are exceptional, it follows that the asymptotic lines have no envelope (but only a cusp locus) in general.

3. If we project the surface orthogonally upon a trope plane, the projections of the asymptotic lines near the trope curve lie in the parts of the plane to which the curve is concave. If the trope is of higher order than the second it will contain points of inflexion. At such points the region of the plane with real asymptotic line-traces crosses the curve. We should expect that another branch of the parabolic curve passes through the point of inflexion. This is the case and is easily verified by working out $rt-s^2$ for the surface

$$z = (x-y^3+\ldots)^2\,(a+bx+cy+\ldots),$$

for which $$rt-s^2 = x\,(px+qy)+\text{higher orders},$$

p, q being constants containing a, b, c, ...

The asymptotic lines at such a point must in general take up the configuration of Fig. 11.

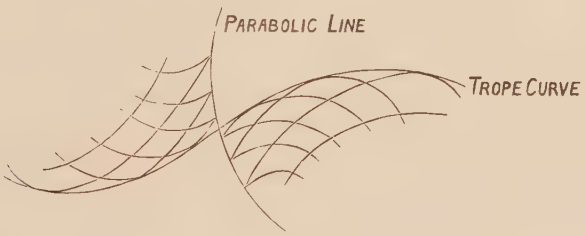

FIG. 11.

Two non-coincident trope planes cannot contain trope curves which

* A trope of class n is the reciprocal of a multiple point of order n. In algebraic geometry then, a conical trope would be as common as a conical node. And the higher the value of n the greater the number of conditions to be satisfied. The same is true in differential geometry.

intersect, unless the intersection be at a multiple point or line of the surface.

The parabolic curve may have a cusp, triple point, &c., by suitably arranging the coefficients in equations (1), (2). We shall not, however, multiply these cases, but pass on to consider the nature of a multiple point in connection with the parabolic line. This point, being the correlative of a trope must be important in connection with the parabolic line as the latter maintains its character under reciprocation.

Moreover this gives a starting point from which to discuss multiple points in the parametric representation of a surface.

Conical Points and Asymptotic Lines.

4. Let a surface be taken as

$$(\xi, \eta, \zeta)_2 + (\xi, \eta, \zeta)_3 + \ldots = 0,$$

where $(\xi, \eta, \zeta)_r$ is a homogeneous function of r dimensions in ξ, η, ζ.

By a suitable choice of axes at the same origin, we may assume

$$xz - y^2 + u_3(x, y, z) + \ldots = 0, \tag{1}$$

to be the surface (projection, &c., not altering asymptotic lines).

Take λ, q two parameters such that $\lambda = 0$ for the point ($x = 0$, $y = 0$, $z = 0$).

Then we may assume for small values of x, y, z,

$$\left.\begin{aligned} x &= \lambda + \frac{\lambda^2}{2} a_2 + \frac{\lambda^3}{3!} a_3 + \ldots \\ y &= \lambda q + \frac{\lambda^2}{2} b_2 + \ldots \\ z &= \lambda q^2 + \frac{\lambda^2}{2} c_2 + \ldots \end{aligned}\right\}, \tag{2}$$

where a, b, c are analytic functions of q.

Forming the second order magnitudes L, M, N, it is easy to shew that

$$\begin{aligned} L &= \theta + \text{terms which vanish when } \lambda = 0, \\ M &= \quad ,, \quad ,, \quad ,, \\ N &= \quad ,, \quad ,, \quad ,, \end{aligned}$$

where θ is a sextic in q and is independent of λ.

In fact θ is $u_3(1, q, q^2)$, the function which appears in equation (1).

Call the roots of $\theta = 0$, $q_1, q_2, \ldots, q_6$. They answer to the six branches of the parabolic line which pass through the node. By examining the equation for asymptotic lines

$$L\,d\lambda^2 + 2M\,d\lambda\,dq + N\,dq^2 = 0,$$

near $\lambda = 0$ (*i.e.*, near the node). we should obtain the following results:— asymptotic lines, approaching the conical point by any direction except the six $q_1, \ldots, q_6$, have a cusp at the point, and if a region between consecutive real roots $q_i q_j$ of θ is synclastic on one side of the conical point, it is anticlastic on the other.

As regards the directions $q_1 \ldots q_6$, it is best to take the equation of the surface as

$$y^2 + z(ax+by+cz) + y^3 + dz^3 + x(ey^2+fyz+gz^2) + x^2(hy+iz) + u_4 + \ldots = 0, \tag{3}$$

where the axis of x gives a root of $\theta = 0$, and the plane $z = 0$ is a tangent plane. This plane cuts the surface in the curve

$$y^2 + y^3 + exy^2 + hx^2y + u_4 + \ldots = 0. \tag{4}$$

This curve has a double cusp along $y = 0$ at the origin, so that the corresponding asymptotic line has a double cusp in the direction of the parabolic curve [which agrees with the ordinary case where these two curves happen to possess a common tangent (§ 14)].

Proceeding to a point P on the parabolic curve at a small distance from O, the conical point, the surface becomes ordinary and the asymptotic curve at P has a cusp, not in the direction OP; unless the branch OP happens to be a trope, in which case the asymptotic curve touches OP at P.

Hence for an isolated conical point, we get a configuration of asymptotic lines, as in Fig. 12.

At all points $I_1 I_2$ of q_i, cusps exist inclined less and less to q_i till at O the cusp osculates the line q_i and points both ways [the curvature of both branches being finite by (4)]. Continuing from O, cusps start at O pointing in all directions between q_i, q_j giving pairs of branches such as aa, till the direction q_j is reached at which a double cusp exists and a transference into the region $q_j q_k$ can be made. The six lines q are, in fact, branches of the parabolic line.

The number of cases is great, depending upon the reality of the roots of the sextic. The simplest is when all the roots are imaginary. Then all the asymptotic curves have cusps at O pointing into one region only.

This is the most easily imagined node and is the only kind which exists on a surface of revolution.

FIG. 12.

Examples and Illustrations.

5. *Kummer's quartic surface* is a good example of some of these properties. The parabolic curve is made up of the sixteen tropes and sixteen nodes. The asymptotic curves run over the regions bounded by nodes, having cusps at the nodes and touching the trope curves, which also traverse these regions, once in the course from one node to the next.

The *surface moulure** gives a simple illustration of the existence of cusp loci and envelopes of the asymptotic lines.

In this case we take

$$\left.\begin{aligned} x &= U\cos v + \int V \sin v\, dv \\ y &= U\sin v - \int V \cos v\, dv \\ z &= \int \sqrt{(1-U'^2)}\, du \end{aligned}\right\} \quad \begin{aligned} &U \text{ being a function of } u; \\ &V \text{ a function of } v; \\ &W = \sqrt{(1-U'^2)}. \end{aligned}$$

This gives

$$\frac{X}{-(U-V)W\cos v} = \frac{Y}{-(U-V)W\sin v} = \frac{Z}{U'(U-V)} = \frac{1}{U-V}.$$

Thus $U-V=0$ may be a singular point or line. Leaving this aside,

$$L = WU''-U'W', \quad M=0, \quad N=-W(U-V).$$

The parabolic line is $W(U-V)(WU''-U'W')=0$,

and asymptotic lines are

$$(WU''-U'W')du^2 - W(U-V)\,dv^2 = 0.$$

(1) $W=0$ is a set of parallel tropes, at points of which (except possibly where $U=V$) the asymptotic lines touch the parabolic line in the common direction $(-\sin v,\ \cos v,\ 0)$.

(2) $WU''-U'W'=0$ is the true parabolic line, at points of which, $v=$ const. gives the direction of the asymptotic cusp, and $u=$ const. that of the parabolic line itself.

An *anchor ring* has real asymptotic lines on the inner portions of its surface, touching the trope circles, and has cusps at the double points on the axis (if they are real). But consider the surface

$$(r-a)^2+(z-b\phi)^2 = c^2 \quad \text{[cylindrical coordinates]},$$

obtained by revolving a circle on a helix instead of on a circle.

The asymptotic lines are given by

$$d\phi^2 . r(r-a) + 2d\phi . dr \frac{b}{r}(z-b\phi) + \frac{c^2 dr^2}{(z-b\phi)^2} = 0.$$

* Darboux, *Théorie des Surfaces*, Vol. I, p. 104.

The discriminant is

$$b(z-b\phi)^2 \pm r^2 c(r-a) = 0, \quad \text{or} \quad r = \text{constant}. \tag{1}$$

At points of this the direction of the asymptotic line is

$$d\phi . r(r-a) + dr \frac{b}{r}(z-b\phi) = 0,$$

or

$$d\phi^2 . r^2(r-a) \pm bcdr^2 = 0,$$

on using the equation (1). But the direction of the curve (1) is that of $r = \text{const}$. These two directions are not the same unless

$$\text{either } r^2(r-a) = 0, \text{ or } b = 0.$$

The former never occurs on the parabolic line; the latter, only for an anchor ring. The spiral surface with $b \neq 0$ gives the true cusp locus as it should do.

THE CONSTRUCTION OF CO-APOLAR TRIADS ON A CUBIC CURVE

By WILLIAM P. MILNE, M.A., D.Sc.

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1. *Introduction.*

In the *Proc. London Math. Soc.*, Ser. 2, Vol. 9, Pt. 3, I discussed the following question :—

Given a cubic curve, to find two triads of points ABC and PQR situated on the curve, such that if X be any point on the curve, the two pencils of lines X[ABC] and X[PQR] shall be apolar for all positions of X on the curve.

In that discussion I shewed that each of the triads ABC and PQR is of a particular type and must satisfy the condition that any point of one triad must lie on the bi-polar of the other two points of the same triad. Such triads were designated "apolar triads," and two apolar triads fulfilling the property of subtending apolar pencils at every point on the cubic were called "co-apolar triads." I also shewed that if one apolar triad be assigned, all the other co-apolar triads can be found by projecting the points of the given apolar triad through a point of the curve on to the curve again, so that the triad thus obtained consists of three collinear points, and then finding all the triangles co-apolar with this collinear triad.

The object of the present paper is to give a method for constructing the triads co-apolar with a given triad without projecting the given apolar triangle into a collinear set of points. It will be found that if ABC be a given apolar triad and P any point on the curve, then two triangles exist co-apolar with ABC and having P as one of their vertices.

2. *Co-Apolar Systems.*

Let ABC be an assigned apolar triangle, and let P be any arbitrarily assigned point on the curve. Let Σ be a conic such that the class-cubic

$P\Sigma$ is co-apolar with the triad ABC. We shall refer to Σ as a conic "complementary to P with respect to the triad ABC."

We shall now prove that *all conics complementary to P form a four-line pencil.*

Let ABC be taken as the triangle of reference and let $P \equiv (1, 1, 1)$.

Let the cubic referred to the apolar triangle ABC be

$$a_2x^2y+a_3x^2z+b_3y^2z+b_1y^2x+c_1z^2x+c_2z^2y = 0. \quad (1)$$

Let the point $P \equiv (1, 1, 1)$ taken in conjunction with the conic

$$\Sigma \equiv Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0$$

form a class cubic co-apolar with ABC.

The apolar locus of ABC and $P\Sigma$ is easily found to be

$$(C+2F)\,x^2y-(B+2F)x^2z+(A+2G)\,y^2z-(C+2G)\,y^2x+(B+2H)\,z^2x \\ -(A+2H)\,z^2y = 0. \quad (2)$$

Hence identifying the equations (1) and (2), we obtain

$$C+2F = \quad \rho a_2,$$

$$B+2F = -\rho a_3, \text{ \&c.}$$

If we solve these six equations for A, B, C, F, G, H, we find that, subject to the condition

$$a_2+a_3+b_3+b_1+c_1+c_2 = 0, \quad (3)$$

Σ is of the form $\quad \lambda(l^2+m^2+n^2-mn-nl-lm) = \Sigma_0, \quad (4)$

where Σ_0 is a certain definite conic.

Now (3) is the condition that P lies on the cubic curve. Also, the left-hand side of (4) is the tangential equation to the two points P', P'', which are the intersections of the polar line, and the polar conic of P with respect to the triangle ABC. Hence (4) shews that the conics complementary to the given point P form a four-line pencil, of which one degenerate member is the point-pair P', P'' as above defined.

3. Before developing the result of § 2 further, it will be convenient to set down some of the properties of such triangles as ABC and $PP'P''$. (Throughout the paper, we shall adhere to this notation for such triads of points.)

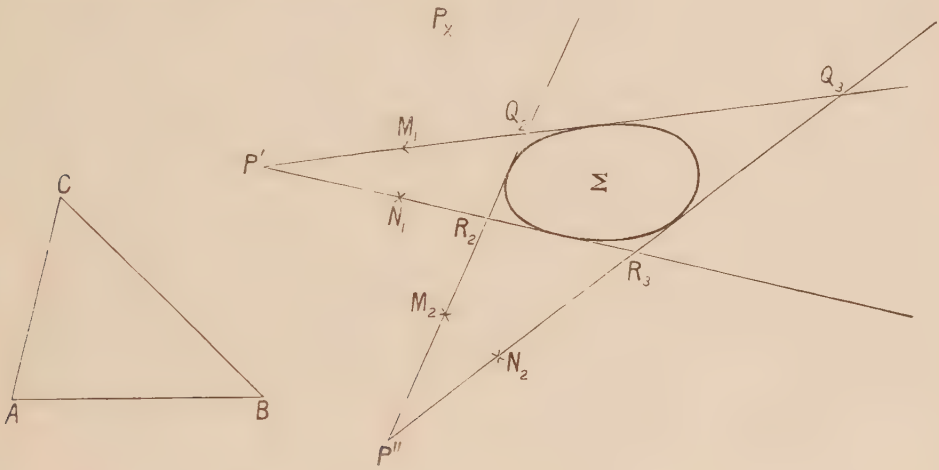
(i) If ABC be chosen as the triangle of reference, and P be taken as $(1, 1, 1)$, then $P' \equiv (1, \omega, \omega^2)$ and $P'' \equiv (1, \omega^2, \omega)$. The relationship between the two triangles is wholly symmetrical, and they are sextuply in

prospective. Such a pair of triangles can always be represented by the apices of an equilateral triangle, its centroid and the circular points at infinity.

(ii) If two triads of points subtend apolar pencils at every point of the plane, the two triads must be of the above type ABC, $PP'P''$. Moreover they are the most general type of triads having this property.

(iii) If A, B, C, X, Y be five given points, there are in general two points L, M, such that LX, LY are the Hessian of the pencil $L[ABC]$; L, M are the poles of $X'Y''$ and $X''Y'$ respectively. There is an infinite number of such points L when A, B, C, X, Y are on a conic, with X, Y as the Hessian points of A, B, C. In such cases X, Y, Z (where Z is the pole of XY with respect to the conic $ABCXY$) form along with ABC two triads of the above type.

We shall now apply these results to the system of four-line conics Σ, complementary to P as obtained in last article.



Let P be a given point on the cubic curve, and let Σ be a conic complementary to P with respect to the apolar triad ABC. Let the tangents from P' to Σ intersect the tangents from P'' to Σ in the points Q_2, Q_3, R_2, R_3 as indicated in the figure. Then, since the point-pairs Q_2, R_3 and Q_3, R_2 are degenerate members of the four-line system Σ (the third degenerate member of which is P', P''), we obtain (by § 2) that Q_2, R_3 and Q_3, R_2 are each dyads complementary to P.

To obtain geometrically the tangents from P' and P'' to Σ we proceed thus:—

Let $P'Q_2Q_3$ cut the cubic curve again at M_1. Then M_1 subtends

apolar pencils at ABC and $P\Sigma$. Hence taking the various positions of Σ we see that M_1P, M_1P' and any other line through M_1 are apolar to $M_1[ABC]$. Thus, M_1P, M_1P' are the Hessian lines of the pencil $M_1[ABC]$. M_1 therefore lies on the conic $ABCPP'$, *i.e.*, the polar conic of P'' with respect to ABC, by (iii) above.

We thus have the following result :—

If ABC be an assigned apolar triad and P any given point on the cubic curve, there are two dyads complementary to P, and these are obtained as the intersections of $P'M_1$, $P'N_1$ with $P''M_2$, $P''N_2$, where M_1, N_1 and M_2, N_2 are the points (other than A, B, C, P) in which the polar conics of P'', P' (with regard to ABC) cut respectively the cubic curve.

4. *Line and Conic Constructions.*

It is to be noted that co-apolar triads can be obtained from their independent elements by means of constructions involving the use of straight lines and conics only, except of course in so far as the cubic curve itself enters into these constructions. The independent elements are two points of one triad and one point of the other. Let A and B be two assigned points of one triad. Then the third point C is either of the two points (other than that on AB) in which the bipolar line of A, B cuts the cubic curve. The intersection of the bipolar line of A, B with AB must be discarded because three collinear points cannot be regarded as forming an apolar triad unless their line of collineation touches the Cayleyan (*Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 235–243). The apolar triad ABC having been thus constructed by means of straight lines and conics only, it is shewn in the last article that being given *one* point P of the co-apolar triad, the complementary dyads of P can be obtained by the use of straight lines and conics only.

5. *Determinantal Systems of Points.*

In conclusion we shall shew that all triads co-apolar with any assigned apolar triad ABC group themselves into closed systems of nine points, which we shall refer to as "determinantal systems of points."

The characteristic property of a determinantal system of points may be described thus :—Let ABC be any assigned triangle, and let $P_1Q_2R_3$ be any other triangle in the same plane. Let P_1 be isolated, and let $P_1'Q_2$ intersect $P_1''R_3$ in Q_3. Also let $P_1'R_3$ intersect $P_1''Q_2$ in R_2. Let Q_2 and R_3 be isolated in turn and the corresponding constructions made. Thus is

obtained a system of points

$$\begin{matrix} P_1, & P_2, & P_3, \\ Q_1, & Q_2, & Q_3, \\ R_1, & R_2, & R_3, \end{matrix}$$

such that if any one of the six triangles defined by the terms of the expansion of the above determinantal form be taken, the other five are obtained by the above method of construction. In fact, the points form a "closed system." The proof of this is easy and need not be detailed here.

Suppose, now, that $P_1Q_2R_3$ is a triad of points on the cubic curve co-apolar with ABC. Let Q_3, R_2 be the dyad (other than Q_2, R_3) complementary to P_1. Similarly, let R_1, P_3 and P_2, Q_1 be complementary to Q_2 and R_3 respectively.

It is plain now from the method of constructing complementary dyads that, *being given an apolar triad the co-apolar triads group themselves in determinantal systems of nine points, each point on the curve defining a unique determinantal set.*

[I am under deep obligations to the referees who have suggested the order of development which has been adopted in the revised form of the paper.]

TWO APPLICATIONS OF GENERAL THEOREMS IN COMBINATORY ANALYSIS: (1) TO THE THEORY OF INVERSIONS OF PERMUTATIONS; (2) TO THE ASCERTAINMENT OF THE NUMBERS OF TERMS IN THE DEVELOPMENT OF A DETERMINANT WHICH HAS AMONGST ITS ELEMENTS AN ARBITRARY NUMBER OF ZEROS

By Major P. A. MACMAHON.

[Received and Read March 9th, 1916.]

1. THE THEORY OF INVERSIONS OF PERMUTATIONS.

In a recent paper* I defined the greater index of a permutation as depending upon the situation of the numbers or letters which *immediately* precede a number or letter which is prior in numerical or alphabetical order. Thus, if the assemblage, say of numbers, be 1, 1, 2, 3, 3, 4, 5, and any permutation

$$\begin{array}{ccccccc} 4 & 1 & 2 & 1 & 5 & 3 & 3 \\ 1 & & 3 & & 5 & & \end{array}$$

we find that the first, third, and fifth numbers immediately precede numbers which are prior in numerical order; the sum of the numbers 1, 3, 5 is 9, which is called the greater index of the permutation. If p be the greater index of a permutation of the numbers in the assemblages

$$1^i 2^j 3^k \ldots,$$

the exponents, of course, denoting repetitions, I shewed that, the summation having regard to every such permutation, we have

$$\Sigma x^p = \frac{(1-x)(1-x^2)\ldots(1-x^{i+j+k+\ldots})}{(1-x)(1-x^2)\ldots(1-x^i).(1-x)(1-x^2)\ldots(1-x^j).(1-x)(1-x^2)\ldots(1-x^k)\ldots},$$

* *American Journal of Mathematics*, Vol. xxxv, No. 3, 1913, "The Indices of Permutations and the Derivation therefrom of Functions of a Single Variable associated with the Permutations of any Assemblage of Objects."

or in Cayley's notation, writing $1-x^s=(s)$,

$$\Sigma x^p = \frac{(1)(2)\dots(i+j+k+\dots)}{(1)(2)\dots(i)\,.\,(1)(2)\dots(j)\,.\,(1)(2)\dots(k)\dots}.$$

This notation is exceedingly illuminating, and is a striking example of the power mathematics has gained by an appropriate notation.*

The similarity of the expression to the number which enumerates the permutations, obtained by putting x equal to unity, is obvious. In my writings I have termed the expression "The Permutation Function." I wish to shew the application of this theorem to the study of inversions of permutations.† A permutation of an assemblage of numbers involves an inversion whenever a number precedes (not necessarily immediately) a number prior to it in numerical order. A permutation therefore involves a definite number of inversions, and the main question is the determination of the number of the permutations, each of which possesses a given number of inversions. The researches set forth in the *Lehrbuch* of Netto are concerned with permutations of assemblages of numbers between which there are no similarities. He indicates by $I_\kappa^{(n)}$ the number of the permutations of n *different* elements, each of which possesses exactly κ inversions, and by purely arithmetical reasoning he derives various relations between the numbers $I_\kappa^{(n)}$.

In the present note we are concerned with the inversions which are possessed by permutations of the general assemblage of numbers

$$1^i\,2^j\,3^k\dots.$$

Consider first the permutation

$$4\ 1\ 2\ 1\ 5\ 3\ 3$$

which we found above to have a greater index equal to 9.

Here 4 precedes 1, 2, 1, 3, 3 giving 5 inversions,

2 ,, 1 ,, 1 ,,

5 ,, 3, 3 ,, 2 ,,

Hence the permutation possesses $5+1+2=8$ inversions, or, as we may say, has an inversion number equal to 8.

Next take the whole of the permutations of the assemblage

$$1,\ 1,\ 2,\ 3\,;$$

* See in this connexion *Napier's Tercentenary Memorial Volume*, Longmans, Green & Co., 1915; J. W. L. Glaisher, on *Logarithms and Computation*, p. 75.

† See Netto's *Combinatorik*, p. 92

there are twelve of them, and we will write down for each permutation the greater index and the inversion number.

Permutation.	Greater Index.	Inversion Number.
1123	0	0
1132	3	1
1213	2	1
1231	3	2
1312	2	2
1321	5	3
2113	1	2
2131	4	3
2311	2	4
3112	1	3
3121	4	4
3211	3	5

It will be observed that in this case the greater indices constitute the same *assemblage* of numbers as the inversion numbers. It will now be established that if p and v be the greater index and inversion number of a permutation of any assemblage of integers

$$\Sigma x^p = \Sigma x^v.$$

We have to shew that Σx^v has the same algebraic expression as has been already found for Σx^p.

Σx^v is a finite and integral function of x, in which the highest exponent of x is $ij+ik+jk+\ldots$, for clearly v in this case is derived from the permutation which is the assemblage written in descending order of magnitude. This is also the highest exponent of x in the algebraic expansion of the expression found for Σx^p; for this is

$$\binom{i+j+k+\ldots+1}{2} - \binom{i}{2} - \binom{j}{2} - \binom{k}{2} - \ldots$$

or

$$ij+ik+jk+\ldots.$$

For all permutations which commence with the integer s, denote Σx^v by $(\Sigma x^v)_s$. Restricting ourselves, for convenience, to the assemblage

$$1^i 2^j 3^k,$$

let $$\Sigma x^v = F_{i,j,k}(x),$$

then we see that $(\Sigma x^v)_1$ for the assemblage $1^i 2^j 3^k$ is equal to Σx^v for the assemblage $1^{i-1} 2^j 3^k$, since the leading integer 1 can add nothing to the inversion number v.

Thus $$(\Sigma x^v)_1 = F_{i-1,j,k}(x),$$

also $$(\Sigma x^v)_2 = x^i F_{i,j-1,k}(x),$$

since the leading integer 2 is prior to i ones, and therefore adds i to every inversion number of permutations of the assemblage $1^i 2^{j-1} 3^k$.

So also $$(\Sigma x^v)_3 = x^{i+j} F_{i,j,k-1}(x),$$

and, finally,

$$\Sigma x^v = F_{i,j,k}(x) = F_{i-1,j,k}(x) + x^i F_{i,j-1,k}(x) + x^{i+j} F_{i,j,k-1}(x).$$

Assuming the law to hold for the three functions

$$F_{i-1,j,k}(x), \quad F_{i,j-1,k}(x), \quad F_{i,j,k-1}(x),$$

we find that

$$\begin{aligned} F_{i,j,k}(x) = & \frac{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}-\mathbf{1})}{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i}-\mathbf{1})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{j})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{k})} \\ & + x^i \frac{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}-\mathbf{1})}{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{j}-\mathbf{1})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{k})} \\ & + x^{i+j} \frac{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}-\mathbf{1})}{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{j})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{k}-\mathbf{1})} \\ = & \frac{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})}{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{j})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{k})}. \end{aligned}$$

Since, by trial, we find the law to obtain for the smaller values of the sum $i+j+k$, we have here an inductive proof of the theorem, and since the process is perfectly general, we find that for the general assemblage

$$1^i 2^j 3^k \dots,$$

$$\Sigma x^v = \Sigma x^p = \frac{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}+\dots)}{(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{i})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{j})\cdot(\mathbf{1})(\mathbf{2})\dots(\boldsymbol{k})\dots}.$$

The theory of the inversion number is identical with that of the greater index, and thus all of the results obtained in the *American Journal* paper, which has been quoted, are equally true of inversion numbers.

For the particular case discussed in Netto's *Combinatorik*,

$$\Sigma x^v = \frac{(\mathbf{1})(\mathbf{2}) \ldots (\boldsymbol{n})}{(\mathbf{1})^n},$$

and the results he gives are readily deduced from this expression.

The Number of Terms in a Determinant which has some Zero Elements.

In a determinant of order n, the number of terms which arises in the development is in general $n!$; but if some of the elements are zeros, we obtain a lesser number which it is the object of this note to determine. If the elements in the principal diagonal are zeros, and there are no other zeros, the determinant has been named by Sylvester "invertebrate," and it is well known that the number of terms is equal to the number of permutations of n different letters, which are such that no letter is in its original place. The number is therefore

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \ldots \pm \binom{n}{n} 0!.$$

G. de Longchamps* also established that if the first q elements in the diagonal are zeros, the number is

$$n! - \binom{q}{1}(n-1)! + \binom{q}{2}(n-2)! - \ldots \pm \binom{q}{q}(n-q)!.$$

The similar question in which the zero elements are arbitrarily chosen has also been considered by de Longchamps and by C. A. Laisant,† but their solutions fell short of what is desirable.‡ The matter may be handled in the following manner :—

In *Phil. Trans. Roy. Soc.*, 1894, A, "A certain Class of Generating Functions in the Theory of Numbers," I shewed that if $X_1, X_2, \ldots, X_n$ be linear functions given by the matricular relation

$$(X_1, X_2, \ldots, X_n) = \begin{vmatrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \\ \cdot & \cdot & \ldots, & \cdot \\ n_1, & n_2, & \ldots, & n_n \end{vmatrix} (x_1, x_2, \ldots, x_n),$$

* *Jour. de Math. Spéc.* (1891).

† C. R. Paris (1891); *Soc. Math. de France Bull.* (1891).

‡ Netto, *Lehrbuch der Combinatorik* (1901), p. 73.

the coefficient of the term

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n},$$

in the development of the product

$$X_1^{\xi_1} X_2^{\xi_2} \dots X_n^{\xi_n},$$

is equal to the coefficient of the term

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n},$$

in the expansion of the algebraic fraction

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2)\dots(1-n_nx_n)|},$$

wherein the denominator is in symbolic form in such wise that on multiplication the products a_1b_2, $a_1b_2c_3$, ... are to be placed in determinant brackets, like $|a_1b_2|$, $|a_1b_2c_3|$, ..., and denote the coaxial minors of the determinant $|a_1b_2c_3 \dots n_n|$, which appertains to the matricular relation. This result I have termed a master theorem in the theory of permutations in the work *Combinatory Analysis*, published 1915, by the Cambridge University Press.

Now Laisant has shewn, in the notation of this note, that considering the determinant $|a_1b_2c_3 \dots n_n|$, if we form the product

$$(a_1x_1+a_2x_2+\dots+a_nx_n)(b_1x_1+b_2x_2+\dots+b_nx_n)\dots(n_1x_1+n_2x_2+\dots+n_nx_n),$$

and therein write $a_\kappa = b_\lambda = c_\mu = \dots = 1$ whenever the element is not zero, and $a_\kappa = b_\lambda = c_\mu = \dots = 0$ whenever the element is zero, the number of terms in the determinant is equal to the coefficient of $x_1x_2 \dots x_n$ in the development of the product.

Hence by the master theorem the number of terms in the determinant is equal to the coefficient of

$$x_1x_2 \dots x_n,$$

in the development of the fraction

$$\frac{1}{|(1-a_1x_1)(1-b_2x_2)\dots(1-n_nx_n)|},$$

when therein the elements are given the values unity or zero in the manner specified above.

Some examples are now given.

Suppose all the elements of the determinant to be zero except those in the principal diagonal. We put

$$a_1 = b_2 = c_3 = \ldots = n_n = 1,$$

and the remaining elements equal to zero; the fraction is then

$$\frac{1}{1-\Sigma x_1+\Sigma x_1 x_2-\ldots+(-)^n x_1 x_2 \ldots x_n}$$

or

$$\frac{1}{(1-x_1)(1-x_2)\ldots(1-x_n)},$$

in which the coefficient of $x_1 x_2 \ldots x_n$ is unity.

This is correct because clearly the determinant has only one term formed from the elements of the principal diagonal.

Again, suppose that the only zero elements are those in the principal diagonal. We put

$$a_1 = b_2 = c_3 = \ldots = n_n = 0,$$

and the remaining elements equal to unity; the fraction is then

$$\frac{1}{1-\Sigma x_1 x_2-2\Sigma x_1 x_2 x_3-3\Sigma x_1 x_2 x_3 x_4-\ldots-(n-1)x_1 x_2 \ldots x^n}$$

(see *Combinatory Analysis*, Vol. I, p. 100).

The coefficient of $x_1 x_2 \ldots x_n$ in the expansion denotes also the number of permutations of $x_1 x_2 \ldots x_n$ which are such that every element is displaced. The expansion gives various properties of these numbers.

When there are no zero elements the fraction becomes

$$\frac{1}{1-x_1-x_2-x_3-\ldots-x_n},$$

in which the coefficient of $x_1 x_2 \ldots x_n$ is $n!$.

Next take the determinant

$$\begin{vmatrix} 0 & 0 & a_3 & a_4 \\ b_1 & b_2 & 0 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix},$$

which, on development, is found to have ten terms.

The fraction, derived from the coaxial minors of the determinant

$$\begin{vmatrix} 0 & 0 & x_3 & x_4 \\ x_1 & x_2 & 0 & x_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix},$$

is

$$\frac{1}{1-x_2-x_3-x_4-x_1x_3-x_1x_4+x_2x_3},$$

and on development we find the term

$$10x_1x_2x_3x_4$$

a verification.

ON THE UNIFORMITY OF GASEOUS DENSITY, ACCORDING TO THE KINETIC THEORY

By S. Chapman.

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Introduction.

1. According to the kinetic theory, gases are composed of molecules in violent and highly inconstant motion. This motion is regarded as subject to dynamical laws, so that if at any instant the state of the gas were fully known, it would be theoretically possible to trace out the motions and configuration of the molecules at any prior or subsequent time. In actual fact, however, we never even approach such complete knowledge of the state of a gas at any instant. We are able to determine, with the means of measurement at our disposal, only the mean values of various characteristics of the gas (such as density, velocity, or energy) averaged over molecular groups which are generally very large in number. The results of such measurements afford merely an approximation to the real facts, though the error committed may be in practice quite unimportant. For instance, experimental research suggests that the density of a gas in the undisturbed state is uniform and constant; this can be reconciled with the contrary conclusion drawn from the kinetic theory only if the variations of density predicted by the latter are beyond our powers of measurement.

The present paper is concerned with an attempt to calculate an upper limit to these variations of density; this problem, which is not without intrinsic interest, was originally suggested to the author by an attempt to work out a mathematically rigorous statement and proof of Maxwell's well-known law of distribution of molecular velocities in the steady state of a gas. Certain difficulties common to both problems are solved in the course of the present investigation, but the proof of Maxwell's law involves further points which have as yet received no solution. A few remarks on these are given at the close of this paper. The main outlines of the theory in each case are due to Jeans, whose work forms the culmination of a long series of attempts to prove Maxwell's law, made by Maxwell

himself, Boltzmann, Burbury, Watson, and others.* While the principles may be regarded as established, the mathematical development of the theory requires further elaboration for the attainment of exactness and rigour, and the present essay is designed to contribute to that end.

Jeans' argument has two main divisions. Any possible state of the system of molecules considered is representable by a point in a generalized space, the coordinates of which correspond to the space or velocity coordinates required to specify the configuration and motions of the molecules. Within a certain volume in this generalized space (the bounds of which are determined by the initial limitations on the distribution or total energy of the molecular system) all positions of the representative point are regarded as equally probable. In the first part of his argument Jeans† proves, by means of Liouville's theorem in analytical dynamics, that the probability of a given range of states—represented by any given volume in the generalized space—is independent of the time. In the second part the most probable state is determined, and the probability of a small divergence from this state is investigated. This is found to be very small, so that, in conjunction with the first part of the argument, the conclusion is drawn that the actual state of a gas is almost certain to coincide very nearly with the most probable state. The problem here considered is the determination of a definite estimate of the probability of a given divergence from this.

A Particular Numerical Case of the Result.

2. The general character of the result may be indicated by a rough statement in numerical terms for a particular case. In a cubic centimetre of a gas at normal temperature and pressure there are $2{\cdot}75.10^{19}$ mole-

* Cf. Jeans, *Dynamical Theory of Gases*, Ch. III, V, where references are also given to the work of the other writers mentioned. Sir J. Larmor has drawn my attention to an interesting paper by Lord Kelvin (*Collected Works*, Vol. VI, p. 11, or *Nature*, Vol. IX, p. 441, 1874), where may be found what is perhaps the first introduction into the kinetic theory of considerations of probability such as were subsequently used by Boltzmann in his theory of entropy, and by Jeans in his proof of Maxwell's law. The paper deals with the probability of uniformity of mixture of two gases, and use was made of Stirling's theorem in the manner now usual.

† The proof given in his treatise would be improved, from a logical standpoint, if the unit of probability were taken to be the probability of the *range* of states represented by an element of unit *volume* in the generalized space, instead of the probability of any one configuration. The latter is represented by a single point in the generalized space, and the number of such points in any volume is the number of the continuum (C), which is the same for all volumes of finite size. On the other convention here suggested, the probability of a given range of states would be directly proportional to the corresponding representative volume. This would obviate the necessity to consider a distribution of discrete representative points; it strengthens the analogy of their motion with that of a continuous fluid; and it is more accordant with the actual investigation of probability in the second part of the proof. It is also at least equally valid on *a priori* physical grounds.

cules. The probable variation of density within this volume may be illustrated by subdividing it into one million cells of equal extent: if $\bar{\rho}$ is the mean density throughout the cubic centimetre, and ρ the actual density in a particular cell at any instant, $(\rho-\bar{\rho})/\bar{\rho}$ will measure the "divergence of density" in that cell at the time considered. The maximum divergence of density (positive or negative) may vary with the time, and may occur in different cells at different times. We cannot set any upper limit to its value save that imposed by the impossibility of packing more than a certain number (or possibly all) the molecules in any one cell. We can, however, determine the probability of a given value being exceeded; this probability is most readily described by regarding it as the fraction of any given interval of time during which the maximum divergence of density will transcend the given limit, though it may not be possible to maintain this interpretation of the probability on grounds of strict logic. If we adopt this form of statement, it appears (cf. § 9) that in no one of the million sub-divisions of our cubic centimetre of gas will the maximum divergence of density exceed $\frac{1}{5200}$ for a greater fraction of any interval of time than is equivalent to one second in more than ten years. The corresponding limit for the mean divergence of density is $\frac{1}{5200000}$, by the ordinary theory of probability. The probability of a given divergence of density would, of course, be smaller or greater if we were to take a smaller or greater number of elementary subdivisions of the whole volume, and it would be greater, also, if the mean density of the gas were less than that considered. It is clear, however, that with all ordinary densities the variations predicted by the kinetic theory are quite beyond our powers of measurement.

We now proceed to consider the general case, making a general reference to Jeans' treatise for a fuller discussion of the theory as a whole.

The most Probable Distribution of Density in a Gas.

3. We shall consider random distributions of N molecules (which for the present will be regarded as mathematical points without extension), within a volume Ω which is subdivided into n cells of equal volume ω, so that $\omega = \Omega/n$. From § 4 onwards, both n and N will be supposed large, n, however, being so much smaller than N that N/n^2 is also large.

If we make no distinction between the individual molecules, the probability θ_a of all possible configurations of the molecules within Ω, such that in the definite cells $1, 2, \ldots, n$, there are definite numbers $a_1, a_2, \ldots, a_n$ of molecules, is given by

$$\theta_a = \frac{N!\, n^{-N}}{a_1!\, a_2! \ldots a_n!}. \tag{1}$$

By the multinomial theorem it is clear that the sum of all such fractions θ_a, corresponding to all possible partitions $(a_1, a_2, \ldots, a_n)$ of N, is unity, as it should be. The numbers $a_1, a_2, \ldots, a_n$ are evidently subject to the condition

$$a_1+a_2+\ldots+a_n = N. \tag{2}$$

By the theory of the Gamma-function we have (accurately)

$$\log\Gamma(1+x) = (x+\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + f(x), \tag{3}$$

where*

$$f(x) = \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) e^{-xt}\frac{dt}{t}. \tag{4}$$

This is true whether x is an integer or not; when x is integral,

$$\Gamma(1+x) = x!.$$

By substitution from (3) and (4) in (1), we readily find that†

$$\theta_a = n^{\frac{1}{2}n}(2\pi N)^{-\frac{1}{2}(n-1)}\, e^{f(N)}\, e^{-P_a - Q_a}, \tag{5}$$

where, for convenience, we have used the notation thus defined:

$$P_a \equiv \sum_{r=1}^{n} (a_r+\tfrac{1}{2})\log(na_r/N), \qquad Q_a \equiv \sum_{1}^{n} f(a_r). \tag{6}$$

Only the last term in (5) varies with different partitions of N, so that in discussing the most probable distribution of the molecules, corresponding to the maximum value of θ_a, we have to investigate the stationary values of P_a and Q_a. We shall show that the latter functions each have only one stationary value, an absolute minimum, corresponding to the partition

$$a_1 = a_2 = \ldots = a_n = N/n. \tag{7}$$

Hence it follows that θ_a has only one maximum value, and that this corresponds to a uniform distribution of the molecules between the n cells into which Ω is divided.

It is convenient for our purpose to regard the a's as continuous variables, though in reality they are necessarily confined to integral values. The expressions (5) and (6) are well adapted to this convention, since they are valid for all values of the a's.

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* Cf. Bromwich, *Infinite Series*, §§ 179, 180.

† In Jeans' proof Q_a and certain portions of P_a are neglected. These parts of P_a ultimately cancel out amongst themselves, as we shall see, but they are of the same order as Q_a, which, in fact, determines the nature of our result.

For the stationary values of P_a, we must have

$$\delta P_a = \sum_1^n \left(\log \frac{na_r}{N} + 1 + \frac{1}{2a_r}\right) \delta a_r = 0,$$

for all variations of the a's subject to

$$\sum_1^n \delta a_r = 0.$$

Hence some value of λ exists such that, corresponding to a stationary value of P_a,

$$\log \frac{na_r}{N} + 1 + \frac{1}{2a_r} + \lambda = 0$$

for all values of r. Consequently all the values of a_r must be equal, and P_a has a single stationary value given by (7). To prove that this is a minimum, we write

$$a_r = (N/n) + \alpha_r \quad (r = 1 \text{ to } r = n), \tag{8}$$

so that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 0. \tag{9}$$

If $|\alpha_r| < N/n$, we may expand P_a as follows

$$\begin{aligned} P_a &= \sum_1^n \left(\frac{N}{n} + \alpha_r + \tfrac{1}{2}\right) \left\{ \sum_{m=1}^{\infty} \frac{-1}{m} \left(-\frac{n\alpha_r}{N}\right)^m \right\} \\ &= \frac{n}{2N}\left(1 - \frac{n}{2N}\right) \sum_1^n \alpha_r^2 - \frac{n^2}{6N^2}\left(1 - \frac{n}{N}\right) \sum_1^n \alpha_r^3 + \ldots \\ &\qquad - \frac{(-n)^m}{m(m+1)N^m}\left(1 - \frac{mn}{2N}\right) \Sigma \alpha_r^{m+1} + \ldots . \end{aligned} \tag{10}$$

This is essentially positive when the α's are sufficiently small, and the one stationary value of P_a is a minimum; since (cf. 6, 7) the value of P_a at that point is zero, it is positive for all other partitions of N.

Again, by (4) and (6), we have

$$Q_a = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) \left(\sum_1^n e^{-a_r t}\right) \frac{dt}{t}.$$

Both $\Sigma e^{-a_r t}$ and the remainder of the integrand are essentially positive. The *product* of the terms in the sum $\Sigma e^{-a_r t}$ is constant, viz., $e^{-\Sigma a_r t}$ or e^{-Nt}; hence, by the theory of the arithmetic and geometric means of a set of positive quantities, $\Sigma e^{-a_r t}$, and consequently also Q_a, has only one stationary value, an absolute minimum, corresponding to a uniform distribution of the molecules between the n cells.

The Probability of a given Divergence of Density.

4. By the theory* of Bernoulli's numbers B_1, B_2, ..., we have

$$\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}=\sum_1 B_m\frac{(-1)^{m-1}t^{2m-1}}{(2m)!},$$

and it is known that the remainder in this series after any term is less than the next succeeding term. Considering only the first two terms, therefore, we have

$$Q_a=\frac{1}{12}\int_0^\infty\left(\sum_1^n e^{-a_r t}\right)dt-\frac{\theta}{360}\int_0^\infty\left(\sum_1^n e^{-a_r t}\right)t^2dt$$

$$=\frac{1}{12}\sum_1^n\frac{1}{a_r}-\frac{\theta}{360}\sum_1^n\frac{1}{a_n^3}.$$

where $0<\theta<1$. In this expression we shall substitute $(N/n)+a_r$ for a_r, and expand a_r^{-1} and a_r^{-3} in powers of na_r/N (which we limit to values less than unity), by means of the binomial theorem. Combining the resulting expression with (10), we may write

$$(11)\quad P_a+Q_a=\frac{k}{12}\frac{n^2}{N}+\frac{1}{2}\frac{n}{N}\left(1-\frac{kn}{2N}\right)\Sigma a_r^2-\frac{1}{6}\frac{n^2}{N^2}\left(1-\frac{kn}{N}\right)\Sigma a_r^3$$
$$+\frac{k}{12}\frac{n^3}{N^3}\Sigma a_r^4-\ldots,$$

where k is used to represent a number differing from unity by a quantity not greater than (say) $\frac{10}{n}$ or $\frac{10n^2}{N}$, the difference not being necessarily always of the same amount, however, even for the various cases of k in the same expression. This notation is convenient, as we shall find it sufficient to neglect products, squares, or higher powers of $\frac{1}{n}$ or $\frac{n^2}{N}$.

5. We next "order" the various partitions of N according to the corresponding values of $\sum_1^n a_r^2$. Thus we shall consider the total probability of all configurations, such that

$$(12)\qquad \sum_1^n a_r^2\leqslant R^2,$$

* Cf. Bromwich, *Infinite Series*, p. 234; it is there shown that $B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$.

subject, of course, to the condition (9), and to the obvious restriction

$$-\frac{N}{n} \leqslant a_r \leqslant \frac{n-1}{n} N \tag{13}$$

(implying that the a's must be positive and not greater than N). Equations (9) and (12) together represent an $(n-1)$-dimensional sphere of radius R, *i.e.*, the section of the n-dimensional sphere (12) by the plane (9). Points in this $(n-1)$-dimensional space which represent integral partitions of N occur at the rate of one per unit volume, and since the direction cosines of the plane $\Sigma a_r = 0$ are all equal to $n^{-\frac{1}{2}}$, integral partitions occur at the rate of $n^{-\frac{1}{2}}$ per unit content of this plane, *i.e.*, per unit volume of the $(n-1)$-dimensional sphere referred to. By the theory of Dirichlet's integrals,* the volume of the latter is

$$\pi^{\frac{1}{2}(n-1)} R^{n-1} / \Gamma\left(\frac{n+1}{2}\right).$$

Consequently the number of integral partitions of N, such that

$$R^2 \leqslant \Sigma a_r^2 \leqslant (R+dR)^2 \quad (\Sigma a_r = 0) \tag{14}$$

is

$$2n^{-\frac{1}{2}} \pi^{\frac{1}{2}(n-1)} R^{n-2} dR / \Gamma\left(\frac{n-1}{2}\right), \tag{15}$$

which we shall denote by $N_a(R)\, dR$.

Now θ_a, the probability of any particular partition of N, will not be constant throughout the spherical shell (14); we will denote its mean value for all these partitions by $\theta_a(R)$. Then for the *total* probability $\phi(R)$ of all partitions within the sphere defined by (9) and (12), we have

$$\phi(R) = \int_0^R \theta_a(R) N_a(R)\, dR. \tag{16}$$

The problem before us is the evaluation of $\phi(R)$: owing to the difficulties in the way of an exact solution, however, we shall be content to find a limit below which $\phi(R)$ does not fall, determining this limit as close to $\phi(R)$ as our analysis will conveniently allow. We shall, in fact, make the rough approximation to $\theta_a(R)$ implied by replacing the mean value of $e^{-P_a - Q_a}$ (which is the only variable part of θ_a) by $e^{-P_a(R) - Q_a(R)}$, where $P_a(R) + Q_a(R)$ is the mean value of $P_a + Q_a$ throughout the spherical shell (14). By so doing we clearly diminish $\theta_a(R)$, since the mean value

* Cf. Williamson's *Integral Calculus*, p. 320.

of $e^{f(x)}$ is clearly greater than the exponential of the mean value of $f(x)$; this is a generalization of the simple formula

$$\tfrac{1}{2}(e^{x_1}+e^{x_2}) > e^{\frac{1}{2}(x_1+x_2)},$$

which is true whatever be the values of x_1 and x_2.

Now the mean value of $\Sigma\alpha_r^{2m+1}$ over the shell (14) is evidently zero, while the mean value of $\Sigma\alpha_r^{2m}$ is*

$$\frac{1.3.5 \ldots (2m-1)}{(n+1)(n+3)\ldots(n+2m-3)}\left(\frac{n-1}{n}\right)^{m-1} R^{2m}.$$

Consequently (cf. 11)

$$(17)\quad P_a(R)+Q_a(R) = \frac{k}{12}\frac{n^2}{N}+\tfrac{1}{2}\frac{n}{N}\left(1-\frac{kn}{2N}\right)R^2+\frac{k(n-1)}{4n(n+1)}\frac{n^3}{N^3}R^4+\ldots,$$

wherein the common ratio of successive terms is of order n^2R^2/N^2 at most: thus, if R^2 never exceeds $2N$ (say), the terms after the third, *i.e.*, that containing R^4, when combined are at most of order n^2/N times the third term, and may therefore be included in the factor k of the latter. Similarly $(n-1)/(n+1)$ will be included under the same factor.

We therefore substitute from (17) in (5), in order to get a lower limit to $\theta_a(R)$, and insert the result in the integrand of (16), so that, with the aid of (15), we have

$$(18)\quad \phi(R) > K\int_0^R e^{-\rho}R^{n-2}dR,$$

where

$$(19)\quad K \equiv (n/2N)^{\frac{1}{2}(n-1)}\,e^{f(N)-kn^2/12N}/\Gamma\left(\frac{n-1}{2}\right),$$

$$(20)\quad \rho \equiv \tfrac{1}{2}\frac{n}{N}\left(1-\frac{kn}{2N}\right)R^2-\frac{kn^2}{4N^3}R^4 \equiv x-\frac{kn^2}{4N^3}R^4.$$

We can simplify K by including $f(N)$, which we have seen to be of order $1/12N$, under the factor k in the second term of the index of the exponential. We shall also alter the variable of integration in (18) to x, as defined by (20); this introduces a factor $(1-kn/2N)^{-\frac{1}{2}(n-3)}$, amongst others, into the integral, and we may replace this by $\{1+(kn^2/4N)\}$, with a slight change in the value of k. Similarly $e^{-(\rho-x)}$ may be replaced

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* Chapman, *Quarterly Journal of Mathematics*, 1916.

by $\{1-(kx^2/N)\}$. Thus, after a little reduction, (18) may be rewritten in the form

(21)

$$\phi(R) > \left\{ e^{-kn^2/12N} \left(1+\frac{kn^2}{4N}\right) \Big/ \Gamma\left(\frac{n-1}{2}\right) \right\} \int_0^X e^{-x} \left\{ x^{\frac{1}{2}(n-3)} - \frac{k}{N} x^{\frac{1}{2}(n+1)} \right\} dx,$$

where $$X = knR^2/2N.$$

The value of the right-hand side of (21) can most readily be estimated when n is an odd integer, a hypothesis which we may clearly adopt without any real loss of generality. Thus, when m is an integer,

$$\int_0^X e^{-x} x^m dx = \Gamma(m+1) \left\{ 1-e^{-X} \left(1+X+\frac{X^2}{2!}+\ldots+\frac{X^m}{m!}\right) \right\},$$

so that the right-hand side of (21) may be written (when n is an odd integer)

$$e^{-kn^2/12n} \left(1+\frac{kn^2}{4N}\right) \left[1-e^{-X} \left\{ 1+X+\ldots+\frac{X^{\frac{1}{2}(n-3)}}{[\frac{1}{2}(n-3)]!} \right\} \right.$$

$$\left. -\frac{k(n+1)(n-1)}{4N} \left\{ 1-e^{-X} \left(1+X+\frac{X^2}{2!}+\ldots+\frac{X^{\frac{1}{2}(n+1)}}{[\frac{1}{2}(n+1)]!}\right) \right\} \right],$$

or as

$$e^{-kn^2/12N} \left(1+\frac{kn^2}{4N}\right) \left[1-\frac{kn^2}{4N} - ke^{-X} \left\{ 1+X+\frac{X^2}{2!}+\ldots+\frac{X^{\frac{1}{2}(n-3)}}{[\frac{1}{2}(n-3)]!} \right\} \right].$$

If $X = \frac{1}{2}\kappa n$, where $\kappa \geqslant 1$, the terms in the last curly bracket steadily increase, so that their sum is less than $\frac{1}{2}n$ times the last term or (by Stirling's formula) than

$$(n/4\pi)^{\frac{1}{2}} (\kappa e)^{\frac{1}{2}n}.$$

Hence $$\phi(R) > e^{-kn^2/12N} \left(1+\frac{kn^2}{4N}\right) \left\{ 1-\frac{kn^2}{4N} - k\left(\frac{n}{4\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(\kappa-1-\log\kappa)n} \right\},$$

or, if $\kappa = 1+\tau$, where τ is positive,

(22) $$\phi(R) > \left\{ 1-\frac{kn^2}{12N} - k\left(\frac{n}{4\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{4}\tau^2 n} \right\},$$

where we have again included some small quantities under the sign k. In this formula $\tau = (kR^2/N)-1$.

6. As regards the mutual compatibility of the various restrictions which we have placed on the quantities employed in our analysis, N and

n have been supposed large, but such that n^2/N is small, and na_r/N has been supposed less than unity: also we have supposed that

$$\Sigma a_r^2 \leqslant R^2 < 2N.$$

This requires that no a_r shall be so great as $N^{\frac{1}{2}}$ (cf. 9), so that na_r/N must be less than $(n^2/N)^{\frac{1}{2}}$; this does not conflict with the earlier restriction on na_r/N, but merely confines us to the case (which is alone of interest in relation to an actual gas) when na_r/N is very small. Thus the convergence of all our expressions is assured.

We have treated R as if it were a continuous variable, which is not really the case. The integration with respect to R should strictly be a summation, but with large values of R the difference between the two is quite unimportant.

7. In making the various approximations indicated by the factor k, care has been taken never to neglect terms of the highest order involved, so that as the final result takes account of terms of order n^2/N, the neglected terms are of order n^4/N^2 at most. The third term in (22) is usually not directly comparable with the second, and is therefore included in the result, though in all cases of physical interest it might be absorbed in the k of the second term.

When $R = \{(n-1)N/n\}^2$, all possible partitions of N are included in the sphere (9, 12), and $\phi(R)$, the probability of all possible configurations of the molecules, must be unity. The approximation (22) to $\phi(R)$, however, never exceeds $1-(kn^2/12N)$, so that however great R may be, our approximation will never reach the maximum true value (unity) of $\phi(R)$; our estimate of $\phi(R)$ is probably a good approximation only for values of R just greater than $N^{\frac{1}{2}}$. Perhaps the constant term $kn^2/12N$ remains in (22) merely because of the neglect of some counterbalancing quantity in $e^{-P_a-Q_a}$, omitted when for the mean value of the latter over the spherical shell R, $R+dR$, we substituted $e^{-P_a(R)-Q_a(R)}$. I think that with sufficient trouble a much closer inequality than (22) might be deduced, but I shall not trouble to attempt this, since (22) is exact enough for any case of physical interest.

Physical Interpretation.

8. The physical interpretation of our analysis in the case of a gas is as follows. Let ρ_r be the density in the r-th cell ω, and $\bar{\rho}$ the mean

density throughout Ω. Then (m denoting the molecular mass)

$$\bar{\rho} = m\frac{N}{\Omega} = m\frac{N}{n\omega},$$

$$\rho_r = m\frac{a_r}{\omega} = \frac{m}{\omega}\left(\frac{N}{n}+a_r\right) = \bar{\rho}\left(1+\frac{na_r}{N}\right) = \bar{\rho}\,(1+\lambda_r),$$

where $$\lambda_r = (\rho_r-\bar{\rho})/\bar{\rho},$$

so that λ_r is what was defined in § 2 as the divergence of density in cell r. Hence the physical meaning of (22) is (neglecting the third term on the right) that in a fraction less than $kn^2/12N$ of all possible distributions of density in the gas will $\Sigma\lambda_r^2$ exceed n^2R^2/N^2, or $k(1+\tau)\,n^2/N$, provided that $\tau > 0$.

9. In the numerical case described in § 2, with the data there given, we have $N = 2{\cdot}75\,.\,10^{19}$, $n = 10^6$, so that

$$\frac{n^2}{12N} = \frac{1}{330{,}000{,}000}.$$

The error involved in treating k as unity is hence quite negligible, and the third term in (22) is many million million times smaller than $n^2/12N$ if $\tau > \frac{1}{100}$, *i.e.*, if $R^2 > \frac{101}{100}N$. This lower limit of τ is quite needlessly large, but is sufficient for our purpose. Corresponding to this limit for R the maximum divergence of density in any one of the million cells is* approximately $\frac{1}{5200}$. Interpreting the probability in terms of time, this maximum divergence of density will be exceeded during less than one second in ten years, since one second is less than $\dfrac{1}{330{,}000{,}000}$ of ten years.

For less dense gases the probability of a given divergence of density from the mean is naturally greater. The following table contains the best information which our result is able to afford on this point; it shows that even for very highly rarefied gases the departure from uniformity of density is extremely small. It should be noticed that in the table both N and n are varied, but the latter only in the ratio of the cube root of the former. This preserves the closest approximation which our analysis will

* Thus, the limit of $\Sigma\lambda_r^2$ is n^2R^2/N^2, or $1/(5200)^2$ approximately. The largest possible value of λ_r, consistent with $\Sigma\lambda_r = 0$, corresponds to the case when one λ is large and positive (λ_0) and all the others are equal and negative, of value $-\lambda_0/(n-1)$. Then $\Sigma\lambda_r^2$ is equal to $\lambda_0^2\{1+(n-1)/(n-1)^2\} = n\lambda_0^2/(n-1)$, and λ_0 is consequently equal to 1/5200 approximately.

yield, but it should be remembered that if n were kept constant the probabilities in the last row would increase more rapidly as N diminishes.

Density (as a ratio of the density at 0°C. and 760 mm. of mercury pressure)	1	$\frac{1}{8}$	$\frac{1}{27}$	$\frac{1}{125}$	$\frac{1}{1000}$	$\frac{1}{64000}$	$\frac{1}{1,000,000}$
Approximate corresponding pressure in mm. of mercury at 0°C.	760	95	28	6	0·76	0·012	0·00076
N	$2{\cdot}75.10^{19}$	$3{\cdot}44.10^{18}$	$1{\cdot}02.10^{18}$	$2{\cdot}20.10^{17}$	$2{\cdot}75.10^{16}$	$4{\cdot}3.10^{14}$	$2{\cdot}75.10^{13}$
Adopted value of n ...	1,000,000	500,000	333,333	200,000	100,000	25,000	10,000
Limit of maximum divergence of density	$\frac{1}{5200}$	$\frac{1}{3676}$	$\frac{1}{3000}$	$\frac{1}{2325}$	$\frac{1}{1643}$	$\frac{1}{822}$	$\frac{1}{520}$
Probability	$\frac{1}{33.10^7}$	$\frac{1}{16{\cdot}5.10^7}$	$\frac{1}{11.10^7}$	$\frac{1}{66.10^6}$	$\frac{1}{33.10^6}$	$\frac{1}{82.10^5}$	$\frac{1}{33.10^5}$
Equivalent of probability expressed as a fraction of the form $\frac{1 \text{ second}}{x \text{ years}} : x =$	10	5	3·3	2	1	$\frac{1}{4}$	$\frac{1}{10}$

Molecules of Finite Size.

10. When the molecules are of finite size, instead of being mathematical points, the above analysis requires revision, since certain molecular configurations which were formerly possible are now precluded by the impenetrability of matter. The necessary modifications of the analysis are, I imagine, of no simple character, and I shall not attempt a discussion of them. It will be sufficient to point out that the finite size of the molecules must *increase* the tendency towards uniformity of density. In the limiting case when the molecules can just be packed into the volume Ω, it is clear that the density must be absolutely uniform. In less extreme cases, also, an upper limit (not present in the former case) must exist to the possible divergence of density in any cell, owing to the impossibility of packing more than a certain number of the molecules into one cell. This must tend to diminish the probability of divergence of density even below the small value above determined.

The Law of Distribution of Velocities.

11. The analogous problem of the distribution of velocities may be regarded as the determination of the probable distribution of N points in a three-dimensional space, the coordinates of any point being (u, v, w), the

velocity components of the corresponding molecule. If there were no restrictions on these velocities, a uniform distribution (throughout any allotted space) would be predicted as most probable, the case being exactly similar to that already considered in the earlier part of this paper. We are concerned in practice, however, only with systems which possess a definite amount of energy E, or whose energy is known within narrow limits. This, incidentally, sets limits to the possible velocity of any molecule, since $u^2+v^2+w^2$ cannot exceed $2E/m$.

The velocity configuration can now be represented by a point on a sphere of radius $(2E/m)^{\frac{1}{2}}$ in a $3N$-dimensional space, the coordinates being the velocity components of the N molecules. As our basis of probability we may regard ranges of velocity configurations corresponding to equal areas on this sphere as equally probable. The difficulty of the problem consists in a concise expression and use of this condition.

If, as before, we partition the range of molecular velocities in the *three* dimensional space first mentioned [which we may take as a sphere of radius $(2E/m)^{\frac{1}{2}}$ and volume Ω] into n equal cells, and if the mean energy of a molecule corresponding to points in cell r is ϵ_r, we may consider the most probable distribution of velocities subject to the conditions (2) and

$$a_1\epsilon_1+a_2\epsilon_2+\ldots+a_n\epsilon_n = E. \tag{23}$$

Here $a_1, a_2, \ldots, a_n$ represent, as before, the number of points falling in each velocity cell. The condition (23) does not restrict the energy of the system even approximately to the value E, however, on account of the variability of position of the points within the single cells; the system may easily have an amount of energy which is enormously greater than E, consistently with (2) and (23).

The most probable distribution of velocities, consistent with (2) and (23), is found by making $\delta(P_a+Q_a)=0$ subject to $\Sigma\delta a_r = 0$ and $\Sigma\epsilon_r\delta a_r = 0$. This leads to the condition

$$\log\frac{na_r}{N}+1+\frac{1}{2a_r}-\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)e^{-a_r t}\frac{dt}{t}+\lambda+\mu\epsilon_r = 0,$$

where λ and μ are certain constants independent of r. As in § 4, this is equivalent to

$$\log\frac{na_r}{N}+1+\frac{1}{2a_r}-\frac{\theta}{12a_r^2}+\lambda+\mu\epsilon_r = 0,$$

where $0<\theta<1$. Consequently the most probable distribution is given by

$$a_r e^{1/2a_r-\theta/12a_r^2} = \frac{N}{n}e^{-1-\lambda}e^{-\mu\epsilon_r}, \tag{24}$$

which, when a_r is large, differs only by a negligible amount from the distribution assigned by Maxwell's law, viz.,

$$a_r = Ae^{-\mu\epsilon_r} = Ae^{-\frac{1}{2}\mu m(u^2+v^2+w^2)}.$$

Only when a_r is a small integer (it is never less than unity) does the distribution given by (24) differ from Maxwell's law by so much as one molecule in a cell.

It appears, therefore, that a law of distribution extremely like that of Maxwell's may be deduced as being most probable under conditions which are much less restrictive than those with which we are really concerned. The further step in the argument, however, by which it is proved that distributions differing much from the most probable distribution are highly improbable, offers considerably greater difficulty; if the energy restriction is used in the form (23), it is far from easy to determine the probability of all or of a given range of configurations, largely owing to the very unsatisfactory nature of the condition (23). In fact, while I have no doubt that, as in the case of the space configuration of the molecules, the most probable velocity configuration, given by Maxwell's law, is hardly ever appreciably departed from, I think it likely that no really satisfactory proof of this will be forthcoming until the energy condition is more accurately and effectively used in the investigation.

ON THE DEDUCTION OF CRITERIA FOR THE CONVERGENCE OF FOURIER'S SERIES FROM FEJÉR'S THEOREM CONCERNING THEIR SUMMABILITY

By S. POLLARD.

Communicated by G. H. HARDY.

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As is well known, the theory of Fourier's series, other than the Riemann theory, has been developed along two lines. The older, or, as it has been termed, the classical theory, is concerned with direct investigations into the convergence of the series, based mainly on the property of Dirichlet's integral. The more modern theory is concerned with the summability rather than the convergence of the series, and is based mainly on the properties of Fejér's and Poisson's integrals. Among the results established the two following may be quoted.

I. (Fejér's Theorem.)—The Fourier's series of $f(x)$ is summable (C1), to sum $\frac{1}{2}\{f(x+0)+f(x-0)\}$, at every point where this expression has a meaning.

II. (Schwarz's Theorem.)—The above result holds if we replace "summable (C1)" by "summable (A)", a series Σu_n being said to be summable (A), to sum s, if

$$\lim_{h\to 1-0}(\Sigma u_n h^n) = s.$$

The interesting problem naturally arises as to how far the results of the classical theory as to the convergence of the series can be deduced from the above results as to its summability.

So far Jordan's result, which states that the Fourier's series converge to $f(x)$ or $\frac{1}{2}\{f(x+0)+f(x-0)\}$, as the case may be, within any interval for which the variation of f is bounded, is the only one which has been so deduced. It can be deduced from I by an application of Hardy's convergence theorem, and from II, by an application of Littlewood's theorem.*

* Hardy's theorem asserts that if Σu_n is summable (C), and $u_n = O(1/n)$, then Σu_n is convergent; Littlewood's that the summability (A) of the series involves its convergence under the same condition. See *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320, and Vol. 9, pp. 434–448.

But Jordan's criterion is comparatively crude, as it deals with convergence throughout an interval, not with convergence at a point.

A more delicate criterion, dealing with convergence at a point, is that of Dini, as follows.

The Fourier's series for $f(x)$ converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at every point for which

$$\int_0^\epsilon |f(x+\theta)+f(x-\theta)-f(x+0)-f(x-0)|\frac{d\theta}{\theta}$$

exists.

It is the object of this paper to shew that a modification of the analysis leading up to Hardy's theorem will enable us to deduce Dini's result from I.

Consider any real series $u_0+u_1+\ldots+u_n+\ldots$ which has the following properties :—

(i) $u_n \to 0$ as $n \to \infty$.

(ii) Σu_n contains an infinity of terms of either sign.*

(iii) Σu_n is summable (C1) to sum s.

Construct an auxiliary series Σa_n as follows :—

Let u_N be the negative term with the greatest modulus, u_{N_1} the negative term with the greatest modulus in $\sum_{n=N+1}^{\infty} u_n$, u_{N_2} that in $\sum_{n=N_1+1}^{\infty} u_n$, and so on.

Then $N_1, N_2, \ldots$ certainly exist, because $u_n \to 0$ as $n \to \infty$.

Take
$$a_n = -u_N \quad (n \leqslant N),$$
$$= -u_{N_1} \quad (N < n \leqslant N_1),$$

and so on.

Then the a_n's form a decreasing sequence of positive terms, such that

$$u_n \geqslant -a_n,$$

for all values of n.

With the customary notation put

$$s_m = \sum_{k=0}^{m} u_k,$$

$$\sigma_m = \frac{1}{m}\sum_{k=0}^{m-1} s_k,$$

so that $\sigma_m \to s$ by hypothesis.

* If (ii) is not true, the terms are ultimately of the same sign, and Σu_n certainly converges to s.

Landau's extension* of Mr. Hardy's theorem is to the effect that if nu_n is bounded below, then Σu_n is convergent, with sum s. It is concerned with cases in which $u_n \to 0$ in a special way.

The theorem on which we base our work is concerned with cases in which $\sigma_m \to s$ in a special way, and is as follows:—

If $a_n \to 0$, and we can find a sequence of integers p_m, such that p_m tends steadily to infinity with m, p_m/m tends steadily to zero, $p_m a_m$ tends to zero, and each of the means

$$\frac{s_{m-p_m}+s_{m-p_m+1}+\ldots+s_{m-1}}{p_m} = \frac{m\sigma_m-(m-p_m)\,\sigma_{m-p_m}}{p_m},$$

$$\frac{s_m+s_{m+1}+\ldots+s_{m+p_m-1}}{p_m} = \frac{(m+p_m)\,\sigma_{m+p_m}-m\sigma_m}{p_m}$$

tends to s, then the series Σu_n is convergent, and has the sum s.

It may be observed that the utility of the theorem lies in the fact that, however slowly a_m tends to zero, a sequence p_m can always be chosen which satisfies the condition that $p_m a_m$ tends to zero, and yet itself tends to infinity with m.

We observe first that, since $p_m/m \to 0$, we have $p_m < \mu$, where μ is the greatest integer in $\frac{1}{2}m$, for all sufficiently large values of m. Hence $m-p_m \geqslant \mu+1$. Also

$$\frac{p_m}{m} < \frac{p_{\mu+1}}{\mu+1}, \qquad p_m < 2p_{\mu+1} \leqslant 2p_{m-p_m};$$

so that $$|p_m a_{m-p_m}| \leqslant 2\,|p_{m-p_m} a_{m-p_m}| \to 0.$$

To proceed with the proof, we have

$$\begin{aligned} s_{m-k} &= s_m-(u_{m-k+1}+\ldots+u_m) \\ &\leqslant s_m+ka_{m-k} \\ &\leqslant s_m+p_m a_{m-p_m}, \end{aligned}$$

when $0 < k \leqslant p_m$; and

$$\begin{aligned} s_{m+k} &= s_m+(u_{m+1}+\ldots+u_{m+k}) \\ &\geqslant s_m-p_m a_m, \end{aligned}$$

when $0 \leqslant k < p_m$. Thus

$$\frac{s_{m-p_m}+\ldots+s_{m-1}}{p_m} \leqslant s_m+p_m a_{m-p_m},$$

$$\frac{s_m+\ldots+s_{m+p_m-1}}{p_m} \geqslant s_m-p_m a_m.$$

* See de la Vallée-Poussin's *Cours d'Analyse*, Ed. 2, Vol. 2, p. 157.

Make $m \to \infty$, and consider lower and upper limits of indetermination respectively. We obtain

$$s \leqslant \underline{\text{Lim}}\, s_m,$$

$$s \geqslant \overline{\text{Lim}}\, s_m.$$

Hence Lim s_m exists and is equal to s.

By applying the theorem to $\Sigma(-u_n)$, which is equivalent to making the auxiliary series Σa_n depend on the positive terms of Σu_n, we can obtain a second criterion for the convergence of Σu_n.

We are now in a position to deduce Dini's condition from Fejér's theorem.

For, if Σu_n is the Fourier series of $f(x)$, we have

$$\sigma_m - s = \frac{1}{m\pi}\int_0^{\frac{1}{2}\pi} \phi(\theta)\left(\frac{\sin m\theta}{\sin\theta}\right)^2 d\theta,$$

where $$\phi(\theta) = f(x+\theta)+f(x-\theta)-f(x+0)-f(x-0).$$

Hence

$$\frac{1}{p}\{m\sigma_m-(m-p)\,\sigma_{m-p}\}-s = \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\phi(\theta)\,\frac{\sin^2 m\theta-\sin^2(m-p)\theta}{p\sin^2\theta}\,d\theta$$

$$= \left(\frac{1}{\pi}\int_\epsilon^{\frac{1}{2}\pi}+\frac{1}{\pi}\int_0^\epsilon\right)\phi(\theta)\,\frac{\sin^2 m\theta-\sin^2(m-p)\theta}{p\sin^2\theta}\,d\theta.$$

The first integral tends to zero if $p \to \infty$, however slowly. The modulus of the second is not greater than

$$\frac{1}{\pi}\int_0^\epsilon |\phi(\theta)|\left|\frac{\sin(2m-p)\,\theta\sin p\theta}{p\sin^2\theta}\right| d\theta$$

$$< \frac{1}{\pi}\int_0^\epsilon |\phi(\theta)|\,\frac{d\theta}{\sin\theta},$$

since $$\left|\frac{\sin p\theta}{\sin\theta}\right| < p.$$

If $\int_0^\epsilon |\phi(\theta)|\,\frac{d\theta}{\theta}$ converges, so does $\frac{1}{\pi}\int_0^\epsilon |\phi(\theta)|\,\frac{d\theta}{\sin\theta}$, and this last integral can be made as small as we please by choice of ϵ.

Thus, if Dini's condition holds,

$$\frac{1}{p}\{m\sigma_m-(m-p)\,\sigma_{m-p}\} \to s$$

however slowly $p \to \infty$ as $m \to \infty$, and in particular if p be the p_m of our theorem.

Since $a_m \to 0$, we can certainly find p_m so that all the conditions of the theorem are satisfied. The series is therefore convergent.

MIQUEL'S THEOREM AND THE DOUBLE SIX

By E. K. WAKEFORD.

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1. From Miquel's theorem that the foci of the five parabolas, each touching four out of five straight lines in a plane are concyclic, may be deduced a proof of the theorem of the double six of lines in space: namely, that if five lines $a_r (r = 1, \ldots, 5)$ meet a line b_6, and five lines $b_r (r = 1, \ldots, 5)$ each meet all the lines a_r, except the one with the same suffix, then a line meets the five lines $b_r (r = 1, \ldots, 5)$. A projective form of Miquel's theorem is found in § 2; in § 3 a subsidiary theorem is proved; although this theorem is well known it is desirable to be quite sure that a proof of the double six does not depend on cubic surfaces. The proof itself follows immediately.

2. We shall first explain a few terms which will be convenient. The adjective "genuine" will be used to denote "not degenerate." If triangles can be drawn circumscribed about a conic S, and inscribed in a conic C, we shall say that S is "triangularly inscribed" in C, or that C is "triangularly circumscribed" about S.

LEMMA I.—*Given four skew lines $l_1 l_2 l_3 l_4$, and two points A, B; to find a twisted cubic passing through A, B, of which $l_1 l_2 l_3 l_4$ are chords.*

Conicoids through A, B, l_1, and l_2, cut l_4 in pairs of points in involution. Similarly conicoids through A, B, l_1 and l_3 cut l_4 in pairs of points in involution. Hence, by taking the common pair P, Q of these two involutions, we can find one conicoid containing A, B, P, Q, l_1 and l_2, and another containing A, B, P, Q, l_1 and l_3. The curve of intersection of these two conicoids consists of l_1 and a twisted cubic of which l_1 is a chord. This cubic must pass through A, B, P, Q, and must have l_2 and l_3 as chords as well as l_1, since they are generators of the same system as l_1 in the two conicoids respectively.

Hence since P, Q are on l_4, we have found a twisted cubic through A, B, with $l_1 l_2 l_3 l_4$ as chords.

LEMMA II.—The projective form of Miquel's theorem which we require is as follows: *Let* $l_r (r = 1, \ldots, 6)$ *denote six lines in a plane, and* $S_r (r, = 1, \ldots, 6)$ *the six conics each of which touches all of these lines except the one with the same suffix as itself. Then any genuine conic C which is triangularly circumscribed about* S_1, S_2 *and* S_3, *is triangularly circumscribed about* S_4, S_5 *and* S_6 *also.*

Project l_6 to infinity and C into a circle.

Then the conics $S_r (r = 1, \ldots, 5)$ become parabolas.

Now, if a circle is triangularly circumscribed about a parabola, it passes through the focus of the parabola, and conversely.

Hence C passes through the foci of S_1, S_2 and S_3.

But by Miquel's theorem the foci of S_1, S_2, S_3, S_4, and S_5, are concyclic.

So C passes through the foci of S_4 and S_5.

Thus C is triangularly circumscribed about S_4 and S_5, and similarly about S_6.

3. If a genuine twisted cubic is projected from a point on it into a conic C, each line in the plane of the conic is the projection of one and only one chord of the cubic. The chords which cut a given line in space are well known to be projected into lines enveloping a conic triangularly inscribed in C, and conversely; any such conic represents a line in space in the sense that its tangents are the projections of chords cutting a certain fixed line in space.

Let us now take the five lines $a_r (r = 1, \ldots, 5)$ cutting b_6, as in § 1, and find a genuine twisted cubic of which they all are chords. This is done by fixing the points of intersection of the cubic with a_1 in such a way that the cubic cannot possibly be degenerate, and afterwards applying Lemma I. It will be found that this is easy provided that, as is of course assumed, a_1 does not meet b_1.

Now project the cubic from a point on it into a conic C, and let the chords a_r be projected into five lines $l_r (r = 1, \ldots, 5)$ circumscribing a conic S_6 which must be triangularly inscribed in C. If we fix four tangents to a conic, and add the condition that the conic is triangularly inscribed in C, we get just two solutions in general, representing the two lines in space which cut all the four chords of the cubic of which the tangents are the projections. If, then, we draw the five conics $S_r (r = 1, \ldots, 5)$ which are triangularly inscribed in C, and each of which touches all the lines l_r except the one with the same suffix, we shall have the conics which represent the five lines $b_r (r = 1, \ldots, 5)$. We shall prove that these five conics have a

common tangent, from which it follows that there is a chord of the cubic cutting all the lines b_r ($r = 1, \ldots, 5$).

For, let the fourth common tangent of S_1 and S_2 other than l_3, l_4, and l_5 be l_6. We observe that each of S_1, S_2, and S_6 touches five of the lines l_1, l_2, l_3, l_4, l_5, and l_6, and that C is triangularly circumscribed about all three. Hence, by Lemma II, C is triangularly circumscribed about the other conics, each of which touches five of the six lines, that is to say, the conics touching l_1, l_2, l_4, l_5, l_6; l_1, l_2, l_3, l_5, l_6; and l_1, l_2, l_3, l_4, l_6 respectively. But S_3 is the only conic other than S_6, which touches l_1, l_2, l_4, l_5, and is also triangularly inscribed in C. Hence S_3 must be the conic touching l_1, l_2, l_4, l_5, and l_6. Similarly S_4 and S_5 are touched by l_6.

Hence as we pointed out above there is a chord of the cubic cutting all b_1, b_2, b_3, b_4, and b_5, and so the theorem of the double six is proved.

THE NODAL POINTS OF A PLANE SEXTIC

By J. HODGKINSON.

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I. *Statement of the Problem considered.*

The general equation of a sextic curve contains 27 arbitrary constants. Hence by the arbitrary assignment of 9 double points the curve is uniquely determined. If, however, the cubic through the 9 points have the equation $C = 0$, $C^2 = 0$ is such a sextic and is therefore the only one. Hence the 9 points cannot be wholly arbitrary, and in this paper I have investigated their interdependence, and some other points of a kindred character.

The problem was noted by Salmon (*Curves*, 3rd edition, Art. 45). In the second edition he states that only seven of the double points may be quite arbitrary, but this remark disappears in the third. My attention was called to it by Mr. A. E. Jolliffe, who kindly pointed out the weaknesses of my earlier efforts towards its solution. I have also had the advantage of some notes by Mr. H. W. Richmond.

I have established the following results.

(1) Given 8 quite arbitrary double points of a sextic, the 9th lies on a curve of the 9th degree, having the 8 points as triple points.

(2) When 8 double points are arbitrarily assigned, and the 9th has been assigned on its locus but otherwise arbitrarily, the number of unicursal sextics having these assigned double points is, at most, 30.

(3) A similar theorem is shown to hold good in the case of $3m$-ic curves having 9 points of multiplicity m.

II. *A Geometrical Method.*

This is due to Mr. Richmond.

Suppose 8 double points of a sextic S to be arbitrarily assigned. If

the sextic have a 9th double point, one and only one cubic can be drawn through the 9 points. For, if the 9 points be the intersections of two cubics C_1, C_2, a member of the family $\lambda C_1+\mu C_2$ can be drawn to pass through these 9 points and any other point arbitrarily chosen on the sextic. This cubic then intersects the sextic in 19 points, and the sextic must therefore be composite. Hence one and only one cubic, C say, can be drawn through the 9 points.

Suppose these points are A, B, C, D, E, F, G, H, K. Through A, B, C, D, E, F, G, H we take another cubic C_1 intersecting the cubic $ABCDEFGHK$ in L.

We have now a sextic S cutting C in A,B,C,D,E,F,G,H,K, each taken twice, and a second sextic C_1^2 cutting C in A,B,C,D,E,F,G,H,L, each taken twice. K taken twice and L taken twice are coresidual, so that the tangents to C at K and L meet on the curve, C at P say. Now from P to C 4 tangents can be drawn, one of which is PL. Hence there are 3 possible positions for K, *i.e.*, given 8 double points of a sextic, and a cubic passing through them, 3 possible positions of a 9th double point lie on the cubic.

The use of elliptic functions leads easily to the same result.

$S=0$ being a sextic with 9 nodes, and $C=0$ the cubic through them, we can express the coordinates of a point upon it as elliptic functions of a parameter u. If the intersections of S and C have parameters u_1, u_2, ..., u_9 (each being taken twice),

$$2(u_1+u_2+\ldots+u_9)\equiv 0,$$

with respect to the periods $2\omega_1$, $2\omega_2$ of the elliptic functions. Therefore

$$u_1+u_2+\ldots+u_9\equiv 0, \text{ or } \omega_1, \text{ or } \omega_2, \text{ or } \omega_3.$$

The first case is not admissible for our purpose, for it fixes the 9 nodes at the intersections of 2 cubics, when, as we have seen, the sextic is composite.

Hence 8 of the points being assigned, the 9th may have any one of 3 positions upon any particular cubic through the 8 points.

If now the locus of the 9th double point, when 8 are arbitrarily assigned, be a curve of degree n, having multiple points of order m at each of the 8 points, and also intersecting a cubic through the 8 points in 3 points, we have

$$8m+3=3n.$$

The solution in integers of this equation is

$$n=8p+1, \quad m=3p.$$

Assuming that this locus is not a composite curve, the maximum number of its double points is $4p(8p-1)$.

Hence
$$4p(8p-1) \geqslant 12p(3p-1),$$
$$8p-1 \geqslant 9p-3,$$
$$2 \geqslant p,$$

leading to
$$n = 1,\ 9,\ 17,$$
$$m = 0,\ 3,\ 6.$$

Even if we do not make this assumption, we can assert that the maximum number of double points is $4p(8p+1)$, a result which leads to

$$4 \geqslant p.$$

The analytical method of the next section will definitely decide that the actual value of p is 1, *i.e.*,

$$n = 9, \quad m = 3.$$

III. *An Analytical Method.*

Suppose 8 points to be arbitrarily assigned as double points of a sextic. The general equation of all such sextics is

$$p_1 S_1 + p_2 S_2 + p_3 S_3 + p_4 S_4 = 0,$$

where S_1, S_2, S_3, S_4 are 4 linearly independent sextics of the system.

We first show that 4 such sextics can be obtained. Draw any 2 cubics C_1, C_2 through the 8 points. For S_2, S_3, S_4, we may write C_1^2, $2C_1C_2$, C_2^2 respectively. We obtain an independent form for S_1 in either of the following ways.

(1) Through the 8 points we take a trinodal quartic with the nodes at 3 of the points, and a conic through the remaining 5. This gives a composite sextic of the type required.

(2) We take 2 unicursal cubics, each having its node at one of the 8 points, and passing through the remaining 6. This also gives a composite sextic.

Call either of these $2S$.

We have now the equation of the family in the form

$$2p_1 S+p_2 C_1^2+2p_3 C_1 C_2+p_4 C_2^2 = 0.$$

Suppose there be a further double point at (x, y, z). We introduce a further, but non-essential, simplification by choosing C_1 as that particular member of the family of cubics through the 8 points which passes the 9th double point (x, y, z).

At this double point, putting $C_1 = 0$,

$$p_1 \frac{\partial S}{\partial x} +p_3 C_2 \frac{\partial C_1}{\partial x} +p_4 C_2 \frac{\partial C_2}{\partial x} = 0,$$

$$p_1 \frac{\partial S}{\partial y} +p_3 C_2 \frac{\partial C_1}{\partial y} +p_4 C_2 \frac{\partial C_2}{\partial y} = 0,$$

$$p_1 \frac{\partial S}{\partial z} +p_3 C_2 \frac{\partial C_1}{\partial z} +p_4 C_2 \frac{\partial C_2}{\partial z} = 0.$$

Hence either (i) p_1, p_3, p_4 all vanish,

or (ii) $C_2 = 0$,

or (iii) $$J = \begin{vmatrix} \frac{\partial S}{\partial x}, & \frac{\partial C_1}{\partial x}, & \frac{\partial C_2}{\partial x} \\ \frac{\partial S}{\partial y}, & \frac{\partial C_1}{\partial y}, & \frac{\partial C_2}{\partial y} \\ \frac{\partial S}{\partial z}, & \frac{\partial C_1}{\partial z}, & \frac{\partial C_2}{\partial z} \end{vmatrix} = 0.$$

The possibility (i) reduces the sextic to $C_1^2 = 0$, and (ii) we have seen to imply that the sextic is composite.

Hence the 9th node of a proper sextic must lie on the curve $J = 0$, which is of the 9th degree.

The properties of Jacobians are demonstrated by Salmon (*Curves*, Art. 396), who shows that if S, C_1, C_2 have any common points, J also passes through them ; and that where, as here, C_1, C_2 are of the same order, and the common points are double points of S, they are also double points of J. Finally he gives, for the values of the second differential coefficients of J at these common points, the expressions

$$x\frac{\partial^2 J}{\partial x^2} = (n-2m)\frac{\partial^2 S}{\partial x^2}, \qquad x\frac{\partial^2 J}{\partial x\,\partial y} = (n-2m)\frac{\partial^2 S}{\partial x\,\partial y}, \qquad \ldots,$$

where n is the order of S, m of C_1 and C_2. Writing $n = 6$, $m = 3$, we find that $\frac{\partial^2 J}{\partial x^2}$, $\frac{\partial^2 J}{\partial x\,\partial y}$, ..., all vanish. The 8 double points are all triple points on the locus of the 9th node.

Further, there are the 12 points associated with the family of cubics through the 8 points known as the critic centres. For these

$$\frac{\partial C_1}{\partial x} : \frac{\partial C_1}{\partial y} : \frac{\partial C_1}{\partial z} = \frac{\partial C_2}{\partial x} : \frac{\partial C_2}{\partial y} : \frac{\partial C_2}{\partial z}.$$

These plainly satisfy $J = 0$. These particulars are more than sufficient to determine the curve.

Although, however, the locus of the 9th node passes through the critic centres, the sextic which has one of them for its 9th node is composite. For C being the unicursal cubic which passes through the 8 given points and has a double point at the particular critic centre chosen, will meet the sextic in 20 points.

IV. *Sextics which touch a Cubic in Nine Points.*

Mr. Richmond points out that if a sextic touches a cubic in 9 points, these 9 points are double points of a sextic (proper or composite), and conversely.

This is at once obvious, for the method of § II above applies exactly. Or, again, supposing S to be a 9-nodal sextic, and C the cubic through the 9 nodes, and Γ any other cubic, then

$$S + C\Gamma = 0$$

is a sextic touching the cubic at the 9 points.

Conversely, if S be a sextic touching a cubic C in 9 points, and Γ a

quite arbitrary cubic, then $$S+C\Gamma = 0$$

is also a sextic meeting C in 9 pairs of coincident points. It will contain 10 arbitrary constants. We choose the constants of Γ, so that

$$\frac{\frac{\partial S}{\partial x}}{\frac{\partial C}{\partial x}} = \frac{\frac{\partial S}{\partial y}}{\frac{\partial C}{\partial y}} = \frac{\frac{\partial S}{\partial z}}{\frac{\partial C}{\partial z}} = -\Gamma$$

at each of the 9 points. This introduces 9 conditions, for we remember that, at a point of contact of S and C, the three fractions are equal. Hence this can always be done and the sextic $S+C\Gamma$ is then a 9 nodal sextic.

V. *Number of Unicursal Sextics having Nine given Nodes.*

The assignment of 8 quite arbitrary double points determines 24 constants. The assignment of a 9th on the node locus determines 2 more, leaving one constant undetermined in the equation of a sextic.

If, in fact, we substitute for the sextic $2S$ used in § III above, a sextic which has a 9th node, it is easy to see that the general equation of the family of sextics having the 9 nodes reduces to

$$2\lambda S+\mu C^2 = 0.$$

If there be a 10th double point (x, y, z)

$$\lambda \frac{\partial S}{\partial x} + \mu C \frac{\partial C}{\partial x} = 0,$$

$$\lambda \frac{\partial S}{\partial y} + \mu C \frac{\partial C}{\partial y} = 0,$$

$$\lambda \frac{\partial S}{\partial z} + \mu C \frac{\partial C}{\partial z} = 0.$$

Hence (x, y, z) is an intersection of the curves

$$\frac{\partial S}{\partial x}\frac{\partial C}{\partial y} - \frac{\partial S}{\partial y}\frac{\partial C}{\partial x} = 0,$$

$$\frac{\partial S}{\partial x}\frac{\partial C}{\partial z} - \frac{\partial S}{\partial z}\frac{\partial C}{\partial x} = 0.$$

These have 49 intersections : 9 of these intersections are the double points,

and 10 others are the intersections of

$$\frac{\partial S}{\partial x} = 0, \quad \frac{\partial C}{\partial x} = 0,$$

which must be rejected because they do not also lie upon

$$\frac{\partial S}{\partial y}\frac{\partial C}{\partial z} - \frac{\partial S}{\partial z}\frac{\partial C}{\partial y} = 0.$$

Hence there are, at most, 30 unicursal sextics which have 9 assigned double points.

VI. *A Similar Theorem concerning Bicursal 3m-ics.*

It appears that a similar theorem holds good in the more general case of a $3m$-ic curve having 9 points of multiplicity m. The deficiency of such a curve is

$$\frac{(3m-1)(3m-2)}{2} - \frac{9m(m-1)}{2} = 1.$$

The number of arbitrary constants contained in the general equation of a $3m$-ic is $\frac{9m(m+1)}{2}$, and the curve is therefore uniquely determined by the arbitrary assignment of the 9 points. Such a $3m$-ic is $C^m = 0$, where $C = 0$ is the cubic through the 9 points. It is therefore the only one. Hence only 8 of the points can be arbitrarily assigned, and the 9th must lie on a certain locus. The equation becomes more complicated with increase of m.

The assignment of 8 m-ic points leaves $\frac{m(m+1)}{2}$ constants undetermined. Hence the general equation of the $3m$-ic having these 8 m-ic points will be

$$\sum_{1}^{\frac{1}{2}m(m+1)+1} p_r S_r = 0.$$

For $(m+1)$ of these curves S, we may take C_1^m, $C_1^{m-1}C_2$, $C_1^{m-2}C_2^2$, ..., C_2^m. As in the simple case, $m = 2$, we choose C_1 as the cubic passing through the 9th point.

The conditions that (x, y, z) be a 9th m-ic point are the set

$$\frac{\partial^{m-1}}{\partial x^{q_1}\partial y^{q_2}\partial z^{q_3}}\{\Sigma p_r S_r\} = 0,$$

where q_1, q_2, q_3 are positive integers and $q_1+q_2+q_3 = m-1$. Now re-

membering that $C_1 = 0$ at (x, y, z), $\frac{\partial^{m-1} C_1^m}{\partial x^{q_1} \partial y^{q_2} \partial z^{q_3}}$ will vanish at (x, y, z), and $\frac{\partial^{m-1} C_1^k C_2^{m-k}}{\partial x^{q_1} \partial y^{q_2} \partial z^{q_3}}$ will contain a factor C_2. Eliminating $p_1, p_2, \ldots$ from the set of equations, we obtain a determinant of $\frac{m(m+1)}{2}$ columns, in which m columns are of degree $2m-2$ in the variables, and the remaining $\frac{m(m-1)}{2}$ are of degree $2m+1$. Hence the degree of the expanded determinant is

$$\frac{m(m-1)(2m+1)}{2} + 2m(m-1) = \frac{m(m-1)(2m+5)}{2}.$$

Now it is plain that this determinant must contain as factors the similar determinants obtained for $(m-1)$, $(m-2)$, ..., 2. For, if $S_{3(m-n)}$ be a $3(m-n)$-ic, having 9 $(m-n)$-ic points, and C the cubic through them, $S_{3(m-n)} C^n$ is a $3m$-ic having 9 m-ic points.

Further, it is easily shown that if a $3m$-ic have 9 m-ic points where a $3(m-n)$-ic also has 9 $(m-n)$-ic points, the $3m$-ic will be composite. Hence we must reject from the locus found above all the factors giving loci for the cases $(m-1)$, $(m-2)$, ..., 2.

If n_m be the degree of the locus of the 9th m-ic point, we have

$$\sum_2^m n_p = \frac{m(m-1)(2m+5)}{2},$$

whence

$$n_m = 3(m^2-1).$$

By the use of elliptic functions, as in § II above, we see at once that, on a cubic through the 8 points, there are, for m prime (the case m not prime requires some further consideration), m^2-1 positions for the 9th m-ic point. Hence, if n be the order of the locus at each of the 8 points,

$$8n + (m^2-1) = 9(m^2-1),$$

$$n = (m^2-1).$$

Hence, when m is prime, the locus of the 9th m-ic point, 8 such points having been arbitrarily assigned, is a curve of order $3(m^2-1)$ having the 8 points as multiple points of order m^2-1.

This result agrees with what we found for $m = 2$. We shall verify it, by the method of residuation, for the case $m = 3$.

If the 8 assigned triple points are A, B, C, D, E, F, G, H, and 2 cubics through them C_1, C_2 intersect in K, and L be a possible position

on C_1 for the 9th triple point, K taken 3 times and L taken 3 times are coresidual.

Describe a conic meeting C_1 in 3 coincident points at K, and also cutting it in X, Y, Z. Then a conic can be described passing through X, Y, Z and also meeting C_1 in 3 coincident points at L.

Taking XYZ as triangle of reference, we transform our plane by the substitution

$$\xi = \frac{1}{x}, \quad \eta = \frac{1}{y}, \quad \zeta = \frac{1}{z}.$$

The cubic C_1 is transformed into a cubic. The conics passing through X, Y, Z are transformed into straight lines, and the number of such conics which meet C_1 in 3 coincident points is therefore the number of points of inflexion of a non-singular cubic, *i.e.*, 9. One of these corresponds to the inadmissible point K. Hence the number of possible positions of the 9th triple point lying on the cubic C_1 is 8. If q be the degree of the locus of the 9th triple point, and p its order at the 8 given points,

$$8p+8 = 3q.$$

This is satisfied by $q = 24$, $p = 8$, according to the formula found above.

When, however, m is not prime, we must reduce the degree of the locus and the order of its multiplicity at the 8 given points still further.

When we use elliptic functions to obtain the number of points of the locus lying on any particular cubic through the 8 points, we find that the possible set of values of the parameters of the elliptic function for these points contains also the set arising when we take the same problem for p, where p is a factor of m.

Hence the locus as obtained above will contain as factors the similar loci obtained with such factors of m.

Thus in finding in this manner the locus of the 9th point of multiplicity 6 of an 18-ic, we obtain as factors the loci of the 9th double point of a 6-ic, and of the 9th triple point of a 9-ic.

We have then to find a number $\phi(m)$, which is such that

$$\phi(m) = m^2-1,$$

where m is prime, and $\quad \Sigma\phi(d) = m^2-1,$

where m is composite, the summation referring to all the divisors of m, including itself.

Let $\quad m = a^\alpha b^\beta c^\gamma \ldots .$

Then $$\sum_{r=0}^{\alpha}\sum_{s=0}^{\beta}\sum_{t=0}^{\gamma}\ldots\phi(a^r b^s c^t\ldots) = a^{2\alpha}b^{2\beta}c^{2\gamma}\ldots -1,$$

$$\sum_{r=0}^{\alpha-1}\sum_{s=0}^{\beta}\sum_{t=0}^{\gamma}\ldots\phi(a^r b^s c^t\ldots) = a^{2(\alpha-1)}b^{2\beta}c^{2\gamma}\ldots -1,$$

and therefore $$\sum_{s=0}^{\beta}\sum_{t=0}^{\gamma}\ldots\phi(a^\alpha b^s c^t\ldots) = m^2\left(1-\frac{1}{a^2}\right).$$

Proceeding in this manner, we finally get

$$\phi(m) = m^2\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{b^2}\right)\left(1-\frac{1}{c^2}\right)\ldots.$$

Hence the locus of the 9th m-ic point of a $3m$-ic, given 8 such points, is a curve of order $3m^2\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{b^2}\right)\left(1-\frac{1}{c^2}\right)$, where $a, b, c, \ldots$ are the different prime factors of m, and its multiplicity at each of the 8 points is $m^2\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{b^2}\right)\left(1-\frac{1}{c^2}\right)\ldots$.

I have verified this for $m=4$ by residuation. Suppose the 8 points are A, B, C, D, E, F, G, H, and the other common point of the family of cubics through them K. We require to find the number of possible positions of a 9th point X of multiplicity 4 lying on any particular cubic through A, B, C, D, E, F, G, H.

Then X and K each taken 4 times are coresidual. Draw tangents at X, K meeting the cubic in Y, L. Then Y and L each taken twice are coresidual. Hence the tangents at Y, L meet at M on the cubic. This gives a construction for X.

The tangent at K determines L uniquely. The tangent at L determines M uniquely. The tangents from M determine 4 positions of Y, one of which is L. The tangents at Y determine 16 positions of X, one of which is K, and three others are those found for the positions of the 9th node of a 6-ic. Hence the number of possible positions for the 9th 4-ic point of a proper 12-ic is 12. Forming our equation as before, we get

$$8p+12 = 3q.$$

This is satisfied by $p=12$, $q=36$, according to the formula above.

I am bound to remark that this does not appear to me to be wholly

satisfactory, for it appears that certain factors may be rejected many times.

Thus suppose we have to find the locus of the 9th 12-ic point of a 36-ic. We discarded factors for the loci of the 9th point of multiplicity of r of a $3r$-ic, for $r = 2, 3, 4, \ldots, 11$. Of these the cases $r = 2, 4, 6, 8, 10$ all contained factors representing the locus of the 9th double point of a sextic. Finally we discarded it again, since 2 is a factor of 12. I am not able to justify this, but the general expression I obtained is easily verifiable by the method of residuation for $m = 2^n$, and I am therefore inclined to think the result is true.

[Since completing the above, I have received the following references: J. E. Campbell, *Messenger of Mathematics*, 1891–2, p. 158; Terracini, *Atti . . . . Torino*, Feb. 20th, 1916. I regret that I have had no opportunity to consult them to ascertain their precise bearing on the subject I have discussed.]

NOTE ON FUNCTIONS OF UPPER AND LOWER TYPE

By W. H. YOUNG.

[Received March 9th, 1916.—Read April 27th, 1916.]

1. I propose in the present note to give simple proofs of certain results I have communicated elsewhere. Starting with simple functions, that is with functions which are constant in each of a finite number of closed or open stretches into which the fundamental segment is divided, we can, by employing suitable sequences which alternately increase and decrease monotonely, obtain every mathematically definable function.*

There will in the case of each function be a minimum to the number† of such monotone sequences required. We may, if this number be denoted by α, call the function a function of the α-th class. Further, inside this class, the functions will be naturally regarded as of two distinct types, according as the last of the monotone sequences employed is descending or ascending, or, what is the same thing, according as the function is approached from above, or from below, by the generating functions of an earlier type, or types.

Functions of the two types differ in fact in an essential particular. We shall find it convenient to call the type which is approached from below *an upper type*, and that which is approached from above *a lower type*.

Thus a semi-continuous function is a function of the first class and is of the upper type, or lower type, in that class, according as it is upper semi-continuous, or lower semi-continuous. A continuous function, on the other hand, is of the first class and of both types.

More generally, with the notation I have already employed for the earlier types of functions, a function which is symbolised by r letters u and l, where r is an integer, is of the r-th class, and is of the upper or lower type in that class according as the first letter is a u or an l.

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* These sequences are, in general, infinite in number, and may follow one another in any well ordered manner.

† Ordinal number in Cantor's sense.

It is often convenient to regard a function properly belonging to a certain class as belonging to all subsequent classes. It will then be indeterminate as to type, except in the class to which it properly belongs. Thus a *lu*-function is a lower function in the second class, and both a lower and an upper function in the third class. This causes no confusion.

2. The classes are, of course, designated in the first instance by the ordinary ordinals. The language employed is, however, still applicable when other and transfinite ordinals are required for the purpose.

Thus in the proof of the following theorems it is nowhere assumed that the class of functions considered is characterised by an ordinary integer n.

THEOREM 1.—*If a succession of functions of an upper type in a certain class has a lower function* which is not greater than any of them, then the lower function is of the same type and class.*

Let the functions be $f_1, f_2, \ldots, f_n, \ldots$, and let L be the lower function of the succession. Also let

$$g_1 = f_1,$$

and let g_n be defined as the function which at each point has the less of the two values of g_{n-1} and f_n there.

Then $g_1, g_2, \ldots$ clearly form a monotone descending (*i.e.*, non-ascending) succession of functions. Moreover, as the functions are all $\geqslant L$, by hypothesis, the functions g_n are also all $\geqslant L$.

Hence, if g is the unique limit of g_n as n increases indefinitely, we have

$$g \geqslant L. \tag{1}$$

Moreover g, as the unique limit of a descending sequence of functions of an upper type, is of this same type necessarily.

Again, since for each index n, we have

$$g_n \leqslant f_n,$$

it follows that this unique limit of g_n is not greater than the lower function of the f_n-succession. In other words,

$$g \leqslant L. \tag{2}$$

* That is the function whose value at each point is the lower limit of the succession.

From (1) and (2) the required result follows, since g has been shown to have the required type.

THEOREM 2.—*If a succession of functions of a lower type in a certain class has an upper function* which is not less than any of them, then the upper function is of the same type and class.*

The proof of Theorem 1 applies *mutatis mutandis.*

We notice *en passant* that these theorems contain as particular cases the theorems which have been used in proving them, viz., those with respect to the nature of the limiting functions of monotone sequences of functions of a certain type. In particular they are the generalisations of the special results that monotone successions of continuous functions have upper or lower semi-continuous functions as limits, according as the monotony is descending or ascending.

3. We now apply the above theorem to a proof of the fundamental theorem in the theory of uniform convergence.

THEOREM 3.—*The limit of a uniformly convergent sequence of functions of a particular type is itself a function of the same type.*

Let $f_1, f_2, \ldots$ be the functions forming the sequence. Then, since the sequence is uniformly convergent, we can, given e, find m, so that, for all values of x,

$$|f(x)-f_n(x)| < e \quad (n \geqslant m).$$

Hence, if

$$e_1 > e_2 > \ldots \to 0$$

be any monotone descending sequence of constants with zero as limit, we can weed out the functions f, and re-number them, so that

$$|f(x)-f_n(x)| < e_n$$

for all values of n.

Hence, if the functions $f_1, f_2, \ldots$, are of an upper type, and we write

$$g_n(x) = f_n(x)+e_n,$$

we shall have

$$\operatorname*{Lt}_{n\to\infty} g_n(x) = f(x),$$

and

$$g_n(x) \geqslant f(x),$$

where $g_n(x)$ is of the same type as $f_n(x)$. The required result then follows by Theorem 1.

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* That is the function whose value at each point is the upper limit of the succession.

If, however, the functions $f_1, f_2, \ldots$ are of a lower type, and we write

$$h_n(x) = f_n(x) - e_n,$$

we shall have $\underset{n\to\infty}{\text{Lt}}\ h_n(x) = f(x),$

and $h_n(x) \leqslant f(x),$

where $h_n(x)$ is of the same type as $f_n(x)$. The required result then follows by Theorem 2.

COR.—*The limit of a uniformly convergent sequence of functions all belonging to one of Baire's classes itself belongs to the same class.**

4. We can, of course, easily generalise Theorem 3, so as to make it applicable to oscillating successions of functions. No further reasoning is, however, required to obtain the corresponding statements for uniform oscillation above and below, of the first and second kinds. The student has merely to use the definitions of such oscillation by means of the monotone sequences belonging to the oscillating succession.

5. It is, however, of importance to notice that in §§ 2, 3, we have not utilised the necessary and sufficient conditions that a function may belong to a certain type in a certain class.

In a preceding paper† I have proved the sufficiency of the condition by employing a somewhat difficult line of reasoning, and I later‡ proved Theorem 3 by means of these conditions. The reverse order is certainly the simpler. We can, in fact, prove the sufficiency of the conditions in a few lines, as soon as we have proved the fundamental theorem in uniform convergence of § 3.

THEOREM.—*If a function $f(x)$ is such that either the points at which $f \geqslant k$, or the points at which $f \leqslant k$, form a set denoted by n letters beginning with an i, then f is of the $(n+1)$-th class, and of the upper type when the inequality is $\geqslant$, or of the lower type when it is $\leqslant$.*

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* H. Lebesgue, "Sur les fonctions représentables analytiquement," 1905, *Jour. de math. pure et appl.*, Ser. VI, Tome I, p. 156.

† "On Functions and their Associated Sets of Points," 1912, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, p. 279.

‡ "Uniform Oscillation of the First and Second Kind," 1913, *ibid.*, p. 357.

It will be sufficient to prove the former of the statements.

Insert n means $k_1, k_2, \ldots, k_n$ between the lower bound K_0 and the upper bound K of f, and let us form functions $g_1, g_2, \ldots, g_n$ as follows:—

$$\begin{aligned}
g_1(x) &= k_1, && \text{where } f \geqslant k_1, && \text{and is zero elsewhere;}\\
g_2(x) &= k_2 - k_1, && \text{,, } f \geqslant k_2, && \text{,,}\\
&\ldots \quad \ldots \quad \ldots\\
g_r(x) &= k_r - k_{r-1}, && \text{,, } f \geqslant k_r, && \text{,,}\\
&\ldots \quad \ldots \quad \ldots\\
g_n(x) &= k_n - k_{n-1}, && \text{,, } f \geqslant k_n, && \text{,,}
\end{aligned}$$

Then these functions are evidently* all of them of the $(n+1)$-th class and of the upper type in that class. Hence, if we write

$$f_n = g_1 + g_2 + \ldots + g_n,$$

this function f_n is of the same type and class.

Also

$$\begin{aligned}
f_n &= 0, && \text{when} && k_0 \leqslant f < k_1,\\
&= k_1 && \text{,,} && k_1 \leqslant f < k_2,\\
&\ldots && \ldots && \ldots \quad \ldots\\
&= k_r && \text{,,} && k_r \leqslant f < k_{r+1},\\
&\ldots && \ldots && \ldots \quad \ldots\\
&= k_n && \text{,,} && k_n \leqslant f < K.
\end{aligned}$$

Now make $n \to \infty$, then evidently f_n converges to f everywhere. Moreover the convergence is plainly uniform, since to any e an m can be found so that

$$|f - f_n| < e \quad (n \geqslant m).$$

We have, in fact, only to choose m, so that

$$(K - K_0)/m < e.$$

Hence by Theorem 3 the result follows. Q.E.D.

6. No alteration is required in the proof of the necessity of the conditions. This necessity follows, in fact, almost immediately from (i) the theorem that, if

$$f_1 \leqslant f_2 \leqslant \ldots \leqslant f_n \to f,$$

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* "On Functions and their Associated Sets Points," p. 238.

then the points at which $f > k$ form an outer limiting set of the sets of points at which, for all values of n, $f_n > k$, and the analogous theorem, and (ii) the obvious circumstance that the points at which $f = k$ form an inner limiting set of the sets of points at which

$$k - e_n \leqslant f < k + e_n \quad (e_n \to 0),$$

for all values of n, and at which

$$k - e_n < f \leqslant k + e_n,$$

for all values of n.

ON THE DERIVATES OF A FUNCTION

By GRACE CHISHOLM YOUNG.

[Received March 9th, 1916.—Read April 27th, 1916.]

1. Theorems relating to perfectly general functions are rare.* In the present connexion there is the theorem, given by myself in the *Acta*,† that *the points, if any, at which the upper derivate on one side is less than the lower derivate on the other side, form a countable set.* In this theorem no restriction whatever is laid on the primitive function, except tacitly that it is everywhere finite, and the derivates may be ordinary or taken with respect to any monotone increasing function $g(x)$ without constant stretches.‡

I propose now to enunciate and prove three fundamental theorems, already known to be true for special classes of functions, concerning the derivates of a function $f(x)$, which, if not perfectly general, is nevertheless, for mathematical purposes, practically unrestricted in character. I only assume that

(*a*) $f(x)$ *is a measurable function ;*

and (*b*) $f(x)$ *is finite everywhere.*

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* I would recall the theorem communicated by my husband to the British Association at Leicester in 1907, and, in an extended form, to the Mathematical Congress at Rome in 1908, that, except at a countable set of points, the limits of $f(x)$ on the right and on the left [that is the limits of $f(x+h)$ and $f(x-h)$, $(0<h)$] are the same, and $f(x)$ lies between the greatest and least of these limits (inclusive) on each side. See W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity," 1907, *Quarterly Journal*, Vol. 39, pp. 77 and 82. "Sulle due funzioni a piu valori costituite dai limite d'una funzione d'una variabile reale a destra e a sinistra di ciascun punto," 1909, *Rend. della R. Acc. dei Lincei*, Ser. 5, Vol. XVII, pp. 582-587. These surprising theorems formed the starting point of our investigations, and suggested that similar general results could be obtained for derivates ; till now, however, we had not been able to justify such a supposition.

† "A Note on Derivates and Differential Coefficients," 1912, *Acta Math.* (1914), Vol. 37, p. 144.

‡ Cp. W. H. Young, "On Integrals and Derivates with respect to a Function," 1914, *Proc. London Math. Soc.*, Ser. 2, Vol. 15, p. 61, footnote. I take this occasion to call attention to a misprint: this footnote (p. 61) and the following footnote (p. 62) have been interchanged.

The enunciations of these theorems are as follows:—

THEOREM 1.—*The points at which the upper derivate on one side is* $+\infty$ *and the lower derivate on the other side is not* $-\infty$, *form a set of content zero.**

In other words, if $f^+(x) = +\infty$, then $f_-(x) = -\infty$, except at a set of content zero; and if $f^-(x) = +\infty$, then $f_+(x) = -\infty$, except at a set of content zero.

THEOREM 2.—*The points at which* $f(x)$ *has an infinite right-hand (left-hand) differential coefficient form a set of content zero.*†

As an immediate corollary we have the generalisation of Lusin's theorem:—

The points, if any, at which a measurable function $f(x)$ *has an infinite differential coefficient, form a set of zero content.*

THEOREM 3.—*The points at which one of the upper derivates and one of the lower derivates of* $f(x)$ *are finite and not equal form a set of content zero.*‡

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* In my essay on "Infinite Derivates," to which was awarded the Gamble Prize at Girton College, Cambridge, last year, I obtained the theorem that, *when* $f(x)$ *is a continuous function, the points at which* $f^+(x) = +\infty$, *and both* $f_-(x)$ *and* $f_+(x)$ *different from* $-\infty$, *form a set of content zero;* the reasoning was subsequently seen by me to give the theorem as it stands above, when $f(x)$ is continuous; in this form it is given by Denjoy for a continuous function in his interesting memoir, "Sur les nombres dérivés des fonctions continues," 1915, *Jour. de Math.*, Sér. 7, Vol. 1, pp. 105–240, *Application III*, p. 195. This appeared almost exactly at the time of the award of the Gamble Prize, for which, of course, the publication of my result had been delayed. After perusing M. Denjoy's memoir, I was unwilling to put the corresponding part of my prize essay into print as it stood; with some slight alterations it is appearing in the *Quarterly Journal of Mathematics*, and the present communication, in which M. Denjoy's results and my own are considerably extended, is a further result. I take this opportunity, however, to call attention to the importance of Denjoy's results; if we assume them, the theorems of the present paper can be deduced with extreme ease by the use of the *C*-property (see below, § 2).

† This theorem, given by Denjoy for a continuous function, *Application I*, p. 187, is an extension of the original theorem by Lusin (1911), *Recueil de la Société math. de Moscou*, Vol. XXVIII, 2, in Russian. Both Denjoy and I use a Lemma, introduced by Lusin in this connexion. For a function of bounded variation this theorem is, of course, included in Lebesgue's theorem quoted in the following footnote.

‡ The original of this theorem would seem to be Lebesgue's theorem that *a continuous function of bounded variation has summable derivates, which are accordingly finite almost everywhere, and possesses a finite differential coefficient, except at a set of content zero* (1901), *Leçons sur l'intégration*, p. 128. W. H. Young removed the restriction as to continuity (1910), "On Functions of Bounded Variation" (1910), *Quarterly Journal*, Vol. 42, § 2, p. 79. Montel, on the other hand, retaining the continuity and the finitude of the derivates almost

In other words the four sets at which

(i) $f^+(x)$ and $f_-(x)$, (ii) $f^+(x)$ and $f_+(x)$, (iii) $f^-(x)$ and $f_-(x)$,

$f^-(x)$ and $f_+(x)$,

have finite unequal values, are all sets of zero content.

The three main theorems enable us to state for any measurable function which is finite everywhere the striking properties enunciated for a continuous function by Denjoy* :—

If we neglect a set of points whose content is certainly zero, the derivates of $f(x)$ at any point x belong to one of the three following cases :—

(I) *There is an ordinary finite differential coefficient $f'(x)$.*

(II) *The upper derivates on each side are $+\infty$ and the lower derivates on each side are $-\infty$.*

(III) *The upper derivate on one side is $+\infty$, the lower derivate on the other side is $-\infty$, and the two remaining extreme derivates are finite and equal.*

Each of these cases may occur at points of a set of any content, as is shown by examples, and all three cases may present themselves simultaneously for a single function $f(x)$, or one or more of the cases may be altogether absent.

From these three main theorems we can draw a number of interesting corollaries, of which the following may be specially mentioned :—

THEOREM 4.—*There is no geometrical distinction of right and left with*

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everywhere, removed the restriction as to bounded variation (1913), *Comptes Rendus*. More generally he enunciates the theorem that the content of the set of points at which all the derivates are finite is the same as that of the set of points where a finite differential coefficient exists (it being tacitly assumed that the function is continuous). Montel's proof consists in an adaptation of a method given by my husband and myself in these *Proceedings*, Ser. 2, Vol. 9, p. 331, for the proof of Lebesgue's theorem. It will be seen that it is this same form of proof, amplified but not altered, which I have used in the present paper. Montel uses, in addition, a Lemma due to Denjoy. Finally, Denjoy, in his recent memoir, gives two theorems which are together equivalent to the above theorem when $f(x)$ is continuous (*Applications II and V*, pp. 188 and 192). It is hardly necessary to say that no previous writer except my husband, from whom the idea is taken, has contemplated other than ordinary derivates.

* Denjoy, *loc. cit.*, p. 105.

regard to the four extreme derivates $f^+(x)$, $f_+(x)$, $f^-(x)$, $f_-(x)$, *except at a set of content zero.**

Also the two following generalisations of Montel's theorems, which are immediate corollaries from Theorem 3 :—

If the derivates of $f(x)$ *are all finite, except possibly at a set of content zero, then* $f(x)$ *has a finite differential coefficient almost everywhere.*

The set of points at which the derivates of $f(x)$ *are finite is the same as the set of points at which the differential coefficient exists and is finite, exception being possibly made of a set of content zero.*

The proofs here given of the main theorems are closely modelled on that formerly given in these *Proceedings* by my husband and myself for the case of a monotone function $f(x)$; they do not involve integration, nor transfinite processes, nor Cantor's numbers.

In conclusion I relax the condition (*b*) laid on our measurable function, and only assume that

(*c*) $f(x)$ *is finite at a set of points of positive content.*

The three main theorems then remain unaltered, provided we always abstract from consideration the points at which the primitive function is infinite.

If we have the condition that

(*c bis*) $f(x)$ *is finite at a set of points of positive content dense everywhere*

(so that the derivates are everywhere determinate), the Theorems 1 and 3, as well as Lusin's theorem, remain true as they stand, and the second theorem takes the following form :—

THEOREM 2 *bis.—The points at which* $f(x)$ *has an infinite right-hand (left-hand) differential coefficient consist of the infinities of* $f(x)$, *and possibly also a set of content zero.*

To obtain these results I require to elaborate slightly the work of Egoroff and Lusin, used in the earlier part of this paper.

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* In 1908 my husband published the theorem that *there is no distinction of right and left with respect to the derivates of a continuous function except at a set of the first category.* We suspected at that time that this set was of content zero, but failed to obtain a proof. Two discussions were given; both require the continuity of the primitive function. This set contains the above set of content zero, which is accordingly of the first category. See W. H. Young, "Oscillating Successions of Continuous Functions," 1908, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 805–7.

2. The generality of the present results is due to the application of Egoroff's theorem.* The enunciation of this important theorem is as follows :—

If a succession of measurable functions converges at all the points of an interval (a, b), *except possibly at a set of content zero, it is always possible to remove from the fundamental interval* (a, b) *a set of content as small as we please, and such that with respect to the complementary set, the succession converges uniformly.*

Hence it immediately follows that it is a necessary and sufficient condition that the limiting function $f(x)$ of a sequence of measurable functions should be finite at every point of the fundamental interval, except possibly at a set of content zero, that we should always be able to find a perfect set, whose content differs from that of the fundamental segment by less than e, with respect to which the sequence converges uniformly, e being any chosen positive quantity.

In consequence of Egoroff's theorem, as Lusin pointed out,† every measurable function which is finite, except possibly at a set of content zero, has what he calls the C-property; that is to say, we can remove a suitable set of points, of content as small as we please, from the fundamental segment, leaving over a complementary set, with respect to which the function is continuous. This property is indeed almost evident for measurable functions assuming only a finite number of values, whence it easily follows for all bounded measurable functions, and again for all measurable functions finite everywhere or almost everywhere, using the theorem, easily proved, that the limit of a sequence of functions possessing the C-property has itself the C-property.

If instead of a fundamental segment we are working only in a set of positive content, say greater than q, the C-property still holds for a measurable function, finite almost everywhere. For we only have to remove a set of content less than q from the continuum, leaving over a complementary set with respect to which our function is continuous, and this complementary set will be bound to have in common with the fundamental set a sub-set of positive content, with respect to which the function

* D. Egoroff, "Sur les suites de fonctions mesurables," 1911, *Comptes Rendus*, Vol. 153. The argument is reproduced in W. H. Young's paper, "On Successions whose Oscillation is usually Finite," 1912, *Quarterly Journal of Math.*, Vol. 44, pp. 131–133. The reader will observe an obvious clerical error of $+\infty$ for $-\infty$ in the enunciation and proof on p. 131, and in the enunciations of the three following theorems, see below, § 13.

† N. Lusin, *loc. cit.*, Summary in "Sur les propriétés des fonctions mesurables," 1912, t. 154, see below, § 17. Conversely a function which has the C-property is easily seen to be a measurable function finite everywhere or almost everywhere.

is continuous. It is by means of this property that I am able to apply to our very general function the reasoning which Lusin, Denjoy, and myself had used in the case of a continuous function.

3. An essential part of our mathematical machinery is a Lemma in the Theory of Sets of Points, introduced by Lusin.*

LUSIN'S LEMMA.—*If S is a set of positive content, it contains a perfect sub-set, dense nowhere,† and throughout of positive content, that is such that the part of it in every interval containing one of its points is of positive content.*

If the given set is not dense nowhere, we can change it into such a set by excluding from it all the points which are internal to a set of intervals, dense everywhere and of sufficiently small content. We shall therefore assume that the set S is nowhere dense, so that this condition as to the required sub-set is certainly fulfilled. If the given set is not closed, we may replace it by its nucleus. Thus we may assume that S is closed and dense nowhere.

Let the intervals whose end-points are the rational points be arranged in countable order $d_1, d_2, \ldots$, and consider all those of these intervals which contain points of S forming a sub-set of zero content; such intervals we shall call *zero intervals*. The content of the sub-set of S inside any finite or infinite number of the zero intervals is then zero, since the content is not greater than the sum of the contents in the separate zero intervals. Hence the sub-set S' of S, consisting of the points of S not internal to the zero intervals, is a sub-set of the same content as S. It is moreover clearly a closed sub-set, since none of its limiting points can be internal to the zero intervals, and they all belong to S, since S is closed.

Now the black intervals of S' are tiled over by the zero intervals, and therefore each contains only a sub-set of S of content zero. Moreover the end-points of these black intervals are not internal to zero intervals. Hence no two of these black intervals can abut, for, if so, any one of the intervals $d_1, d_2, \ldots$, one of whose end-points lay in one of the two abutting black intervals, and the other in the other, would be a zero interval, and

* Enunciated without proof in the paper quoted above on p. 361. He gave me a proof in MS. based on Lebesgue integration. D. Mirimanoff gave me another, on the lines of his elegant proof of the theorem of Cantor-Bendixson, "Sur quelques points de la théorie des ensembles," 1914, *Enseignement Math.*, Vol. 16, pp. 29, 30. Denjoy sketches a proof on p. 131 of his recent memoir. All the proofs are perfectly straightforward. The one here given is taken from my Gamble Prize Essay.

† In the present paper the property of being dense nowhere is not required.

would contain the common end-point, which, as we saw, is impossible. Thus the black intervals are non-abutting, and therefore S' is a perfect set.

Now, if P is any point of S', and d any interval containing P, there will be one of the intervals $d_1, d_2, \ldots$, containing P inside d, and this will not be a zero interval, and will therefore contain a sub-set of S of positive content. Thus also the part of S' in d has positive content. This proves that S' is throughout of positive content.

We have thus proved the Lemma.

4. The adaptation to our present purposes of the proof, to which reference has already been made, requires something to take the place of Dini's theorem that *the upper and lower bounds of any derivate are the same as those of the incrementary ratio, provided the function itself is continuous.* Generalisations of this theorem have lately been given in a communication to this Society,* but they are not sufficiently general for our present purpose. The Lemmas I use are fourfold. It will, however, only be necessary to enunciate one of them; the others are got by interchanging right and left.

LEMMAS (*vice* Dini's Theorem).—

I. *If S is a perfect set, with respect to which a function† $f(x)$ is continuous, and at each point x of which*

$$f^+ < A,$$

then we can find an interval (a, b), containing a part of the set S, such that

$$[f(x+h)-f(x)]/h = R(x, x+h) \leqslant A \quad (h > 0),$$

provided x and $x+h$ both lie in (a, b), and x is a point of the set S.

II. *If S is a perfect set, with respect to which a function $f(x)$ is continuous, and at each point x of which*

$$B < f_+(x),$$

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* W. H. Young, "On Integrals and Derivates with respect to a Function," 1914, *Proc. London Math. Soc.*, Ser. 2, Vol. 15, p. 43.

† Unrestricted in character. If we are forming the derivates of $f(x)$ with respect to a monotone function $g(x)$ without constant stretches, we must assume here that both $f(x)$ and $g(x)$ are continuous with respect to the set S. This means, at most, the avoidance of a countable set of points in assigning S, and does not therefore affect the efficiency of the Lemmas in their application.

then we can find an interval (a, b), *containing a part of the set* S, *such that*

$$[f(x)-f(x-h)]/h = R(x, x-h) \geqslant B \quad (h>0),$$

provided x *and* $x-h$ *both lie in* (a, b), *and* x *is a point of the set* S.

It will be sufficient to prove the former of these lemmas.

Suppose it is not true. Then taking any interval (a_1, b_1) of length $< e$, containing part of the set S, there must be inside it a pair of points x_1 and x_1+h_1, of which x_1 is a point of S, such that

$$R(x_1, x_1+h_1) > A, \quad (h_1>0). \tag{2}$$

Since $f(x)$ is continuous at the point x_1 with respect to the perfect set S, there is an interval d_1, with x_1 as one of its end-points (left or right, according as x is or is not a right-hand end-point of a black interval of the set S), containing part of the set S, and such that for each point x inside d_1,

$$R(x, x_1+h_1) > A, \tag{3}$$

d_1 being of length less than (x_1, x_1+h_1), and lying in (a_1, b_1), so that

$$0 < x_1-x < 2e. \tag{4}$$

Now in d_1 take an interval (a_2, b_2) of length less than $\frac{1}{2}e$, containing part of S, and repeat the argument, and so on. We get a sequence of intervals (a_1, b_1), (a_2, b_2), (a_3, b_3), ..., each contained in the preceding, and with lengths which diminish constantly down towards zero. There is therefore a single point X internal to all of them.

X must be a point of the perfect set S, since in the successive intervals (a_r, b_r) we always have a part of S. Also for each index r, we have a point x_r+h_r such that, by the relations corresponding to (3) and (4),

$$R(X, x_r+h_r) > A, \tag{5}$$

and

$$0 < x_r+h_r-X < 2^{-r+1}e. \tag{6}$$

That is we have a sequence of points on the right of X with X as limiting point, for which the incrementary ratio to the point X is greater than A. Hence, by the definition of the right-hand upper derivate,

$$f^+(x) \geqslant A.$$

But this is contrary to (1), therefore our supposition is untenable, and the statement first made is true.

5. We shall require one more lemma, but it is well known, and need not be proved here. We only recall the enunciation:—

YOUNG'S FIRST LEMMA.*—*If with each point of an interval as left-hand end-point we are given an interval, or several intervals forming a finite or infinite set, we can find a finite number of these, nowhere overlapping, and such that the sum of the complementary intervals is less than any pre-assigned positive quantity e.*

6. We now go on to the enunciation and proof of the first of our main theorems:—

THEOREM 1.—*If $f(x)$ is any finite measurable function, the points at which the upper right-hand derivate $f^+(x)$ has the value $+\infty$ are identical with the points at which the lower left-hand derivate has the value $-\infty$, exception being made, at most, of a set† of content zero.*

The points in question are none other than the points at which

$$f^+(x) = +\infty, \quad f_-(x) > k_i, \tag{1}$$

for all integers i, where $k_1 > k_2 > k_3 > \ldots$

is a monotone decreasing sequence of constants with $-\infty$ for limit.

Denoting the set of points at which (1) holds by G_i, we see that the set G_i contains G_{i-1}, for each integer i, and the set G, which we are going to consider, is the outer limiting set of the sets G_i. Therefore, it will be sufficient to prove that G_i always has the content zero, since if this is true for all the sets G_i, it is also true for the outer limiting set.

Now if we write $g(x) = f(x) - k_i x,$

we have $g^+(x) = f^+(x) - k_i, \quad g_-(x) = f_-(x) - k_i.$

Therefore the set G_i is also the set of points at which

$$g^+(x) = +\infty, \quad g_-(x) > 0.$$

Also the function $g(x)$ is, like $f(x)$, measurable and finite. Thus our problem is reduced to proving that, when k_i is zero, the content of G_i is zero. We proceed therefore to prove that *the points at which* $f^+(x) = +\infty$, $f_-(x) > 0$ *form a set of content zero.*

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* "On the Existence of a Differential Coefficient," *loc. cit.*, p. 327. Another proof is given in "The Reduction of Sets of Intervals," 1914, *Proc. London Math. Soc.*, Ser. 2, Vol. 14, p. 121.

† This set is of course measurable.

Assume, if possible, that these points form a set S_1 of positive content. Then since $f(x)$ has the C-property (§ 2), we can remove a sub-set of sufficiently small content, leaving over a complementary sub-set S_2, with respect to which $f(x)$ is continuous.

By Lusin's Lemma (§ 3), we can now find a perfect sub-set S_3 of S_2, which is throughout of positive content. At every point of S_2, we have

$$f_-(x) > 0,$$

by (2); hence, by one of the lemmas, *vice* Dini's theorem (§ 4), we can find an interval (a, b), containing a part E of S_2 (which is accordingly of positive content E), such that the incrementary ratio of $f(x)$ over any interval in (a, b) whose right-hand end-point belongs to the set E is $\geqslant 0$.

We may clearly choose a and b to be points of E, then $f(b)$ and $f(a)$ are finite. Now adjoin to each point x which is a left-hand end-point, or an internal point, of a black interval of the set E, the part r_x of that black interval on the right of the point x. For each such interval r_x then the incrementary ratio of $f(x)$ is $\geqslant 0$, and therefore also the increment of $f(x)$ is $\geqslant 0$.

To each of the remaining points of (a, b) other than b, we adjoin all the intervals $(x, x+h)$ to the right of the point x and in (a, b), such that

$$R(x, x+h) > Q, \tag{1}$$

where Q is any chosen positive quantity, as great as we please, say

$$Q > 2[f(b)-f(a)]/E. \tag{2}$$

Now we apply Young's First Lemma. We get a finite number of the intervals $(x, x+h)$ and r_x, nowhere overlapping, and such that the sum of the complementary intervals, say t_x, is less than e, where e is any chosen positive quantity, say $< \frac{1}{2}E$.

Since a and b, as well as the left-hand end-points of each of the chosen intervals, belong to the set E, it follows that the right-hand end-point of each t_x belongs to E; hence the increment of $f(x)$ over each t_x is > 0.

But the intervals t_x with the chosen intervals r_x and $(x, x+h)$ form a finite number of abutting intervals, reaching from a to b, therefore the sum of the increments of $f(x)$ over them is $f(b)-f(a)$. Omitting therefore the increments over the intervals t_x and r_x, as being $\geqslant 0$, we have, by (1),

$$\begin{aligned} f(b)-f(a) &\geqslant \Sigma\{f(x+h)-f(x)\} \\ &\geqslant \Sigma hR(x, x+h) > Q\Sigma h > Q(E-e) > \tfrac{1}{2}QE, \end{aligned} \tag{3}$$

since the sum of the intervals t_x is less than e, and the intervals r_x contain no point of E inside them, so that the intervals $(x, x+h)$ contain a sub-set of E of content greater than $E-e$, that is greater than $\frac{1}{2}E$.

But the last inequality (3) is in contradiction to (2), and is therefore impossible. Thus our assumption is untenable, which proves the theorem.

7. The second main theorem is an immediate result of the preceding in conjunction with my theorem from the *Acta*.

THEOREM 2.—*If $f(x)$ is a measurable function which is finite, except possibly at a set of content zero, the points, if any, at which it has an infinite right (left) hand differential coefficient (with determinate sign), form a set of content zero.*

It is evidently sufficient to prove at length that the points at which

$$f^+(x) = f_+(x) = f'_+(x) = +\infty \tag{1}$$

form a set G of content zero. The other parts of the theorem result at once by the exchange of right and left, upper and lower, $+\infty$ and $-\infty$.

Now, by my theorem in the *Acta*, except at a countable set S_2 of points of G,

$$f^-(x) = +\infty, \tag{2}$$

since

$$f_+(x) = +\infty.$$

Hence, by (2) and the preceding theorem, except at a set S_1 of content zero,

$$f_+(x) = -\infty;$$

but this latter relation is not true at *any* point of G, therefore G itself is a set of content zero, which proves the theorem.

8. We next proceed to prove the third theorem :—

THEOREM 3.—*If $f(x)$ is a finite measurable function, the points, if any, at which one of the upper derivates and one of the lower derivates are finite and different from one another, form a set of content zero.*

In virtue of my theorem from the *Acta* (§ 1), the proof is the same in all the different cases which may occur. In fact, if the upper derivate in question is not on the same side as the lower derivate, *e.g.*, if we are concerned with $f_-(x)$ and $f^+(x)$, we only have to remove a countable set of points, and at the remaining points we shall have

$$f_-(x) \leqslant f^+(x),$$

precisely as if the two derivates referred to the same side.

In the course of the proof we use the Lemmas *vice* Dini's Theorem, and it will, of course, be an appropriate pair of these Lemmas which have to be called into play.

We shall accordingly only give the proof in the case when the two derivates are both on the right.

Assume, if possible, that the set S of points at which $f^+(x)$ and $f_+(x)$ are finite and distinct has positive content.

Let S_r denote the sub-set of S at whose points $f^+(x)$ and $f_+(x)$ lie between $-r$ and r, both bounds excluded, r being any positive integer. Then each set S_r contains its predecessor S_{r-1}, and the outer limiting set is S itself. Therefore for some value of r the content of S_r must be positive, in order that the content of S, which is $\underset{r\to\infty}{\text{Lt}}\ S_r$, should be positive. Let r denote the least integer for which this is the case.

Now write $$f(x) = g(x) + rx,$$

then the derivates of $g(x)$ are got from the corresponding derivates of $f(x)$ by adding r; also $g(x)$ is a function of the same character as $f(x)$. Therefore, replacing $f(x)$ by $g(x)$, and then altering the nomenclature and denoting $g(x)$ by $f(x)$, we see that S_r is the set of points at which

$$0 < f_+(x) < f^+(x) < A, \qquad (\alpha)$$

where $A = 2r$.

Since $f(x)$ possesses the C-property (§ 2), and S_r is of positive content, we can remove a sub-set of sufficiently small content, leaving over a complementary sub-set S_r', of positive content, with respect to which $f(x)$ is continuous.

By Lusin's Lemma (§ 3), we can then find a perfect sub-set S_r'' of S_r', which is throughout of positive content.

Finally, since at each point of S_r'' we have

$$f^+(x) < A, \quad \text{and} \quad f_+(x) > 0,$$

we can, by the Lemmas *vice* Dini's Theorem (§ 4), choose a fundamental interval (a, b), so as to contain a part G of S_r'', and such that for each pair of points x and $x+h$ in it, we have

$$\left.\begin{aligned} R(x, x+h) &\leqslant A \quad (h>0) \\ 0 &\leqslant R(x, x+h) \quad (h>0) \end{aligned}\right\}, \qquad (\beta)$$

provided x belongs to the set G.

We may clearly assume a and b to be points of G; thus $f(b)$ and $f(a)$ are both finite.*

* By these preliminary reductions we have reduced the problem to precisely the same as in the case discussed in "The Existence of a Differential Coefficient," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 329, 330, when $f(x)$ is a monotone increasing function. The remainder of the proof may be taken, word for word, from this earlier paper, the tacit use of Dini's theorem on p. 330, line 14, being replaced by the lemmas of § 4 of the present memoir.

Let G_k denote the sub-set of G at which

$$f^+(x) - f_+(x) > k. \tag{1}$$

Then, if k assumes in succession the values A, $\frac{1}{2}A$, $\frac{1}{4}A$, ..., each set G_k is contained in the following, and the outer limiting set is G. Hence again, in order that our assumption should be tenable G_k must, for one of these values of k, have positive content. Let k then have the greatest such value.

Now the set G_k is itself the sum of the finite number of sets $H_{k,y}$, at which, besides (1), we have

$$\tfrac{1}{2}(y-1)\,k \leqslant f_+(x) < \tfrac{1}{2}yk, \tag{2}$$

y denoting any positive integer up to that for which $\frac{1}{2}(y-1)k = A$. Hence at least one of these sets must have positive content; let y have the least of the values for which this is true. Then there is a perfect sub-set of $H_{k,y}$ of positive content; let us call this E.

Let e be any chosen small positive quantity satisfying the inequality

$$\tfrac{1}{2}kE > e[(y+1)k + 2A]. \tag{3}$$

Now divide the fundamental interval (a, b) into a finite number of compartments, namely,

(i) *Black intervals of E, so chosen that the sum of the remaining black intervals is less than e; and*

(ii) *Complementary compartments*, whose sum is accordingly $\geqslant E$ and $\leqslant E+e$, since they contain the whole set E as well as these remaining black intervals. Let there be n of these compartments. We shall consider each of these separately.

To each point x which is an end-point of a black interval of E in the compartment considered, we adjoin that black interval as interval r_x. To each point x of the compartment considered, which is a left-hand end-point or internal point of a black interval of the set E, we adjoin the part r_x of that black interval on the right of the point x. For each such interval r_x then the incrementary ratio of $f(x)$ is $\geqslant 0$ and $\leqslant A$, since the right-hand end-point of r_x belongs to the set E.

At each of the remaining points of the compartment, since it is a point of E, (1) and (2) hold, therefore

$$f_+(x) < \tfrac{1}{2}yk, \tag{4}$$

and

$$\tfrac{1}{2}(y+1)\,k < f^+(x). \tag{5}$$

Therefore we can find intervals $(x, x+h_1)$ and $(x, x+h_2)$, on the right of x, and in our compartment, such that

$$f(x+h_1)-f(x) < \tfrac{1}{2}ykh_1, \tag{6}$$

and

$$\tfrac{1}{2}(y+1)kh_2 < f(x+h_2)-f(x). \tag{7}$$

By Young's First Lemma we can choose out a finite number of the intervals $(x, x+h_1)$ and r_x, nowhere overlapping, and such that the sum of the complementary intervals, say $t_{x,1}$, filling up the compartment, is less than e/n. Then the incrementary ratio of $f(x)$ over each r_x or $t_{x,1}$ is $\leqslant A$, since the right-hand end-point of each such interval belongs to the set E.

Let us do this in each compartment, then the sum of the chosen intervals r_x is, by our choice of the compartments, less than e, and so is the sum of the intervals $t_{x,1}$, since there are n compartments in which they lie, and in each the sum of these intervals is less than e/n.

Now let P_1, p_1, and P denote respectively the sum of the increments of $f(x)$ over the chosen intervals $(x, x+h_1)$, over all the chosen intervals r_x and the intervals $t_{x,1}$, and over the compartments (i). Then, since all these intervals together form a finite number of abutting intervals reaching from a to b,

$$f(b)-f(a) = P_1+p_1+P.$$

But, by (6),

$$P_1 < \tfrac{1}{2}yk(E+e),$$

since the content of the chosen intervals $(x, x+h_1)$ is less than that of the compartment (ii) in which they lie, that is less than $E+e$. Also

$$p_1 < 2Ae,$$

since, as we saw the incrementary ratio over each interval is $\leqslant A$, and the sum of the intervals is less than $2e$.

Hence

$$f(b)-f(a) < \tfrac{1}{2}yk(E+e)+2Ae+P. \tag{8}$$

Similarly, working with the intervals $(x, x+h_2)$, instead of $(x, x+h_1)$, and denoting the sum of the increments of $f(x)$ over the chosen intervals $(x, x+h_2)$, and over the chosen intervals r_x and the intervals $t_{x,2}$, respectively, by P_2 and p_2, we have

$$f(b)-f(a) = P_2+p_2+P.$$

But, by (7),

$$\tfrac{1}{2}(y+1)k(E-e) < P_2,$$

since these intervals contain all the points of E, except a sub-set of content $< e$.

Also $$0 \leqslant p_2,$$

since, the right-hand end-point of each of the intervals r_x and $t_{x,2}$ belonging to the set E, the incrementary ratio over each of these intervals is $\leqslant 0$. Hence $$\tfrac{1}{2}(y+1)\,k(E-e)+P < f(b)-f(a). \qquad (9)$$

Combining (8) and (9),

$$\tfrac{1}{2}(y+1)\,k(E-e) < \tfrac{1}{2}yk(E+e)+2Ae,$$

whence, *a fortiori*, $$\tfrac{1}{2}kE < e[(y+1)k+2A],$$

which is in contradiction to (3).

Thus our assumption is untenable.*

This proves the theorem.

9. If we now examine the cases which may occur at any point x with regard to the extreme derivates $f^+(x)$, $f_+(x)$, $f^-(x)$ and $f_-(x)$ of a measurable function, which is finite almost everywhere, we may distinguish five cases :—

(1) *All the four derivates are finite :* in this case either there is an ordinary finite differential coefficient, or else, by Theorem 3, the point x belongs to a certain set of content zero S_3 ; in the former case there is no distinction of right and left, since all four derivates are equal and finite.

(2) *One, and only one, of the derivates is infinite :* in this case, by Theorem 1, the point x belongs to a certain set S_1 of content zero.

(3) *Two, and only two, of the derivates are infinite :* in this case, unless the point x belongs to S_1, it must be the upper derivate on one side which is $+\infty$, and the lower derivate on the other side which is $-\infty$; the remaining two derivates being both finite, must be equal, unless the point x belongs to S_3 ; we notice that, if x does not belong to S_1 or S_3,

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* As an example of the simple verbal changes required in the other cases of the enunciation, suppose it is $f_-(x)$ and $f^+(x)$ that are to be considered. The relations (β) then become

$$R(x, x-h) \leqslant A, \quad R(x, x+h) > 0 \quad (h > 0).$$

In the construction of the set of intervals (6), however, we must change right into left, and the relation (6) becomes

$$f(x)-f(x-h) < \tfrac{1}{2}kh_1 \quad (h > 0).$$

Thus the intervals $t_{x,1}$, have their *left*-hand end-points belonging to the set S, and therefore, as before, it is the former of the two inequalities (β) which applies. We get the same inequality (8) as before, and the rest of the argument is unaltered.

there is again no distinction of right and left; in fact, the extreme derivates on the right are represented geometrically by the vertical through the point x, and a certain other direction, and those on the left by the vertical and the same other direction; on both sides the vertical is directed upwards or downwards, as the case may be, but this gives no geometrical distinction of right and left.

(4) *Three, and only three, of the derivates are infinite:* in this case the point x belongs to S_1 or a countable set S_2, by Theorems 1 and 2.

(5) *All four derivates are infinite:* in this case, if x does not belong to S_1 or S_2, the two upper derivates are $+\infty$, and the two lower derivates are $-\infty$; this latter case again presents no geometrical distinction of right and left.

10. Disregarding the set of content zero constituted by S_1, S_2, and S_3, we have, by the preceding article, only three possibilities at a point x:—

(1) *The upper derivates on each side are* $+\infty$, *and the lower derivates* $-\infty$, *that is,*

$$f^+(x) = f^-(x) = +\infty, \quad f_+(x) = f_-(x) = -\infty.$$

If $f(x)$ is a continuous function, every possible direction going out from the point x on the right or on the left, including the upward and downward verticals, will be the limiting directions of chords in the vicinity of the point x; such a point I call *an inextricable knot or tangle* on the locus $y = f(x)$.

(2) *The upper derivate on one side is* $+\infty$, *the lower derivate on the other side is* $-\infty$, *and the two remaining extreme derivates are finite and equal;* that is either

$$f^+(x) = +\infty, \quad f_-(x) = -\infty, \quad f_+(x) = f^-(x) = c,$$

or

$$f^-(x) = +\infty, \quad f_+(x) = -\infty, \quad f_-(x) = f^+(x) = c.$$

If $f(x)$ is a continuous function, the limiting directions will then on each side fill up the angle between the upward (downward) vertical and a certain other line; such a point I call *a partial knot or tangle* on the locus.

(3) *There is an ordinary finite differential coefficient:* such a point is *an ordinary point* on the locus.

11. We see then also that we have the following theorem as an imme-

diate corollary to our main theorems :—

Theorem 4.—*There is no geometrical distinction of right and left with regard to the four extreme derivates* $f^+(x)$, $f_+(x)$, $f^-(x)$, $f_-(x)$, *of a measurable function* $f(x)$, *finite almost everywhere, except possibly at a set of content zero.*

Indeed not only is there no distinction of right and left, as pointed out in § 8, except at the points of the sets S_1, S_2, and S_3, but at certain points of S_2 also there is no distinction of right and left ; these are the points at which both a right-hand and a left-hand differential coefficient exist, both infinite and opposite in sign. If $f(x)$ is continuous, such a point is *a cusp* on the locus $y = f(x)$, with vertical tangent pointing upwards or downwards ; this tangent is the unique limiting direction of chords from the point x to neighbouring points on the locus.

We notice that, when $f(x)$ is continuous, there is no distinction of right and left not only with respect to the extreme derivates, but also with regard to the intermediate derivates at an ordinary point, a cusp, or an inextricable knot, but that this is not the case at a partial knot.

12. In connexion with these general theorems I examined* anew the well-known continuous and non-differentiable function of Weierstrass

$$y = f(x) = \sum_{n=0}^{\infty} b^n \cos a^n x\pi,$$

where a is an odd integer, and b a positive quantity less than unity, these two parameters being connected by the inequality

$$ab > 1+3\pi/2.$$

Weierstrass had indeed shown that at *every* point $f(x)$ has one of its upper derivates $+\infty$ and the lower derivate on the other side $-\infty$. Thus a proper differential coefficient, whether finite, or infinite with determinate sign, cannot exist at any point.

By our general theory, however, it is possible that an infinite right-hand or left-hand differential coefficient might exist. Such a point might be a cusp, with vertical tangent, upward or downward, but, by my theorem in the *Acta*, such cusps can, at most, be countably infinite. The remaining points, by Theorem 2, can only form a set of content zero.

I proved that in point of fact such cusps do exist, and form a countably

* These results were communicated to the Société Helvétique on the occasion of its Centenary at Geneva last August. (See *L'Enseignement Math.*, Vol. xvii, p. 348.) The details of the work were incorporated in my Gamble Prize Essay.

infinite set dense everywhere; they are given by

$$x = Ma^{-m},$$

where M and m are any positive integers.

I showed, moreover, that at *every* point except at the cusps, *both* the upper derivates are $+\infty$, or *both* the lower derivates are $-\infty$. The set S_3 is therefore the null set.

Now in the case when $f(x)$ is a continuous function, W. H. Young* has proved that the points at which the two upper derivates are not equal to one another, and the two lower derivates not equal to one another, form a set of the first category. This set, being in Weierstrass's case, the set S_1+S_2, and in the general case of a continuous $f(x)$ the sets $S_1+S_2+S_3$, is therefore a set of the first category and content zero.

I showed that, in Weierstrass's case, this exceptional set is either identical with or contained in a certain set, which is easily proved to be of the first category and content zero. The points of this set are characterised by

$$x = j_0 \cdot j_1 \cdots j_{m-1} j_m j_{m+1} \ldots \textit{ad inf.}$$

when x is expressed in the scale whose base is the parameter a, when for some integer m, depending on x, all the digits $j_{m+1}, j_{m+2}, \ldots$ are even.

It appears from this investigation that, regarded as a curve $y = f(x)$, Weierstrass's curve is not really a curve without tangents, but has, in every interval an infinity of cusps with vertical tangents.

Cellérier, on the other hand, who found independently of Weierstrass, and possibly even before him,† a non-differentiable function, has defined for us *a curve entirely without tangents.* The equation to this curve is

$$y = f(x) = \sum_{n=1}^{\infty} a^{-n} \sin a^n x,$$

where a is a large positive even integer.‡

12. Now let us examine what modifications are necessary in our theorems, if we remove from $f(x)$ the conditions (a) and (b). We must

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* See the paper quoted above in § 1, and also "On the Derivates of Non-Differentiable Functions," 1908, *Messenger of Mathematics*, Vol. 38, pp. 65–69.

† C. Cellérier, "Note sur les principes fondamentales d'analyse" (date uncertain, Prof. Raoul Pictet of Geneva says, according to his memory of a conversation with Cellérier, it must have been as early as 1860), published after the author's death, *Bull. des Sc. Math.* (2), Tome 14 (1890).

‡ It will be seen that I had thus already solved the problem left to the reader by Denjoy on p. 210 of his memoir, and cleared up the point he leaves in doubt as to the possession of unilateral differential coefficients in the case of Weierstrass's function.

then, of course, revise the definitions of the derivates, since the expression for the incrementary ratio has no meaning when $f(x)$ and $f(x+h)$ are both infinite with the same sign. I shall therefore at once put aside all functions which are infinite at every point of any interval. Thus *the set of points at which $f(x)$ is finite is supposed to be dense everywhere.*

Considering then only such pairs of points x and $x+h$ for which the incrementary ratio $R(x, x+h)$ has a definite value, the derivates may be defined as before, and we shall still have everywhere

$$f_+(x) \leqslant f^+(x), \quad f_-(x) \leqslant f^-(x). \tag{1}$$

The discussion given by me in the *Acta* then only requires this modification that the points at which $f(x)$ is infinite form an exception. At such a point we have evidently

$$f(x) = +\infty, \quad f_+(x) = f^+(x) = -\infty, \quad f_-(x) = f^-(x) = +\infty, \tag{2}$$

or $$f(x) = -\infty, \quad f_+(x) = f^+(x) = +\infty, \quad f_-(x) = f^-(x) = -\infty.$$

We have accordingly the following extended form of my theorem in the *Acta*:—

THEOREM.—*The set of points at which the upper derivate on one side is less than the lower derivate on the other side contains the set of points at which the primitive function $f(x)$ is infinite; the remaining points, if any, form a countable set.*

Thus excepting the points considered in (2), and possibly also a countable set of points at which $f(x)$ is finite, we have

$$f_+(x) \leqslant f^-(x), \quad f_-(x) \leqslant f^+(x). \tag{3}$$

13. If we now examine the arguments of the present paper, it appears that, having so defined the derivates, our three main theorems remain unaltered, provided $f(x)$ is only infinite at a set of content zero. Thus, without any further ado, *we can change the condition (b) into the wider condition that $f(x)$ should be finite almost everywhere.*

14. Now let us examine what modifications are necessary in our three main theorems, if we further relax the condition (*b*) laid on our measurable function $f(x)$, and only demand that

(*c*) *$f(x)$ should be finite at a set of positive content.*

It appears on examination that we have nowhere in our reasoning used the fact that $f(x)$ is, except possibly at a set of content *precisely zero*,

finite, except in applying the C-property. If instead of the fundamental interval, we work with the fundamental set obtained by abstracting from the fundamental interval the points at which $f(x)$ is infinite, the former reasoning would still apply, provided the C-property still held in this fundamental set, which, by our condition (c), is a set of positive content.

It will be seen that this reasoning is perfectly valid, as our function does really possess the C-property in this fundamental set.

Thus it appears that our three main theorems remain true in this fundamental set.

15. To obtain these results it is necessary to adapt the work of Egoroff and Lusin referred to in § 2. We have first of all the following auxiliary theorem :—

THEOREM.—*If $f_1(x)$, $f_2(x)$, ... is a succession of finite functions,* which diverges properly to $-\infty$ at a set of points S (which may be the null-set), and $G_{n,e}$ denotes the set of points x at which the upper bound $\bar{R}_n(x)$ of*

$$f_m(x)-u(x) \quad (m \geqslant n),$$

is greater than e, then the set $G_{n,e}$ contains the set $G_{n+1,e}$, for all values of n, and the inner limiting set of the sets $G_{n,e}$ is the set S.

The set $G_{n,e}$ is, in other words, the set such that, corresponding to each of its points x, there is a value of p, such that

$$f_{n+p}(x)-u(x) > e. \tag{1}$$

Hence, if x is a point of $G_{n+1,e}$, there is a value of p, such that

$$f_{n+1+p}(x)-u(x) > e,$$

which shows that x is a point of $G_{n,e}$, and that the integer $p+1$ serves our purpose to prove this.

Now at a point of the set S, the left-hand side of (1) is always $+\infty$, so that the inequality is satisfied for all integers p. Thus the set S is certainly contained in all the sets $G_{n,e}$. It remains therefore only to show that every point common to all the sets $G_{n,e}$ is a point of S.

Let then k be any point of all the sets $G_{n,e}$. Then for a certain increasing succession of integers m, we must have

$$f_m(x)-u(x) > e. \tag{2}$$

Now let us proceed to the limit with m. By the definition of $u(x)$, the

* Not necessarily measurable.

upper limit of $f_m(x)$ cannot be greater than $u(x)$. Also, by (2), it cannot be less than $u(x)$. Finally, by (2), it cannot be equal to $u(x)$ and finite. There only remains the possibility that it is equal to $u(x)$ and infinite. But in this latter case the infinite value cannot be positive, or the left-hand side of (2) would always be negative, which is a contradiction. Hence the infinite value must be negative, which proves that x is a point of the set S.

This proves the theorem.

Similarly we have the alternative theorem :—

THEOREM.—*If $l(x)$ is the lower function of $f_1(x), f_2(x), \ldots$, a succession of finite functions, which diverges properly to $+\infty$ at a set of points Z (which may be the null-set), and $H_{n,e}$ denotes the set of points x at which the upper bound of*

$$l(x)-f_m(x) \quad (m \geqslant n),$$

is greater than e, then the set $H_{n,e}$ contains the set $H_{n+1,e}$, for all values of n, and the inner limiting set of the sets $H_{n,e}$ is the set Z.

16. We can now prove the extension of Egoroff's theorem :—

THEOREM.—*If a succession of finite measurable functions converges at a set T of positive content, we can find a sub-set of T of content as small as we please, such that, with respect to the complementary sub-set of T, the succession converges uniformly, this complementary sub-set being moreover perfect.*

Let $u(x)$ and $l(x)$ be the upper and lower functions of the succession, which are known to be equal and finite at the points of T, and let their common value at each such point be denoted by $f(x)$. Let S and Z denote, as before, the sets at which $u(x) = -\infty$ and $l(x) = +\infty$ respectively. Then S and Z have no common points, and neither of them has points common with T.

Let $G_{n,e}$ denote, as before, the set of points at which the upper bound $\bar{R}_n(x)$ of

$$f_m(x)-u(x) \quad (m \geqslant n),$$

is greater than e. Then, by the first theorem of the preceding article, S is a sub-set of $G_{n,e}$, and, if the remaining sub-set be denoted by $G'_{n,e}$, each set $G'_{n,e}$ contains the next $G'_{n+1,e}$, and there is no point common to all the sets $G'_{n,e}$.

Now since the functions $f_m(x)$ are measurable, so is $u(x)$, and therefore so is $\bar{R}_n(x)$; therefore the sets $G_{n,e}$ are measurable sets. Hence the

content* $G'_{n,e}$ has as limit, when n increases indefinitely, the content of the inner limiting set of the sets $G'_{n,e}$, that is zero.

Therefore if e assume in succession the values $e_1, e_2, \ldots$, forming a monotone descending sequence with zero as limit, we can, corresponding to each index i, determine an integer n_i, so that the content

$$G'_{n_i, e_i} < 2^{-k-i-1},$$

where k is as large a positive integer as we please.

Adding together all these sets G'_{n_i, e_i}, we get a set G, whose content† is not greater than the sum of the content of the composing sets G'_{n_i, e_i}, that is, less than 2^{-k}, which is as small as we please.

If x is any point not belonging to G' nor to S, it is not a point of G'_{n_i, e_i} nor of S, and is therefore not a point of G_{n_i, e_i}: that is

$$f_m(x) - u(x) \leqslant e_i \quad (m \geqslant n_i), \tag{i}$$

and this is true for all indices i.

Similarly, using the second theorem of the preceding article, we obtain a set H' of content as small as we please, such that, if x is any point not belonging to H' nor to Z,

$$l(x) - f_m(x) \leqslant e_i \quad (m \geqslant N_i), \tag{ii}$$

and this is true for all indices i.

Now if x is a point not belonging to any of the sets S, Z, G', and H', both the inequalities (i) and (ii) hold for all values of $m \geqslant m_i$, where m_i is the greater of n_i and N_i. Also if x is a point of the set T, the succession converges to $f(x)$. We have therefore

$$-e_i \leqslant f(x) - f_m(x) \leqslant e_i \quad (m \geqslant m_i), \tag{iii}$$

and this is true for all indices i. Thus the double inequality (iii) holds at all points of T which do not belong to the sets G' and H', of content as small as we please. Since T is by hypothesis of positive content, the double inequality (iii) therefore certainly holds at a sub-set of T of positive content, as near as we please to that of T.

Since this sub-set is of positive content, it contains a perfect sub-set (its nucleus) of the same content. Therefore the inequality (iii) holds at every point of this perfect sub-set.

But the double inequality (iii) expresses exactly the uniform conver-

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* *Qud* inner content.

† *Qud* outer content. This and the preceding footnote show why the attempt to prove this theorem for non-measurable functions fails.

gence of the succession to $f(x)$ with respect to this perfect sub-set. This therefore proves the theorem.

17. Hence we may state the extended form of the necessary and sufficient condition of Weyl-Egoroff that a succession should converge almost everywhere.

CONDITION.—*A necessary and sufficient condition that a succession of finite measurable functions should converge except possibly at a set of content k, less than that of the fundamental segment, is that it should be possible to find a set of content less than $k+e$, with respect to whose complementary set the succession converges uniformly, whatever sufficiently small positive quantity e may be.*

The necessity of the condition is indeed proved in the preceding theorem. To prove the sufficiency, suppose the condition fulfilled, and let $e_1 > e_2 > \ldots$ be any monotone descending sequence with zero as limit. Then we can find sets G_1, G_2, ..., respectively of content less than $k+e_1$, $k+e_2$, ..., with respect to whose complementary set the succession converges uniformly. If E denote the set of all the points not belonging to any of these sets, each point of E will belong to the complementary set of at least one of them, so that the succession will converge there. On the other hand, the content of the set complementary to E, which is a set contained in all the sets G_1, G_2, ..., is not greater than the content of each of these sets, and is therefore $< k+e_i$, for all indices i, that is, it is $\leqslant k$. This proves the sufficiency of the condition.

18. I denote by the (C, k)-property of a function $f(x)$ that we can remove from the fundamental segment a suitable set of content less than $k+e$, where e is any sufficiently small positive quantity, and with respect to the remaining set the function $f(x)$ is continuous.

If $f(x)$ *is a measurable function, and infinite at a set of content k*, less than the content of the fundamental segment, the set of its infinities must of course be included in the set removed, if $f(x)$ possesses the (C, k)-property. Thus for such a function the (C, k)-property is the same as the C-property in the fundamental set obtained by removing from the continuum the infinities of $f(x)$.

What we require, therefore, is simply to prove that *such a function $f(x)$ possesses the (C, k)-property.*

Let then $f(x)$ denote a measurable function, finite except possibly at a set S of content k, less than the length of the fundamental segment (a, b).

Let $f_i(x)$ denote the function which is equal to $f(x)$, wherever

$|f(x)| \leqslant i$, and has the value $+i$ or $-i$ elsewhere, the sign being the same as that of $f(x)$. Then the function $f_i(x)$ is certainly measurable, for with the exception of the values $+i$ and $-i$, the sets of points at which it has any value are the same as for $f(x)$, and the sets at which it has the value $+i$ or $-i$ are measurable, since they are the sets at which $f(x)$ is $\geqslant +i$ and $\leqslant -i$ respectively.

Also giving i all positive integral values, we get a sequence of bounded measurable functions converging or diverging to $f(x)$ everywhere. Hence we may apply the extension of Egoroff's theorem given above (§ 16), T being now the set of points where $f(x)$ is finite. We can, therefore, find a sub-set T_e of T of content less than $\frac{1}{2}e$, where e is as small as we please, such that with respect to the remaining sub-set of T the sequence converges uniformly, this remaining set, say U, being moreover perfect.

If therefore we show (as Lusin has already done), that a bounded measurable function has the C-property, that is the $(C, 0)$-property, it follows at once that our function $f(x)$ has the (C, k)-property, for it will be continuous with respect to a sub-set U' of U, of content as near as we please to that of U, with respect to which each of the functions $f_i(x)$ is continuous. Indeed we only have to find a set S_i of content less than $2^{-i-1}e$ corresponding to $f_i(x)$, in accordance with the C-property, and form the set of all the sets S_i, for all indices i, and denote the sub-set of U containing no point of this latter set by U'.

19. It remains therefore only to give Lusin's proof of the fact that any bounded measurable function possesses the C-property.

Now, as is well known, if $f(x)$ is a bounded measurable function, it is the limit of a monotone ascending sequence of measurable functions, each having only a finite number of values. For instance, if K be the lower and K' the upper bound of $f(x)$, and we write

$$e_n = (K-K')/n,$$

and

$$k_r = K+re_n \quad [r = 0, 1, \ldots, (n-1)],$$

we may define the constituent $f_n(x)$ of the required sequence as follows:—

Wherever $k_{n-1} \leqslant f < K'$, we have $f_n = k_{n-1}$,

$k_{n-2} \leqslant f < k_{n-1}$, ,, $f_n = k_{n-2}$,

and so on, finally, where

$K \leqslant f < k_1$, ,, $f_n = K$.

It is unnecessary to discuss this further, as it is familiar.

Hence, again, as before, it is only necessary to prove that such constituent functions possess the C-property.

But a measurable function which has only a finite number of values is evidently the sum of a finite number of functions each having only the values 0 and 1, each multiplied by a suitable constant. Hence the property only requires to be proved for such a simple function.

Now such a function is 1 at a measurable set S_1, and 0 at the complementary measurable set S_0. In S_1 take a closed set S_1' of content as near as we please to that of S_1, and in S_0 a closed set S_0', of content as near as we please to that of S_0. Then with respect to the set $S_1'+S_0'$ our function is continuous, which proves that it has the C-property.

This completes the proof of the assertion that a measurable function $f(x)$, which is infinite at a set of content k, less than that of the fundamental segment, possesses the (C, k)-property, and therefore possesses the C-property in the set got by removing from the fundamental segment all the infinities of $f(x)$.

20. Using the (C, k)-property, therefore, in place of the C-property, we get, without change in the reasoning, extensions of our three main theorems, in which the points referred to are not points at which $f(x)$ is infinite.

As, however, at a point at which $f(x) = +\infty$ $(h > 0)$,

$$[f(x+h)-f(x)]/h = -\infty,$$

and

$$[f(x)-f(x-h)]/h = +\infty$$

$[f(x+h)$ being different from $+\infty]$, it follows that, supposing $f(x)$ not to be infinite throughout any interval, at a point where $f(x) = +\infty$, we have the right-hand differential coefficient $-\infty$, and the left-hand differential coefficient $+\infty$. At a point where $f(x) = -\infty$, the signs of these infinities are, of course, reversed. Thus such points do not affect the results in Theorem 1, Theorem 3, and Lusin's Theorem, which remain true for our more general function. Theorem 2, however, takes the following form.

THEOREM 2, *bis.*—*The points at which $f(x)$ has an infinite right-hand (left-hand) differential coefficient consist of the infinities of $f(x)$ and possibly of an additional set of content zero.*

THE CREMONA TRANSFORMATIONS OF A CERTAIN PLANE SEXTIC

By HILDA P. HUDSON.

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I. *Degenerate Homaloids.*

A plane Cremona transformation of degree n transforms the straight lines of the first plane into a homaloidal family in the second plane; that is, into a doubly infinite family of rational curves of degree n, such that two general curves of the family meet in one and only one variable point of intersection. The remaining intersections are all accounted for by the base points common to the whole family. The straight lines of the second plane are transformed into a homaloidal family in the first plane, also with a set of base points.

To every point p in the first plane there corresponds a unique point P in the second plane: except that to a base point a there corresponds not a single point but a curve J. The set of curves J corresponding to the whole set of base points, taken with proper multiplicities, constitute the Jacobian of the second homaloidal family.

To a straight line l passing through a there corresponds a homaloid which breaks up into J, corresponding to the single point a, and a rational curve ϕ of degree less than n, corresponding to all the rest of l. Then J meets ϕ in one point other than base points; J is exactly determined by its passages through base points, and ϕ has one degree of freedom, corresponding to the one degree of freedom of the straight line l through a. We have thus a singly infinite family (ϕ) of rational curves, which are transformed into straight lines by a Cremona transformation of degree higher than that of ϕ.

If l joins the base points a_1, a_2, the corresponding homaloid breaks up into J_1, J_2, ϕ, where J_1, J_2 are Jacobian curves, and ϕ corresponds properly to l. Each of J_1, J_2 meets ϕ in one point other than base points, and ϕ is exactly determined by its passages through base points. If three or more of the base points a lie on a straight line l, to this there corresponds

properly a rational curve ϕ, which is more than determined by its passages through base points, which means that the conditions presented to it, by the required multiplicities at the base points, are not independent: but are related to one another in the same way as are the conditions presented to the straight line l by the collinear points a.

Now, if we are given the rational curve ϕ, determined or over-determined by the conditions of having certain base points of assigned multiplicities, the question arises whether we can always regard it as the effective part of a degenerate homaloid of higher degree. The answer is negative, and in the third part of this paper an example is given of a rational curve which cannot be transformed into a straight line by a Cremona transformation of however high degree.

Assume that ϕ is the effective part of a degenerate homaloid of degree $n+\nu$, the remaining part consisting of an aggregate J_ν of degree ν of k curves J belonging to the Jacobian of the homaloidal family. Then J_ν has no multiple points except the base points of the family, and no two curves J intersect elsewhere; each J meets ϕ at one other point. Let Σ extend to all the base points of the family, including all the assigned multiple points of ϕ and possibly other points. Let ϕ, J_ν have r, ρ branches respectively through a specimen base point; for the additional points introduced into Σ, the value of r is 0 or 1. There are a series of geometrical facts expressed by equations between these numbers.

ϕ is rational: $\quad \Sigma \tfrac{1}{2} r(r-1) = \tfrac{1}{2}(n-1)(n-2).$

ϕ can be transformed into a straight line l passing through the k base points in the other plane which correspond to the k curves J; if all the conditions, presented to ϕ by its assigned multiple points, were independent, it would have $2-k$ degrees of freedom. We shall say that the *apparent freedom* is $2-k$; if $k>2$, the conditions cannot be independent:

$$\Sigma \tfrac{1}{2} r(r+1) = \tfrac{1}{2} n(n+3) - 2 + k.$$

The multiplicity $r+\rho$ of $\phi . J_\nu$ at each base point is the same as that of a homaloidal family of degree $n+\nu$; this family is rational:

$$\Sigma \tfrac{1}{2}(r+\rho)(r+\rho-1) = \tfrac{1}{2}(n+\nu-1)(n+\nu-2),$$

and has two degrees of freedom:

$$\Sigma \tfrac{1}{2}(r+\rho)(r+\rho+1) = \tfrac{1}{2}(n+\nu)(n+\nu+3) - 2.$$

J_ν is an aggregate of k rational curves, and is therefore of genus

$-(k-1)$; all its singularities occur in Σ:

$$\Sigma \tfrac{1}{2}\rho(\rho-1) = \tfrac{1}{2}(\nu-1)(\nu-2)-1+k;$$

and J_ν is exactly determined:

$$\Sigma \tfrac{1}{2}\rho(\rho+1) = \tfrac{1}{2}\nu(\nu+3).$$

All but k of the $n\nu$ intersections of ϕ, J_ν fall at base points:

$$\Sigma r\rho = n\nu-k.$$

These are equivalent to five independent equations:

$$\Sigma r^2 = n^2-1+k, \qquad \Sigma \rho^2 = \nu^2+k,$$

$$\Sigma r = 3n-3+k, \qquad \Sigma \rho = 3\nu-k,$$

$$\Sigma r\rho = n\nu-k.$$

Now in the case studied below, of a sextic with ten double points,

$$n = 6, \quad r = 2, 1 \text{ or } 0, \quad \Sigma r\rho \leqslant 2\Sigma\rho,$$

$$6\nu-k \leqslant 2(3\nu-k), \qquad k \leqslant 0,$$

which is impossible if k is positive. It will be shewn that this curve actually exists; and it cannot be transformed into a straight line by any plane Cremona transformation whatever.

It can, of course, be mapped on a straight line by a Riemann transformation; for example, a quartic curve of the singly infinite family, determined by the ten double points and three fixed simple points of the sextic, meets the sextic in one variable point P, whose coordinates are therefore rational functions of the parameter of the family, which may be taken as the coordinate, on a straight line, of a variable point P' corresponding to P.

The sextic can also be transformed into a curve of lower degree by a Cremona space transformation; the sextic is the projection of a twisted sextic with one actual double point, which is the residual intersection of a sextic cone with ten double edges, standing on the plane curve, and a cubic monoid, with the same vertex as the cone, passing through nine of these edges. The twisted sextic is projected from its double point into a plane quartic. This succession of projections can be brought about by a single Cremona space transformation, which does not, however, transform the rest of the plane of the sextic into the plane of the quartic.

II. *Extension of Noether's Theorem.*

Noether* has proved that every Cremona transformation is the result of compounding a series of quadratic transformations; because the sum of the three highest multiplicities of a homaloidal family at base points is always greater than the degree of the family, and a quadratic transformation, with these three points as fundamental triad, lowers the degree of the family. So instead of seeking a Cremona transformation of higher degree, which transforms ϕ into a straight line, we may seek a quadratic transformation which lowers the degree of the curve.

Noether's proof rests on a manipulation of the equations which express that a homaloidal family is rational:

$$\Sigma \tfrac{1}{2} r(r-1) = \tfrac{1}{2}(n-1)(n-2),$$

and that it has two degrees of freedom:

$$\Sigma \tfrac{1}{2} r(r+1) = \tfrac{1}{2} n(n+3)-2,$$

or the equivalent pair

$$\Sigma r^2 = n^2-1, \qquad \Sigma r = 3n-3,$$

where n is the degree of the family, Σ extends to all its base points, and r is the multiplicity of one of them.

Assume that there are at least three multiple base points; the other cases are trivial, and require separate treatment. Let x, y, z be the three highest values of r, and let Σ' denote a summation from which these three points are excluded. Let

$$x = z+\lambda, \qquad y = z+\mu;$$

then

$$\lambda \geqslant \mu \geqslant 0, \qquad z \geqslant r, \qquad z > 1,$$

$$\Sigma' r^2 = n^2-1-x^2-y^2-z^2,$$

$$\Sigma' r = 3n-3-x-y-z.$$

Multiplying the second of these by z and subtracting the first from it,

$$\Sigma' r(z-r) = z(3n-3-x-y-z)-(n^2-1-x^2-y^2-z^2),$$

which can be put in the form, obtained by Prof. Baker in a more general

* *Math. Annalen*, Bd. 3, p. 167; Bd. 5, p. 635.

case,

$$(x+y+z-n)(n+\lambda+\mu) = \Sigma' r(z-r)+2z(\lambda+\mu)+2\lambda\mu+3(z-1)+2.$$

Now $n+\lambda+\mu$, $z-1$, 2 are positive; r, $z-r$, λ, μ are positive or zero; the right-hand side of the last equation is positive, and

$$x+y+z > n.$$

We generalize this as follows: for any curve ϕ of degree n having assigned multiplicities r at the points included in Σ, let

$$\Sigma \tfrac{1}{2} r(r-1) = \tfrac{1}{2}(n-1)(n-2)-p,$$

$$\Sigma \tfrac{1}{2} r(r+1) = \tfrac{1}{2} n(n+3)-f,$$

whence

$$\Sigma r^2 = n^2+1-p-f = n^2-i \text{ say},$$

$$\Sigma r = 3n-1+p-f.$$

Then p is the genus of ϕ; it cannot be negative if ϕ is a proper curve; f is the apparent freedom of the family having the same multiple points as ϕ; if the conditions which these points present are not independent, f can be negative. If there are at least two curves of the family, $i = f+p-1$ is the number of their intersections which do not fall at base points; it can be negative if there is only one curve ϕ having the assigned multiple points.

As before, let x, y, z be the three highest values of r, and let these three points be excluded from Σ'. We find the relation

$$(x+y+z-n)(n+\lambda+\mu)$$
$$= \Sigma' r(z-r)+2z(\lambda+\mu)+2\lambda\mu+(f-p+1)(z-1)+2-2p. \quad \text{(I)}$$

The conclusion $x+y+z > n$ follows in a great variety of cases besides that of the homaloidal family, $p = 0$, $f = 2$, just considered. It applies to:

any family of curves of genus 1, $p = 1$, $f \geqslant 1$;

any family of rational curves, $p = 0$, $f = 1$ or 2;

any rational curve exactly determined, $p = 0$, $f = 0$;

any rational curve for which one of the conditions is a consequence of the others, so that the apparent freedom is -1,

$$p = 0, f = -1.$$

It also applies, but not so obviously, when $p=0$, $f=-2$. The right-hand side of (I) is then

$$\Sigma' r(z-r)+z(2\lambda+2\mu-1)+2\lambda\mu+3,$$

and is certainly positive unless $\lambda=\mu=0$, $x=y=z$. Now Σ' extends to multiple points for which $z\geqslant r\geqslant 2$. The terms for which $r=z$ are each zero; let β be the number of effective terms, for which $r<z$; then $\beta\geqslant 0$, and since

$$r(z-r)=(r-1)(z-r-1)+(z-1),$$

therefore $$\Sigma' r(z-r)\geqslant \beta(z-1),$$

and the right-hand side of (I)

$$\geqslant(\beta-1)(z-1)+2,$$

and is positive unless $\beta=0$, $z\geqslant 3$. Then $r=z=x=y$, and all the multiple points are of the same order z; let their number be a; the original equations give

$$az^2=n^2+3, \qquad az=3n+1;$$

therefore $$z=\frac{n^2+3}{3n+1}, \qquad z-3=\frac{n(n-9)}{3n+1}.$$

Since n and $3n+1$ have no common factor, $3n+1$ is a factor of $|n-9|$, which could only be if either

(i) $n=1$ or 2, $\quad z=1$,

which is impossible since the base points are multiple, or

(ii) $n=9$, $\quad z=3$,

but then a is not integral.

If $p=0$, $f=-3$, the right-hand side of (I) is

$$\Sigma' r(z-r)+2z(\lambda+\mu-1)+2\lambda\mu+4;$$

if this is not positive, $\lambda=\mu=0$, and it becomes

$$\Sigma' r(z-r)-2z+4\geqslant(\beta-2)(z-1)+2,$$

which is positive unless either

(i) $\beta=0$, $\quad x=y=z=r\geqslant 2$,

or (ii) $\beta=1$, $\quad x=y=z\quad\geqslant 3$.

(i) If $\beta = 0$, as before,

$$z = \frac{n^2+4}{3n+2}, \qquad z-2 = \frac{n(n-6)}{3n+2},$$

and since n, $3n+2$ can only have the factor 2 in common, $3n+2$ is a factor of $2\,|\,n-6\,|$. This can only be if $n = 1$, 2 (which do not apply) or 6; hence $n = 6$, $z = 2$, $a = 10$, and we are led again to the sextic with ten double points.

(ii) If $\beta = 1$, there are $a-1\,(\geqslant 3)$ base points of multiplicity $z\,(\geqslant 3)$, and one of lower multiplicity $r\,(\geqslant 2)$, then

$$(a-1)\,z^2+r^2 = n^2+4 \geqslant 31, \qquad n \geqslant 6, \qquad r^2 \leqslant n^2-23,$$

$$(a-1)\,z + r = 3n+2;$$

therefore $$z = \frac{n^2+4-r^2}{3n+2-r}.$$

If $\left.\begin{array}{rl} n = 6, & r = 2 \text{ or } 3 \\ 7, & 2 \ldots 5 \\ 8, & 2 \ldots 6 \\ 9, & 2 \ldots 7 \\ 10, & 2 \ldots 8 \end{array}\right\}$, and the only case in which z is an integer above 2 is

$$n = 9,\ r = 2,\ z = 3,\ a = 10.$$

If $f = -3$, then whatever the genus,

$$\Sigma \tfrac{1}{2} r(r+1) = \tfrac{1}{2} n(n+3)+3$$

$$= 3a_2+6a_3+10a_4+15a_5+21a_6+28a_7+\ldots,$$

where a_r is the number of base points of multiplicity r. If, as before, $x+y+z \leqslant n$, then $n \geqslant 6$, since $z \geqslant 2$.

If $n = 6$, $x = y = z = 2$, $a_2 = 10$, the case already referred to.

If $n = 7$ or 8, $r \leqslant 7$, and casting out the threes, $a_4+a_7 \equiv 2 \pmod 3$; there are at least two fourfold or higher points, and $x+y+z > 8$.

Thus the rational sextic with ten double points is the simplest curve, as regards genus, degree and freedom, which cannot be transformed into a curve of lower degree by any plane Cremona transformation. The next curve is of degree 9 at least.

III. *The Sextic with Ten Double Points.*

The total intersection of a quadric and a cubic surface is a twisted sextic curve with six apparent double points. If the two surfaces touch

at four points the sextic has, in addition, four actual double points, and its projection from any point of space on to any plane is a plane sextic with ten double points. This way of generating the curve was suggested by Mr. Richmond.

Take homogeneous coordinates $x_1x_2x_3x_4$, the four points of contact $A_1A_2A_3A_4'$ being the corners of the tetrahedron of reference. The general quadric through these points is

$$Q \equiv a_1x_2x_3+a_2x_3x_1+a_3x_1x_2+a_1'x_1x_4+a_2'x_2x_4+a_3'x_3x_4 = 0, \qquad (1)$$

or, say, $2Q \equiv x_1u_1+x_2u_2+x_3u_3+x_4u_4,$

where $u_1 \equiv a_3x_2+a_2x_3+a_1'x_4, \ \ldots, \ u_4 = a_1'x_1+a_2'x_2+a_3'x_3,$

and $u_1 \ldots u_4$ are the tangent planes to Q at $A_1 \ldots A_4'$.

The general cubic surface touching the same planes at the same points is

$$C' \equiv c_1x_1^2u_1+c_2x_2^2u_2+c_3x_3^2u_3+c_4x_4^2u_4+b_1x_2x_3x_4+b_2x_1x_3x_4+b_3x_1x_2x_4 \\ +b_4x_1x_2x_3 = 0, \qquad (2)$$

and the total intersection is a twisted sextic S with four actual double points $A_1 \ldots A_4'$.

(i) S is in general a *proper* sextic. It also lies on the surface

$$C'-Q(c_1x_1+\ldots+c_4x_4) \equiv b_1'x_2x_3x_4+\ldots+b_4'x_1x_2x_3 = 0, \qquad (3)$$

where $b_1' \ldots b_4'$ are other constants, and its projection from A_4' on the plane $A_1A_2A_3$ is given by eliminating x_4 between (1) and (3). We find

$$(b_1'x_2x_3+b_2'x_3x_1+b_3'x_1x_2)(a_1x_2x_3+a_2x_3x_1+a_3x_1x_2)-b_4'x_1x_2x_3u_4 = 0,$$

which is a proper plane quartic if $a_1 \ldots b_4'$ have general values. Hence, if S broke up, it could only be into one or two straight lines through A_4' and a proper residual of degree 5 or 4; but by the same argument, the straight lines would have to pass through each of $A_1 \ldots A_4'$, which is impossible; therefore S does not break up.

(ii) S has *six apparent double points.* Take any two points $Y(y_1, y_2, y_3, y_4)$, Z, of space, and express the condition that the straight line YZ shall meet S in X; then X lies upon Q and C'. We may put $x = \lambda y+\mu z$ for all suffixes; let the result of substitution be

$$\lambda^2Q_{11}+\lambda\mu Q_{12}+\mu^2Q_{22} = 0,$$

$$\lambda^3C_{111}'+\lambda^2\mu C_{112}'+\lambda\mu^2C_{122}'+\mu^3C_{222}' = 0.$$

Eliminate $\lambda : \mu$; this gives the equation

$$F \equiv \begin{vmatrix} 0, & 0, & Q_{11}, & Q_{12}, & Q_{22} \\ 0, & Q_{11}, & Q_{12}, & Q_{22}, & 0 \\ Q_{11}, & Q_{12}, & Q_{22}, & 0, & 0 \\ 0, & C'_{111}, & C'_{112}, & C'_{122}, & C'_{222} \\ C'_{111}, & C'_{112}, & C'_{122}, & C'_{222}, & 0 \end{vmatrix} = 0,$$

of degree 6 in y, 6 in z. If Y is fixed, this equation in z represents the sextic cone projecting S from Y and *vice versa*.

If Y is a point of S, then $Q_{11} = C'_{111} = 0$: we can divide by μ before eliminating, and obtain

$$K' \equiv \begin{vmatrix} 0, & Q_{12}, & Q_{22} \\ Q_{12}, & Q_{22}, & 0 \\ C'_{112}, & C'_{122}, & C'_{222} \end{vmatrix} = 0,$$

of degree 2 in y, 5 in z. If Y is fixed, this equation in z represents the quintic cone projecting S from Y. If Z is fixed, this equation in y represents a quadric, meeting S in twelve points Y, which are such that each of the cones projecting S from these points passes through Z; in other words, such that each of the straight lines YZ meets S again, in another of the twelve points of intersection of the quadric $K'(y)$ with S. These twelve points therefore fall into six pairs, each collinear with Z; as viewed from the arbitrary point Z, the curve S has six apparent double points.

(iii) The six apparent double points of S are *in addition to* its four actual double points. If the coordinates (1, 0, 0, 0) of A_1 are substituted for y,

Q_{12} becomes $a_3z_2+a_2z_3+a'_1z_4 \equiv u_z$ say,

C'_{112} ,, c_1u_z,

C'_{122} ,, $2c_1z_1u_z+c_2a_3x_2^2+c_3a_2x_3^2+b_2z_3z_4+b_3z_2z_4+b_4z_2z_3$,

and K' breaks up into u_z and another factor, but does not vanish. Hence $K'(y)$, which contains the ends of the six chords of S through Z, does not contain the four actual double points, which are therefore in addition to the apparent double points.

(iv) The six chords of S through any point *lie on a quadric cone.* If we put $y = z$, Q_{12} becomes $2Q_{22}$, and C'_{112}, C'_{122} each become $3C'_{222}$; then

$$K' = Q_{22}^2 C'_{222} \begin{vmatrix} 0, & 2, & 1 \\ 2, & 1, & 0 \\ 3, & 3, & 1 \end{vmatrix} = -Q_{22}^2 C'_{222},$$

which does not vanish; $K'(y)$ does not in general pass through Z, and the six chords of S are not generators of $K'(y)$, unless Z lies either on Q or on C'.

This investigation holds if instead of C' we take $C \equiv C' + Q.P$, where P is any plane, for C is another cubic surface through S, and touches the same planes as C at $A_1 \ldots A'_4$. We have thus a whole family of quadrics (K) such as K' through the same twelve points of S, which is, in fact, the family $K' + \lambda Q$. In particular, if P is chosen so that C passes through Z, we have $C_{222} = 0$, and K vanishes when we put $y = z$. This quadric K passes through Z, meets each of the six chords of S through Z in three points, and therefore contains them altogether. Hence K is a quadric cone on which the six chords lie.

Since now $C_{222} = 0$, we have identically

$$K = Q_{22} \begin{vmatrix} Q_{12}, & Q_{22} \\ C_{112}, & C_{122} \end{vmatrix} = Q_{22}.K_2, \text{ say,}$$

$$F = Q_{22} \begin{vmatrix} 0, & Q_{11}, & Q_{12}, & Q_{22} \\ Q_{11}, & Q_{12}, & Q_{22}, & 0 \\ 0, & C_{111}, & C_{112}, & C_{122} \\ C_{111}, & C_{112}, & C_{122}, & 0 \end{vmatrix} = Q_2.F_6, \text{ say,}$$

where

$$\begin{aligned} F_6 &= K_2(Q_{12}C_{111} - Q_{11}C_{112}) + (Q_{11}C_{122} - Q_{22}C_{111})^2 \\ &= K_2.K_4 + K_3^2, \text{ say,} \end{aligned}$$

the suffixes shewing the degrees in y. Here F_6, K_2 are cones containing the six chords of S through Z, which are double edges on F_6; this identity shews that they also lie on the cubic surface K_3. They are therefore double on K_3^2 as well as on F_6; therefore, by the same identity, they are double on $K_2.K_4$. Since they are simple but not double edges on K_2, they lie on K_4.

The four actual double points $A_1 \ldots A'_4$ of S are double points on F_6; they lie on Q, C, and their coordinates make Q_{11}, C_{111} vanish; they therefore lie on K_3, K_4, but not on K_2. They are double on $F_6 - K_3^2$, and therefore double on $K_2.K_4$; since they do not lie on K_2, they are double on K_4.

The point Z lies on C but not on Q. It is six-fold on F_6, and double on K_2; the substitution of z for y make C_{111}, C_{112}, C_{222} all vanish, so Z lies on K_3 and K_4. It is therefore triple at least on $F_6 - K_2.K_4$, and therefore triple on K_3^2; this can only be if it is double on K_3. It is then fourfold on $F_6 - K_3^2$, and therefore fourfold on $K_2.K_4$; since it is double on K_2, it is double on K_4.

These multiplicities can be set out in a table:

Surface.	The Six Chords.	$A_1 \dots A_4'$	Z
F_6	2	2	6
K_2	1	0	2
K_3	1	1	2
K_4	1	2	2

Take the sections of these four surfaces by the plane $A_1A_2A_3$, by putting $y_4 = 0$. Let the six chords meet the plane in $B_1 \dots B_6$, and let $A_4'Z$ meet it in A_4. The four plane curves are:

a sextic f_6 with ten double points at A_1, A_2, A_3; A_4; $B_1 \dots B_6$; this will be indicated by $f_6(A_1 \dots A_4 B_1 \dots B_6)^2$;

a conic $k_2(B_1 \dots B_6)^1$;

a cubic $k_3(A_1A_2A_3B_1 \dots B_6)^1$;

a quartic $k_4(A_1A_2A_3)^2 (B_1 \dots B_6)^1$;

with the relation $$f_6 \equiv k_2.k_4 + k_3^2.$$

The existence of f_6 presents three conditions to the ten points $A_1 \dots B_6$; k_2 gives a single condition, among $B_1 \dots B_6$: we shall call it a *condition curve*; k_3 gives no condition, it is exactly determined by its nine simple points; k_4 is another condition curve, for it is exactly determined by its three double points $A_1A_2A_3$ and five simple points $B_1 \dots B_5$: there is therefore one condition that B_6 should lie on this quartic.

There exists another cubic k_3' determined by passing through the nine points $A_2A_3A_4B_1 \dots B_6$. Then $f_6 + \lambda k_3'^2$ is a family of sextics $k_6(A_2A_3A_4B_1 \dots B_6)^2$, and each meets k_2 in $B_1 \dots B_6$ counted twice and no other point. If we use λ to make k_6 pass through another point of k_2, it must break up into k_2, which passes through $B_1 \dots B_6$ once and not through $A_2A_3A_4$, and a quartic $k_4'(A_2A_3A_4)^2 (B_1 \dots B_6)^1$. Similarly there exist two other condition quartics $k_4''(A_1A_3A_4)^2 (B_1 \dots B_6)^1$ and $k_4'''(A_1A_2A_4)^2 (B_1 \dots B_6)^1$: the point A_4 is on exactly the same footing as $A_1A_2A_3$.

By employing other exactly determined curves instead of these cubics, we can prove that there exist an unlimited number of condition curves of higher degree determined by the ten double points of the sextic: see below. Each of these curves, whose existence gives a single condition, is of apparent freedom -1.

To obtain the actual equation of the plane sextic, put $y_4 = 0$ in the determinant F_6. In order to simplify the expressions, take the coordinates of Z to be (1, 1, 1, 1); there is no loss of generality, as Q, C have no metrical properties. Also when we replace C' by $C \equiv C' + Q.P$, we can use the four coefficients of P to make the new cubic surface not only pass through Z, but also have double points at $A_1A_2A_3$. The effect of this is the same as if in C' we assume

$$c_1 = c_2 = c_3 = 0, \quad c_4(a_1' + a_2' + a_3') + b_1 + b_2 + b_3 + b_4 = 0,$$

and the plane curve is

$$f_6 \equiv \begin{vmatrix} 0, & q_{11}, & q_{12}, & q_{22} \\ q_{11}, & q_{12}, & q_{22}, & 0 \\ 0, & c_{111}, & c_{112}, & c_{122} \\ c_{111}, & c_{112}, & c_{122}, & 0 \end{vmatrix} = 0,$$

where

$$q_{11} \equiv a_1y_2y_3 + a_2y_3y_1 + a_3y_1y_2,$$

$$q_{12} \equiv y_1(a_2 + a_3 + a_1') + y_2(a_3 + a_1 + a_2') + y_3(a_1 + a_2 + a_3'),$$

$$q_{22} \equiv a_1 + a_2 + a_3 + a_1' + a_2' + a_3',$$

$$c_{111} \equiv b_4y_1y_2y_3,$$

$$c_{112} \equiv y_2y_3(b_1 + b_4) + y_3y_1(b_2 + b_4) + y_1y_2(b_3 + b_4),$$

$$c_{122} \equiv y_1(c_4a_1' + b_2 + b_3 + b_4) + y_2(c_4a_2' + b_3 + b_1 + b_4) + y_3(c_4a_3' + b_1 + b_2 + b_4).$$

The equation of f_6 contains the eleven constants $a_1a_2a_3a_1'a_2'a_3'b_1b_2b_3b_4c_4$, subject to one condition; as the first ten enter homogeneously, there are nine independent parameters. In the plane, we have fixed the coordinate system by taking $A_1A_2A_3$ as the triangle of reference and assigning the coordinates (1, 1, 1) to A_4. The twelve coordinates of the remaining six double points $B_1 \ldots B_6$ are functions of the nine parameters, and are therefore connected by three independent relations. It is probable that the existence of three condition curves, for example k_2, k_4, k_4', ensures the existence of f_6; but I have no proof that this is so.

IV. *Transformations of the Sextic.*

A quadratic transformation, with any three of the ten double points A, B as fundamental triad, transforms f_6 into another plane sextic with ten double points, but these may be differently grouped. If the fundamental triad is $B_4B_5B_6$,

$f_6(A_1 \dots A_4 B_1 \dots B_6)^2$	becomes	$h_6(A_1 \dots A_4 B_1 \dots B_6)^2$,
$k_2(B_1 \dots B_6)^1$	,,	$h_1(B_1B_2B_3)^1$,
$k_3(A_1A_2A_3B_1 \dots B_6)^1$	,,	$h_3(A_1A_2A_3B_1 \dots B_6)^1$,
$k_4(A_1A_2A_3)^2(B_1 \dots B_6)^1$	,,	$h_5(A_1A_2A_3B_4B_5B_6)^2(B_1B_2B_3)^1$,

and for these new curves there is the identity $h_6 \equiv h_1h_5 + h_3^2$. Apparent freedom is not altered by the transformation; h_3 is exactly determined, and h_1, h_5 are condition curves. The ten double points of h_6 are grouped into three collinear points $B_1B_2B_3$, and seven others, instead of into six points on a conic, and four others.

There are seven cubics such as h_3, each exactly determined by passing through the three points of the first group and six out of the seven points of the second group; four of these cubics arise by transformation from the four curves such as k_3; the other three, such as $h_3'(A_1 \dots A_4B_1 \dots B_5)^1$, arise from the exactly determined quartics such as

$$k_4(B_4B_5)^2(A_1 \dots A_4B_1B_2B_3B_6)^1.$$

As above, a discussion of the family $h_6 + \lambda h_3^2$ proves that one of its members breaks up into h_1 and h_5; and so we can prove the existence of the four condition curves such as h_5; in just the same way, a discussion of $h_6 + \lambda h_3'^2$ proves the existence of three condition curves h_5', passing once through $B_1B_2B_3$, and twice through the remaining sets of six points out of the seven of the second group. $h_5'(A_1 \dots A_4B_4B_5)^2(B_1B_2B_3)^1$ arises from $k_6'(B_4B_5)^3(A_1 \dots A_4)^2(B_1B_2B_3B_6)^1$, which is a condition curve connected with the original sextic, and whose existence could have been proved independently. Since k_6' divides the six B's into two sets of two and four points, there are fifteen such curves; the remaining twelve give by transformation other condition curves, connected with the second type of sextic which has three collinear double points. These two principles, of transformation and of symmetry, lead to an infinite number of condition curves connected with either of the two types of sextic with ten double points.

Instead of the triad BBB in the auxiliary quadratic transformation, we can use a triad ABB, AAB, or AAA, and we are led to other types of

sextics, each with ten double points grouped in some way, and a series of condition curves connected with each. In particular, a transformation using $A_1A_2B_1$, followed by another using $A_3A_4B_1$ (equivalent to a single cubic transformation) transforms the original $k_2(B_1 \dots B_6)^1$ into a $k_4(B_1)^3(A_1 \dots A_4 B_2 \dots B_6)^1$, and transforms the original k_4's into other quartics with triple points; there is now no grouping of the ten double points, which are all on the same footing for this type of sextic.

If we apply quadratic transformations systematically, using every possible variety of triads, to every type of sextic that arises, we obtain five different types and no more, given by the following table.

Reference Number.	Grouping of Double Points.	Simplest Condition Curves.	Some other Condition Curves.
1	A_1A_2; $B_1 \dots B_8$	$k_0(A_1A_2)^1$	$k_6(A_1^3A_2)(B_1 \dots B_7)^2$ $k_8(B_1B_2)^4(A_1^3A_2)(B_3 \dots B_8)^2$
2	$A_1A_2A_3$; $B_1 \dots B_7$	$k_1(A_1A_2A_3)^1$	$k_5(B_1 \dots B_6)^2(A_1A_2A_3)^1$
3	$A_1 \dots A_4$; $B_1 \dots B_6$	$k_2(B_1 \dots B_6)^1$	$k_4(A_1A_2A_3)^2(B_1 \dots B_6)^1$ $k_6(B_1B_2)^3(A_1 \dots A_4)^2(B_3 \dots B_6)^1$ $k_8(A_1)^4(B_1 \dots B_4)^3(A_2A_3A_4)^2(B_5B_6)^1$ $k_{10}(B_1)^5(A_1A_2)^4(B_2 \dots B_5)^3(A_3A_4)^2(B_6)^1$
4	$A_1A_2A_3$; $B_1 \dots B_7$	$k_3(A_1)^2(B_1 \dots B_7)^1$ (three such)	$k_5(B_1)^3(A_1A_2A_3)^2(B_2 \dots B_7)^1$
5	$A_1 \dots A_{10}$	$k_4(A_1)^3(A_2 \dots A_{10})^1$ (ten such)	

In the first type, what is for the sake of uniformity called k_0, a curve of zero degree, is a single point. Two of the double points, A_1A_2, coincide, and the sextic has a tacnode there; we may think of A_2 as the point adjacent to A_1 along the tacnodal tangent. Each of the condition curves k_6, k_8 has three branches through A_1, one of which passes through A_2, that is, touches the tacnodal tangent.

The existence of any of the condition curves in the last column can be proved, if we assume the sextic and any other condition curve of the same row, by considering a suitable curve exactly determined, corresponding to k_3 in the original discussion, and obtained from it by some series of transformations. For example, in the original case (3), to prove the existence of k_6 we assume k_2 and consider the exactly determined quartic $k_4(B_1B_2)^2(A_1 \dots A_4B_3 \dots B_6)^1$. Then $f_6.[k_1(B_1B_2)^1]^2+\lambda k_4^2$ is a singly

THE LONDON MATHEMATICAL SOCIETY.

BALANCE SHEET, 31st October, 1916.

Liabilities.	£	s.	d.
Lieut.-Col. Campbell's Fund	499	7	5
Life Compositions Fund	2296	0	0
Invested Surplus Fund	700	0	0
Sundry Creditors	6	7	4
De Morgan Medal Fund	6	9	3
Revenue Account balance in hand	107	0	10
	£3615	4	10

Assets.	£	s.	d.
London & North Western Railway £385 Four per Cent. Guaranteed Stock at cost ...	499	7	5
New South Wales £2236. 8*s.* 6*d.* 3½ per Cent. Inscribed Stock 1918 at cost	2201	10	0
India Stock £700. 7*s.* 1*d.* 3 per Cent. at cost	700	0	0
War Stock £100 4½ per Cent. at cost ...	99	9	4
Cash at Bank	114	18	1
	£3615	4	10

REVENUE ACCOUNT for the year 1st November, 1915, to 31st October, 1916.

	Year 1914-15					
	£	s.	d.	£	s.	d.
Balance from last year	69	16	6	*53*	*13*	*6*
Income tax returned (3 years)		—		*30*	*7*	*8*
Dividends on—						
Lord Rayleigh's Fund ...	36	6	6	*38*	*2*	*2*
The Society's Investments	95	5	8	*101*	*15*	*6*
Entrance Fees	11	11	0	*7*	*7*	*0*
141 Annual Subscriptions	148	1	0	*153*	*6*	*0*
Sales of *Proceedings*, &c.	147	19	5	*119*	*12*	*0*
	£509	0	1	*504*	*3*	*10*

	Year 1914-15.					
	£	s.	d.	£	s.	d.
Printing and Circulating *Proceedings*, &c.	327	14	6	*330*	*15*	*3*
Purchases for Library	5	13	9	*19*	*11*	*9*
Rent	50	0	0	*50*	*0*	*0*
Attendance...	7	7	0	*8*	*8*	*0*
Teas at Meetings	1	11	6	*1*	*11*	*6*
Postages and Sundries	9	12	6	*9*	*0*	*10*
Fee for preparing Royal Society's Index		—		*15*	*0*	*0*
Balance in hand on 31st October, 1916...	107	0	10	*69*	*16*	*6*
	£509	0	1	*504*	*3*	*10*

REVENUE ACCOUNT of the DE MORGAN MEDAL FUND.

	£	s.	d.
Balance on 31st October, 1915	4	0	2
Dividends received	2	9	1
	£6	9	3

	£	s.	d.
Balance in hand 31st October, 1916	6	9	3
	£6	9	3

STATEMENT OF TRUST FUNDS, 31st October, 1916.

Lord Rayleigh's Fund consists of the following Investments:

Great Indian Peninsula Railway £50. 8*s.* 8*d.* Annuity, Class B.
Do. do. do. do. £200 Guaranteed 3 per Cent. Stock.

This Fund is held on trust to apply the income as part of the general income of the Society.

The *De Morgan Medal Fund* consists of Great Western Railway £60 Five per Cent. Preference Stock.

2nd November, 1916.

Signed on behalf of the Council:—

JOSEPH LARMOR,
A. E. WESTERN.

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

ALLAN CUNNINGHAM,
Lt.-Col. late R.E.,
Auditor.

21st November, 1916.

infinite family of octavics $k_8(B_1B_2)^4(A_1 \dots A_4B_3 \dots B_6)^2$, each meeting k_2 in the equivalent of sixteen points at $B_1 \dots B_6$, and in no other point. One member of the family therefore breaks up into k_2 and another condition curve $k_6(B_1B_2)^3(A_1 \dots A_4)^2(B_3 \dots B_6)^1$. All the properties of any one type follow from the existence of any one of its condition curves; if there exists a condition quartic with three double points, the type is (3), and so on.

In order to prove that a certain quadratic transformation changes a certain type into a certain other type, we only have to prove that it changes the simplest condition curve of the first type into some condition curve of the second.

Applied to type (1), a quadratic transformation using $B_1B_2B_3$ does not alter the type; using $A_1B_1B_2$, it gives type (2), from which (1) was originally derived. A specialized transformation, with two fundamental points coinciding at A_1A_2, does not alter the type.

Whatever the triad, the k_1 and k_2 of types (2) and (3), and suitable ones of the k_3's of type (4), and of the k_4's of type (5), are all transformed into rational curves of degree $\leqslant 4$. If the degree is 0, 1, 2, or 3, the type obtained is (1), (2), (3), or (4); while the two kinds of rational quartics, with three double points or one triple point, give types (3) and (5) respectively.

All these condition curves are rational; the only other curves of degree 6 or less, of any genus determined or over-determined by ten or fewer base points, are such that if we identified the base points with the double points of f_6, the number of intersections with f_6 would exceed the maximum, and f_6 would either break up or coincide with the condition curve; as, for example, $k_3(10)^1$ or $k_6(9)^2(1)^1$.

The curve $k_6(1)^3(7)^2(1)^1$ only appears in the particular case when the simple point is adjacent to the triple point, type (1). This can be proved directly: assume the existence of $f_6(AB_1 \dots B_7CD)^2$ and $k_6(A)^3(B_1 \dots B_7)^2(C)^1$. Then $f_6+\lambda[k_3(AB_1 \dots B_7C)^1]^2$ is a family of sextics $k_6'(AB_1 \dots B_7C)^2$, each meeting k in 36 points. If we use λ to make k' contain any other point of k, it coincides with it entirely. For this value of λ therefore,

$$k \equiv f_6+\lambda k_3^2 \equiv k'(AB_1 \dots B_7C)^2,$$

shewing that k has a double point at C and not merely a simple point as assumed. But k is already rational, and cannot have an additional double point; therefore C coincides with one of the other singularities, either A or B, where f_6 has a tacnode, two distinct branches touching the same tangent line l, and k_3 touches l also; while all the family k', including k, have two branches touching l at C. If C coincided with B, then k would

have a tacnode, which reduces the genus by two, so that k would break up; hence C coincides with A. Then k has a triple point with one branch touching l; the other two branches form a curve having a double point at A, which curve must be considered as touching any straight line through A, and therefore touching l; and k must be considered as having two branches through A touching l. But the triple point is not specialized, and does not reduce the genus by more than before.

The equation of the singly infinite family k' having a tacnode at $A(1, 0, 0)$, both branches touching $x_2 = 0$, has the form:

$$(a+b\lambda)\, x_1^4 x_2^2 + x_1^3 x_2 \{(c_1+d_1\lambda)\, x_2^2 + (c_2+d_2\lambda)\, x_2 x_3 + (c_3+d_3\lambda)\, x_3^2\}$$

terms of lower degree in $x_1 = 0$,

and k is the member of the family for which $\lambda = -a/b$, the first group of terms is absent, and k has an ordinary triple point, one branch touching x_2.

There probably exist other types of the sextic, which are such that no combination of the three conditions among the ten double points expresses the existence of a curve. But we can shew, generally, that if a condition curve exists, of any degree, genus or negative freedom, then the sextic is of one of the five types enumerated.

Let a proper curve k_ν of degree ν be over-determined by passing ρ_k times through the point A_k ($k = 1 \ldots 10$, $\rho \geqslant 0$); let its genus be p and its freedom f:

$$\Sigma \tfrac{1}{2}\rho(\rho-1) = \tfrac{1}{2}(\nu-1)(\nu-2) - p \quad (p \geqslant 0),$$
$$\Sigma \tfrac{1}{2}\rho(\rho+1) = \tfrac{1}{2}\nu(\nu+3) - f \quad (f \leqslant -1).$$

Double and subtract:

$$\Sigma\, 2\rho = 6\nu + 2\,(p-f-1).$$

If the A's are the double points of $f_6(10)^2$, then $\Sigma 2\rho$ is the number of intersections of k_ν, f_6 which fall at A's; if the curves do not break up nor coincide,

$$\Sigma\, 2\rho \leqslant 6\nu, \quad p-f-1 \leqslant 0,$$

which can only be so if

$$p = 0, \quad f = -1,$$

and k_ν is a rational curve of freedom -1; and as we saw in Part I, it can be transformed into a straight line $k_1\,(3)^1$ by a series of quadratic transformations, using triads of its multiple points, which changes k_6 into another sextic with ten double points, which is of type (2) because of the existence of k_1. It was therefore formerly of one of the types which arise from (2) by quadratic transformations, which are the five types enumerated above.

NORMAL COORDINATES IN DYNAMICAL SYSTEMS

By T. J. I'A. BROMWICH.

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THE main idea of the following paper is that of the solution of dynamical problems by means of complex integrals. Such a solution was suggested by Cauchy long ago, but I am not aware that he gave any method by which the arbitrary constants of the solution can be directly and immediately expressed in terms of the initial conditions.

In § 1 this problem is solved for a system free from impressed forces, but subject to given initial disturbances: and in § 2 for given impressed forces, the initial disturbances being zero. It is proved in § 3 that the results so obtained are equivalent to those derived by the method of normal coordinates, *when the latter can be applied.* But it must be understood that the method of § 1 can be used in cases when there is no recognized method of treatment by normal coordinates.

It appears that the general method of normal coordinates, as stated in § 3, was worked out independently by Routh and by Heaviside. Starting from this, Heaviside subsequently developed a method to which he gave the name of "the method of resistance-operators": it is proved in § 4 that this method is really equivalent to the method of § 2, for dealing with constant impressed forces.

In § 5 is given a statement of the extension of the method to deal with continuous systems; and it is verified that in certain simple problems the results agree with those found by known methods (such as Fourier's series). In § 6 is given an independent process for the interpretation of the complex integrals of § 5; it appears that for a vibrating uniform string (and allied problems) this method leads to the same conclusions as the superposition of positive and negative waves.

In § 7 it is shewn that we can apply the general methods of §§ 5, 6 very easily to deal with problems of advancing waves with a wave-front. The problems solved here refer to waves of sound: but similar methods are equally effective for electromagnetic waves. This would be antici-

pated from the fact that such electromagnetic problems were solved first by Heaviside, using his operational methods.

In § 8 an outline is given (for one special type of continuous systems) of a proof of the accuracy of the general method of § 5: and in § 9 the general method is compared with the method of normal functions, and it is proved that the results obtained are identical. In § 9 certain general results on normal functions are proved which I have not met with previously, although they are natural extensions of known results.

On account of the great generality of the method, I have thought it desirable to include a number of illustrative examples: most of them are taken from well-known sources, but the methods of treatment are novel.

I hope to indicate in another paper how the present method can be applied to problems in Conduction of Heat and other types of diffusion: examples of this kind have accordingly been excluded from the present paper.

1. *Solutions when Impressed Forces are absent.*

We begin by considering the solution of a system of simultaneous differential equations with constant coefficients, represented by

$$\left.\begin{aligned} e_{11}x_1+e_{12}x_2+\ldots+e_{1n}x_n &= 0 \\ e_{21}x_1+e_{22}x_2+\ldots+e_{2n}x_n &= 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \\ e_{n1}x_1+e_{n2}x_2+\ldots+e_{nn}x_n &= 0 \end{aligned}\right\}, \tag{1}$$

where e_{rs} denotes the differential operator

$$e_{rs} = a_{rs}D^2+b_{rs}D+c_{rs}, \tag{2}$$

following the notation used by Lord Rayleigh (*Theory of Sound*, Vol. I, § 82). Here the symbol D denotes differentiation with respect to the time t, and the coefficients a_{rs}, b_{rs}, c_{rs} are constants.

In the ordinary dynamical interpretation the coefficients a_{rs} are those of the kinetic energy, b_{rs} those of the dissipation function, and c_{rs} those of the potential energy: but in what follows we shall have no occasion to assume that the symmetrical relations

$$a_{rs} = a_{sr}, \qquad b_{rs} = b_{sr}, \qquad c_{rs} = c_{sr}$$

hold good, and accordingly the solution obtained can be applied to gyrostatic systems and also to systems in which the forces are not derived from a potential energy function.

The solution is found by substituting the contour integrals

$$x_r = \frac{1}{2\pi\iota}\int e^{\lambda t}\xi_r d\lambda, \tag{3}$$

in the equations (1), where $\xi_1, \xi_2, \ldots, \xi_n$ are certain functions of λ to be found, and the path of integration is a closed path in the λ-plane which encloses the poles of these functions ξ_r. The result of the substitution is

$$\int e^{\lambda t}p_1 d\lambda = 0, \quad \int e^{\lambda t}p_2 d\lambda = 0, \quad \ldots, \quad \int e^{\lambda t}p_n d\lambda = 0, \tag{4}$$

where

$$\left.\begin{aligned} p_1 &= \lambda_{11}\xi_1+\lambda_{12}\xi_2+\ldots+\lambda_{1n}\xi_n \\ p_2 &= \lambda_{21}\xi_1+\lambda_{22}\xi_2+\ldots+\lambda_{2n}\xi_n \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ p_n &= \lambda_{n1}\xi_1+\lambda_{n2}\xi_2+\ldots+\lambda_{nn}\xi_n \end{aligned}\right\}, \tag{5}$$

and

$$\lambda_{rs} = a_{rs}\lambda^2+b_{rs}\lambda+c_{rs}. \tag{6}$$

If the equations (4) hold, the path of integration must contain no poles of the functions $p_1, p_2, \ldots, p_n$: and, in virtue of the equations (5), it is clear that the functions p_r have no poles other than those of the functions ξ_r. Consequently, since the path contains the poles of ξ_r, it follows that the functions p_r can have no poles: and the simplest assumption to make is that the functions p_r are simple polynomials. We have now to see that we can adjust the choice of these polynomials so as to correspond to any given initial displacements (u_r) and velocities (v_r).

On putting $t = 0$, in the integrals (3), and in their differential coefficients, we find the equations for the initial displacements and velocities

$$u_r = \frac{1}{2\pi\iota}\int \xi_r d\lambda, \qquad v_r = \frac{1}{2\pi\iota}\int \lambda\xi_r d\lambda, \tag{7}$$

where now the path of integration may be supposed to be a large circle in the λ-plane; for, by taking a sufficiently large radius, we can certainly enclose all the poles of the functions $\xi_1, \xi_2, \ldots, \xi_n$, since these functions are simply rational fractions in λ. Now, when $|\lambda|$ is sufficiently large, any rational fraction in λ can be expanded in descending powers of λ; and if we have the expansion*

$$\xi_r = \frac{X_r}{\lambda}+\frac{Y_r}{\lambda^2}+\frac{Z_r}{\lambda^3}+\ldots,$$

* It is unnecessary to include any positive powers of λ in ξ_r: for such terms will contribute zero to our original solutions (3).

it follows from equations (7) that

$$u_r = X_r, \qquad v_r = Y_r.$$

Thus equations (7) lead to the inference that, when $|\lambda|$ is large,

$$\xi_r = \frac{u_r}{\lambda} + \frac{v_r}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right).* \tag{8}$$

Accordingly we can now determine the polynomials $p_1, p_2, \ldots, p_n$ by substituting for ξ_r from (8) in (5), and retaining only the positive powers of λ. This gives at once

$$\begin{aligned} p_1 = \{(a_{11}\lambda + b_{11})u_1 + a_{11}v_1\} + \{(a_{12}\lambda + b_{12})u_2 + a_{12}v_2\} \\ + \ldots + \{(a_{1n}\lambda + b_{1n})u_n + a_{1n}v_n\}, \end{aligned} \tag{9}$$

with similar formulæ for $p_2, p_3, \ldots, p_n$.

In actual calculations it is generally easier to remember the rule to substitute from (8) in (5), and to retain only positive powers of λ, than to quote the formulæ (9).

We have still to prove that the formulæ (9) do as a matter of fact lead to the equations (7): for this purpose we write the solution of (5) in the form

$$\xi_r = \frac{1}{\Delta(\lambda)} \{F_{r1}(\lambda)\, p_1 + F_{r2}(\lambda)\, p_2 + \ldots + F_{rn}(\lambda)\, p_n\}, \tag{10}$$

where $F_{rs}(\lambda)$ denotes the minor† of λ_{sr} in the determinant

$$\Delta(\lambda) = \begin{vmatrix} \lambda_{11}, & \lambda_{12}, & \ldots, & \lambda_{1n} \\ \lambda_{21}, & \lambda_{22}, & \ldots, & \lambda_{2n} \\ \ldots & \ldots & & \ldots \\ \lambda_{n1}, & \lambda_{n2}, & \ldots, & \lambda_{nn} \end{vmatrix}. \tag{11}$$

Now, if we observe that (9) can be written in the form

$$p_1 = \lambda_{11}\left(\frac{u_1}{\lambda} + \frac{v_1}{\lambda^2}\right) + \lambda_{12}\left(\frac{u_2}{\lambda} + \frac{v_2}{\lambda^2}\right) + \ldots + \lambda_{1n}\left(\frac{u_n}{\lambda} + \frac{v_n}{\lambda^2}\right) + \frac{q_1}{\lambda^2}, \tag{12}$$

where q_1 is another expression of the first degree (at most) in λ, and that

* In accordance with recent practice, the symbol $O(1/\lambda^3)$ is used to denote a function of λ, which is *of order* $1/\lambda^3$, when $|\lambda|$ is large; more precisely, if the function is multiplied by λ^3, the product remains finite, when $|\lambda|$ tends to infinity.

† Note that the order of the suffixes is reversed, intentionally, in F_{rs} and λ_{sr}: in problems for which $\lambda_{rs} = \lambda_{sr}$, the order of the two suffixes is immaterial.

$p_2, p_3, \ldots, p_n$ can be written similarly, we see that (10) leads to the result

$$\dot{\xi}_r = \frac{u_r}{\lambda} + \frac{v_r}{\lambda^2} + \frac{1}{\lambda^2 \Delta(\lambda)} \{F_{r1}(\lambda)\, q_1 + F_{r2}(\lambda)\, q_2 + \ldots + F_{rn}(\lambda)\, q_n\}. \qquad (13)$$

Now consider the expression in the last bracket of (13); its degree in λ is at most $(2n-1)$, since each minor F_{rs} is of degree $2(n-1)$ at most. Also the degree of $\Delta(\lambda)$ is $2n$, *provided that the determinant*

$$A = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{vmatrix}, \qquad (14)$$

is not zero. Consequently the formula (13) leads back to the equations (8), provided that the determinant A is not zero.

When the determinant A is zero, we can deduce from equations (1) at least one equation which contains no term in D^2. Since this equation holds for $t = 0$, there must be at least one relation amongst the initial displacements (u_r) and velocities (v_r); and accordingly it is no longer possible to solve equations (1) with a perfectly arbitrary set of displacements and velocities. This fact may suffice to indicate that the restriction that A is not to be zero does correspond to a real difficulty in the process of solution.

The question may now be naturally asked :—If A is zero, and the corresponding relations are satisfied by the u's and v's, will the general method given here still prove successful?

To this question I cannot give a complete answer; there are certain algebraic complications which have hitherto baffled me and have prevented me from constructing a general proof. But an examination of a number of special examples has convinced me that the general method is still adequate, even when $A = 0$. One of these examples is given below (Ex. 4, p. 409).

Accordingly, *the solution of equations* (1) *subject to assigned initial values of the velocities and displacements is given by the integrals* (3), *combined with equations* (5), (9) *and* (10).

To obtain the final solution in an explicit form, it is necessary to know the roots of the equation $\Delta(\lambda) = 0$; and this is the way in which the period-equation of the problem presents itself here. Knowing these roots, we have simply to calculate the sum of the residues of $e^{\lambda t}\dot{\xi}_r$ at the roots of $\Delta(\lambda) = 0$, in order to have the formula for x_r. Thus, if $\lambda = \alpha$ is any non-repeated root of $\Delta(\lambda) = 0$, we see from (10) that the corresponding contribution to x_1 will be

$$\frac{e^{\alpha t}}{\Delta'(\alpha)} \{F_{11}(\alpha)\, \bar{p}_1 + F_{12}(\alpha)\, \bar{p}_2 + \ldots + F_{1n}\bar{p}_n\}, \qquad (15)$$

where $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n$ denote the results of inserting the value $\lambda = \alpha$ in the expressions for $p_1, p_2, \ldots, p_n$.

There is no difficulty in carrying out the corresponding calculations

for a repeated root: but it is hardly worth while to write out a general formula corresponding to (15). The method of procedure will be easily seen from the examples given below (Exs. 2 and 3): and in many of the ordinary dynamical problems where repeated roots occur, it will be found that the corresponding roots only give *simple** poles in ξ_r (see Ex. 3).

It may, however, be well to point out explicitly that the present method is not affected by the complications which are introduced by the presence of repeated roots in $\Delta(\lambda) = 0$, when we are dealing with the closely associated problem of reducing two quadratic forms to sums of the same squares.†

It is easy to see how to modify the method of solution when the equations (1) are of the first degree only in D: it will suffice to state the corresponding results without fresh investigation.

Suppose, then, that in (1) the differential operators are given by

$$e_{rs} = a_{rs}D + b_{rs}. \tag{2'}$$

Then the equations (3), (5) remain unchanged in form, but we write

$$\lambda_{rs} = a_{rs}\lambda + b_{rs}. \tag{6'}$$

We shall now need only the initial values of $x_1, x_2, \ldots, x_n$; and these will be denoted, as before, by $u_1, u_2, \ldots, u_n$. Then, in place of (9), we find

$$p_1 = a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n, \tag{9'}$$

with similar forms for $p_2, p_3, \ldots, p_n$. This solution will satisfy the required conditions, provided that A, the determinant defined in (14), is not zero.

When A is zero, we find that at least one relation must exist amongst the initial displacements.

Here also (provided that the requisite relations hold), I have found that the general method leads to correct conclusions in all special examples which I have examined: but I have failed to construct a general proof of the property.

Examples of the Process of § 1.

The following examples have been selected to illustrate the method; they are not of types likely to occur in actual dynamical or physical examples, but are easy to work out in detail.

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* Because the corresponding invariant-factors of $\Delta(\lambda)$ are *linear*; so that if $(\lambda-\alpha)^r$ is a factor of $\Delta(\lambda)$, then $(\lambda-\alpha)^{r-1}$ is a factor of *every* first minor $F_{rs}(\lambda)$. Thus $(\lambda-\alpha)$ appears only to the first degree in the denominator of ξ_r.

† For the algebraic side of this problem, reference may be made to my tract on *Quadratic Forms* (No. 3 of the Cambridge University Press series).

Ex. 1.—[Routh's *Rigid Dynamics*, Vol. 2, 1892, Art. 367]

$$\left.\begin{aligned}(D^2-4D)\,x-\ (D-1)\,y&=0,\\ (D+6)\,x+(D^2-D)\,y&=0.\end{aligned}\right\}$$

We solve by taking

$$x=\frac{1}{2\pi i}\int \xi e^{\lambda t}\,d\lambda, \qquad y=\frac{1}{2\pi i}\int \eta e^{\lambda t}\,d\lambda,$$

where

$$\left.\begin{aligned}(\lambda^2-4\lambda)\,\xi-\ (\lambda-1)\,\eta&=p,\\ (\lambda+6)\,\xi+(\lambda^2-\lambda)\,\eta&=q,\end{aligned}\right\}$$

corresponding to (5) above.

Assuming that the initial values of x, y, and Dx, Dy, are represented by x_0, y_0, and x_1, y_1, the expansions corresponding to (8) will be

$$=\frac{x_0}{\lambda}+\frac{x_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right), \qquad \eta=\frac{y_0}{\lambda}+\frac{y_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right).$$

On substituting and rejecting negative powers of λ, we find that

$$\left.\begin{aligned}p&=(\lambda-4)\,x_0+x_1-y_0,\\ q&=x_0+(\lambda-1)\,y_0+y_1.\end{aligned}\right\}$$

Then, on solving, we see that

$$=\frac{p\,(\lambda^2-\lambda)+q\,(\lambda-1)}{(\lambda^2-1)(\lambda-2)(\lambda-3)}=\frac{p\lambda+q}{(\lambda+1)(\lambda-2)(\lambda-3)},$$

and

$$\eta=\frac{-p\,(\lambda+6)+q\,(\lambda^2-4\lambda)}{(\lambda^2-1)(\lambda-2)(\lambda-3)}.$$

It is now an easy matter to evaluate x and y completely by writing down the sum of the residues of $\xi e^{\lambda t}$ and of $\eta e^{\lambda t}$ at all the poles. We shall content ourselves with writing out the value of x only.

Corresponding to $\lambda=-1$, we have the residue

$$\begin{aligned}e^{-t}\left(\frac{-p+q}{12}\right)&=\tfrac{1}{12}e^{-t}\left\{-(-5x_0+x_1-y_0)+(x_0-2y_0+y_1)\right\}\\ &=\tfrac{1}{12}e^{-t}\,(6x_0-x_1-y_0+y_1)\,;\end{aligned}$$

and corresponding to $\lambda=+2$, we have

$$\begin{aligned}e^{2t}\left(\frac{2p+q}{-3}\right)&=-\tfrac{1}{3}e^{2t}\left\{2\,(-2x_0+x_1-y_0)+(x_0+y_0+y_1)\right.\\ &=-\tfrac{1}{3}e^{2t}\,(-3x_0+2x_1-y_0+y_1).\end{aligned}$$

Corresponding to $\lambda=3$, we find

$$\begin{aligned}e^{3t}\left(\frac{3p+q}{4}\right)&=\tfrac{1}{4}e^{3t}\ \left\{3\,(-x_0+x_1-y_0)+(x_0+2y_0+y_1)\right\}\\ &=\tfrac{1}{4}e^{3t}\,(-2x_0+3x_1-y_0+y_1).\end{aligned}$$

The complete formula for x is then the sum of these three residues; it is easy to verify that as a matter of fact the initial values of x and Dx are then x_0 and x_1. It will be seen also that the first and second terms agree with those found by Routh (*l.c.*) using the "method of isolation".

Ex. 2.—To illustrate *the effect of a multiple root in the determinant* $\Delta(\lambda)$, consider the equations

$$\left.\begin{aligned}(D^2-2D)\,x-\ \ y&=0\\ (2D-1)\,x+D^2y&=0\end{aligned}\right\}$$ [Routh, *l.c.*, Art. 373].

Proceeding as in Ex. 1 above, we find the equations

$$\left.\begin{aligned}(\lambda^2-2\lambda)\,\xi- \quad \eta &= p = (\lambda-2)\,x_0+x_1,\\ (2\lambda-1)\,\xi+\lambda^2\eta &= q = 2x_0+\lambda y_0+y_1.\end{aligned}\right\}$$

Hence, on solving, we have

$$(\lambda-1)^3(\lambda+1)\,\xi = \lambda^2 p+q = (\lambda^3-2\lambda^2+2)\,x_0+\lambda^2 x_1+\lambda y_0+y_1,$$

with a similar formula for η.

To obtain the residue corresponding to the multiple root $\lambda = 1$, we write $\lambda = 1+\mu$, and arrange ξ in the form

$$\frac{M_2}{\mu^3}+\frac{M_1}{\mu^2}+\frac{M_0}{\mu}+\frac{N}{2+\mu}.$$

Thus on multiplying up by $\mu^3(2+\mu)$, we find the identity

$$2M_2+\mu\,(2M_1+M_2)+\mu^2\,(2M_0+M_1)+(M_0+N)\,\mu^3$$
$$\equiv (1-\mu+\mu^2+\mu^3)\,x_0+(1+2\mu+\mu^2)\,x_1+(1+\mu)\,y_0+y_1.$$

Hence $\qquad 2M_2 = x_0+x_1+y_0+y_1, \qquad 2M_0+M_1 = x_0+x_1,$

$$2M_1+M_2 = -x_0+2x_1+y_0, \qquad M_0+N = x_0.$$

Then the residue for x, corresponding to $\lambda = 1$, is found from the coefficient of $1/\mu$ in the expansion of

$$\left(\frac{M_2}{\mu^3}+\frac{M_1}{\mu^2}+\frac{M_0}{\mu}+\frac{N}{2+\mu}\right)e^{t(1+\mu)},$$

in powers of μ. This coefficient is seen to be

$$(M_0+M_1t+\tfrac{1}{2}M_2t^2)\,e^t,$$

agreeing with Routh's result (*l.c.*).

The complete expression for x is seen to be

$$(M_0+M_1t+\tfrac{1}{2}M_2t^2)\,e^t+Ne^{-t}.$$

Ex. 3.—Another example of repeated roots in $\Delta(\lambda)$ is given by

$$(D^2-1)\,x+y+z = 0, \qquad x+(D^2-1)\,y+z = 0, \qquad x+y+(D^2-1)\,z = 0,$$

[Routh, *l.c.*, Art. 396].

Proceeding as above we use three symmetric equations, of which the first is

$$(\lambda^2-1)\,\xi+\eta+\zeta = p = \lambda x_0+x_1.$$

Hence $\qquad (\lambda^2+1)(\xi+\eta+\zeta) = p+q+r;$

and so, finally, $\qquad 3\xi = \dfrac{1}{\lambda^2+1}(p+q+r)+\dfrac{1}{\lambda^2-2}(2p-q-r).$

On calculating the residues of $\xi e^{\lambda t}$, we find

$$3x = (x_0+y_0+z_0)\cos t+(x_1+y_1+z_1)\sin t+(2x_0-y_0-z_0)\cosh(t\sqrt{2})+(2x_1-y_1-z_1)\sinh(t\sqrt{2})/\sqrt{2}.$$

The result may be verified at once by noticing that the given equations lead to

$$(D^2+1)(x+y+z) = 0, \qquad (D^2-2)(x-y) = 0, \qquad (D^2-2)(x-z) = 0.$$

This example illustrates the effect of *linear* invariant factors: the value of ξ has only simple poles, and consequently the value of x is of a simpler character than in such a problem as Ex. 2 above.

Ex. 4.—To illustrate *the effect of supposing the determinant A to be zero*, let us consider the equations

$$\left.\begin{aligned}(D^2+1)\,x+(D^2-2D)\,y &= 0,\\ (D^2+D)x+\quad D^2y \quad &= 0.\end{aligned}\right\}$$

Here
$$A = \begin{vmatrix} 1, & 1 \\ 1, & 1 \end{vmatrix} = 0;$$

and there must accordingly be a relation connecting the initial displacements and velocities. If we subtract the first equation from the second, we see that

$$(D-1)x+2Dy = 0.$$

Hence
$$x_1-x_0+2y_1 = 0$$

is the required relation, since the last equation is true for $t = 0$, as well as for all other values of t.

We shall now shew that (subject to this relation amongst the initial values) our general process leads to a correct conclusion. In fact the auxiliary equations are seen to be

$$\left.\begin{aligned}(\lambda^2+1)\,\xi+(\lambda^2-2\lambda)\eta &= \lambda x_0+x_1+(\lambda-2)\,y_0+y_1,\\ (\lambda^2+\lambda)\,\xi+\quad \lambda^2\eta \quad &= (\lambda+1)\,x_0+x_1+\lambda y_0+y_1.\end{aligned}\right\}$$

Hence
$$\lambda(\lambda+3)\,\xi = (\lambda+2)\,x_0+2x_1+2y_1,$$
$$\lambda^2(\lambda+3)\eta = (\lambda+1)\,x_0-(\lambda-1)\,x_1+\lambda(\lambda+3)\,y_0-(\lambda-1)\,y_1.$$

Thus, when $|\lambda|$ is large, we obtain the expansions

$$\left.\begin{aligned}\xi &= \frac{x_0}{\lambda}+\frac{-x_0+2x_1+2y_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right),\\ \eta &= \frac{y_0}{\lambda}+\quad\frac{x_0-x_1-y_1}{\lambda^2}\quad+O\left(\frac{1}{\lambda^3}\right).\end{aligned}\right\}$$

At first sight, these expressions do not appear to agree with (8); but nevertheless, on writing $x_0 = x_1+2y_1$ (in accordance with the relation already found amongst the initial values), we find that

$$\xi = \frac{x_0}{\lambda}+\frac{x_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right), \qquad \eta = \frac{y_0}{\lambda}+\frac{y_1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right),$$

so that as a matter of fact they do agree with (8).

The final solution is found to be

$$\left.\begin{aligned} x &= (x_0+\tfrac{1}{3}x_1)-\tfrac{1}{3}x_1 e^{-3t}, \\ y &= (y_0-\tfrac{2}{9}x_1)+(y_1+\tfrac{2}{3}x_1)\,t+\tfrac{2}{9}x_1 e^{-3t}, \end{aligned}\right\}$$

where the relation $x_0 = x_1+2y_1$ must be remembered.

2. *Effect of Impressed Forces.*

We pass next to the consideration of cases in which the equations contain impressed forces: in practically all physical problems these forces are either constants or simple harmonic functions of t, with or without damping factors. Any force of this type is expressible as the sum of terms of the type $Pe^{\mu t}$, where P and μ are real or complex constants. We consider, therefore, instead of (1), the equations

$$\left.\begin{aligned} e_{11}x_1+e_{12}x_2+\ldots+e_{1n}x_n &= P_1 e^{\mu t} \\ e_{21}x_1+e_{22}x_2+\ldots+e_{2n}x_n &= P_2 e^{\mu t} \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots & \\ e_{n1}x_1+e_{n2}x_2+\ldots+e_{nn}x_n &= P_n e^{\mu t} \end{aligned}\right\}. \tag{16}$$

As in § 1, we substitute the complex integrals (3), and we obtain the results

$$\frac{1}{2\pi\iota}\int e^{\lambda t}p_r\,d\lambda = P_r e^{\mu t}, \tag{17}$$

which replace (4); the values of $p_1, p_2, \ldots, p_n$ being defined by (5) as before.

In order to satisfy (17), the functions $p_1, p_2, \ldots, p_n$ must have $\lambda = \mu$ as a pole, and the path must surround this point. The simplest hypothesis accordingly is to write simply,

$$p_r = P_r/(\lambda-\mu). \tag{18}$$

If the path of integration encloses only the pole $\lambda = \mu$ and none of the roots of $\Delta(\lambda) = 0$, the solution (18) simply represents the *forced oscillation*. Of course this solution may be obtained almost at a glance by more elementary methods. But another solution which is often of more physical interest is *the solution corresponding to zero initial displacements and velocities; the forces being applied at* $t = 0$.

This solution is given by the same algebraic formulæ (5) and (18): but now the path of integration must enclose all the roots of $\Delta(\lambda) = 0$, as well as $\lambda = \mu$.

For, if the path is so chosen, the initial displacements and velocities are again given by the integrals (7), and the path of integration can be taken to be a large circle in the λ-plane.* If we now consider the result of substituting from (18) in the solution (10), and apply reasoning similar to that used on p. 405 above, we see that, when $|\lambda|$ is large, the expression for ξ_r takes the form

$$\xi_r = \frac{X_r}{\lambda^3} + \frac{Y_r}{\lambda^4} + \frac{Z_r}{\lambda^5} + \dots,$$

provided that the determinant A is not zero. Thus

$$\int \xi_r d\lambda = 0, \qquad \int \lambda \xi_r d\lambda = 0,$$

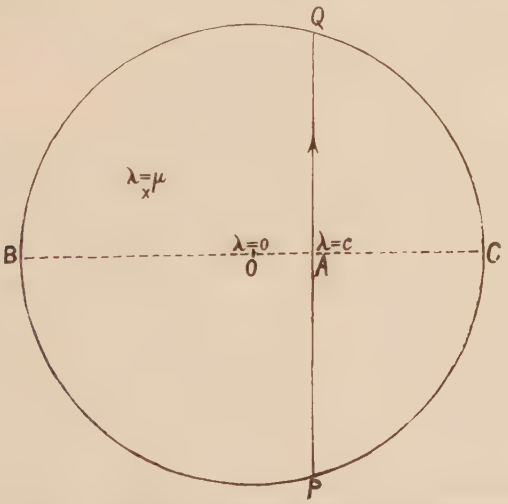
and accordingly the initial displacements and velocities, given by (7), are zero as stated.

The special case of *constant impressed forces* is of interest (see also § 4 below): the solution is then found by putting $\mu = 0$, or

$$p_r = P_r/\lambda, \tag{18a}$$

where the path of integration must now enclose $\lambda = 0$, as well as the roots of $\Delta(\lambda) = 0$.

In dealing with problems which involve impressed forces, it is often more convenient to replace the closed paths hitherto used by a straight line: this line will be parallel to the imaginary axis and on the *positive* side of the origin, so that the limits of integration can be written as $c - \iota\infty$ to $c + \iota\infty$, where c is any *positive constant*.



* Because *all* the poles of the integrands lie within this path of integration: and so the path can be extended as far out as we please, in virtue of Cauchy's theorem.

To prove that this straight path is equivalent to the closed paths, we start from a circle, with its centre at $\lambda = 0$, as in the figure; by taking a sufficiently large circle, we can ensure that all the poles to be considered* lie within the region $PAQBP$; and so we can replace the circle $PCQBP$ by the path $PAQBP$. Now *when t is positive*, it is easy to prove that any integral of the type $\int \xi_r e^{\lambda t} d\lambda$, taken along the arc QBP, will tend to zero as the radius tends to infinity.† Accordingly we are left with the limit of the path PAQ, and the solution takes the form

$$x_r = \frac{1}{2\pi\iota} \int_{c-\iota\infty}^{c+\iota\infty} e^{\lambda t} \xi_r d\lambda \quad (c > 0). \tag{3'}$$

The form (3′) can also be applied to the problem of § 1, except when $t = 0$; for, if ξ_r is of the type (8), it is easy to see that

$$\frac{1}{2\pi\iota} \int_{QBP} \xi_r d\lambda = \tfrac{1}{2} u_r,$$

and that the integral $\dfrac{1}{2\pi\iota} \displaystyle\int_{QBP} \lambda \xi_r d\lambda$ is divergent. Thus, *to get the correct values of u_r, v_r, we must not write $t = 0$ in* (3′); *but we must take the limit of* (3′) *as t tends to zero through positive values.*

By using (3′) we can extend our method, so as to include impressed forces which are not expressed explicitly in the forms assumed in (16) above. Suppose that they are given as functions of t, say $\Phi_r(t)$, and that they are applied from $t = 0$ to $t = \tau$; then Fourier's theorem enables us to write‡

$$\left.\begin{aligned} \Phi_r(t) &= \frac{1}{2\pi\iota} \int_{c-\iota\infty}^{c+\iota\infty} p_r e^{\lambda t} d\lambda \\ \text{where} \qquad p_r &= \int_0^\tau e^{-\lambda\theta} \Phi_r(\theta) d\theta \end{aligned}\right\}. \tag{19}$$

Thus the value of p_r defined by (19), used in connexion with the form (3′), enables us to solve the equations with the impressed forces $\Phi_r(t)$, by exactly the same algebraic process as in the simpler cases given in (16).

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* In practically all cases of interest, the poles have real parts which are either *zero* or *negative*, corresponding to undamped or to damped simple harmonic vibrations.

† Because ξ_r is at most of order $1/\lambda^3$, when $|\lambda|$ is large.

‡ The formulæ (19) may be deduced, as a matter of simple formal transformation, from the ordinary form of Fourier's theorem. Formulæ substantially equivalent were obtained, in this way, by Riemann in his paper on the distribution of primes (*Ges. Werke*, p. 140): and the actual formulæ (19) were deduced similarly by Macdonald (these *Proceedings*, Vol. xxxv, 1902, p. 428). From the point of view of the Theory of Functions of the Complex Variable, more complete discussions have been given by Pincherle and Mellin; see, for instance, a paper by the latter in the *Math. Annalen*, Bd. 68, p. 305, where references to earlier investigations will be found.

Examples of the Effect of Impressed Forces.

To avoid unnecessary algebra, we take two examples involving only a single coordinate. In more complicated problems, the method of procedure is precisely similar.

Ex. 5.—*A dynamical system with one degree of freedom is acted on by a force having the same damping and free period as the system.*

We can represent the equation of motion by

$$\frac{d^2x}{dt^2} + 2\nu\frac{dx}{dt} + (\nu^2+n^2)x = Fe^{-\nu t}\cos(nt+\omega),$$

where $2\pi/n$ is the period, and $e^{-\nu t}$ is the damping factor. The force is supposed applied at $t = 0$, when the system is at rest in an undisturbed position. Then the solution is

$$x = \frac{1}{2\pi\imath}\int \xi e^{\lambda t}\,d\lambda,$$

where
$$\left\{(\lambda+\nu)^2+n^2\right\}\xi = \frac{F}{2}\left\{\frac{e^{\imath\omega}}{\lambda+\nu-\imath n} + \frac{e^{-\imath\omega}}{\lambda+\nu+\imath n}\right\},$$

or
$$\xi = \frac{F\left\{(\lambda+\nu)\cos\omega - n\sin\omega\right\}}{\left\{(\lambda+\nu)^2+n^2\right\}^2}.$$

The residues of $\xi e^{\lambda t}$ have to be calculated at the points $\lambda = -\nu \pm \imath n$, and on addition we obtain

$$x = \frac{Fe^{-\nu t}}{2n^2}\left\{nt\sin(nt+\omega) + \cos(nt+\omega) - \cos nt\cos\omega\right\}$$

$$= \frac{Fe^{-\nu t}}{2n^2}\left\{nt\sin(nt+\omega) - \sin\omega\sin nt\right\}.$$

When this formula has been obtained, it is easy to verify the result by elementary methods: but a direct determination of the arbitrary constants is somewhat tedious, although perfectly straightforward. The result for the special case $\omega = 0$ was given by Bjerknes (*Annalen der Physik*, Bd. 55, 1892, p. 132).

Ex. 6.—*The system is acted on by a force having the same period; but the force is applied for a half period only.*

Here we omit the damping factors (for brevity merely), and take the equation

$$\frac{d^2x}{dt^2} + n^2x = F\sin nt,$$

which holds from $t = 0$ to π/n. If we apply (19), with $\tau = \pi/n$, we find that

$$p = F\int_0^{\pi/n} e^{-\lambda\theta}\sin n\theta\,d\theta = \frac{Fn}{\lambda^2+n^2}(1+e^{-\lambda\pi/n}).$$

Thus
$$\xi = \frac{Fn}{(\lambda^2+n^2)^2}(1+e^{-\lambda\pi/n}).$$

Calculating the residues of $\xi e^{\lambda t}$ at $\lambda = \pm\imath n$, we obtain the result

$$x = -\frac{F\pi}{2n^2}\cos nt \quad (t > \pi/n).$$

This conclusion is easily verified by observing that Ex. 5 (with $\nu = 0$, $\omega = -\frac{1}{2}\pi$) gives the solution up to $t = \pi/n$, and that x and dx/dt are continuous at $t = \pi/n$.

3. *Comparison with the Solution in Terms of Normal Functions.*

The customary method of solving the problem proposed in § 1 above, is to use *normal functions*. That is, corresponding to each root $\lambda = \alpha$ of

the determinantal equation $\Delta(\lambda) = 0$, a solution $l_1, l_2, \ldots, l_n$ is found for the equations

$$\left.\begin{aligned} \lambda_{11} l_1 + \lambda_{12} l_2 + \ldots + \lambda_{1n} l_n &= 0 \\ \lambda_{21} l_1 + \lambda_{22} l_2 + \ldots + \lambda_{2n} l_n &= 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots & \\ \lambda_{n1} l_1 + \lambda_{n2} l_2 + \ldots + \lambda_{nn} l_n &= 0 \end{aligned}\right\}, \tag{20}$$

where $$\lambda = a.$$

It is then evident that if the initial displacements (u) and velocities (v_r) are adjusted so that

$$\left.\begin{aligned} \frac{u_1}{l_1} = \frac{u_2}{l_2} = \ldots = \frac{u_n}{l_n} &= A \\ \text{and} \quad \frac{v_1}{l_1} = \frac{v_2}{l_2} = \ldots = \frac{v_n}{l_n} &= Aa \end{aligned}\right\}, \tag{21}$$

a possible solution of the problem in § 1 is given by

$$x_1 = A l_1 e^{at}, \quad x_2 = A l_2 e^{at}, \quad \ldots, \quad x_n = A l_n e^{at}.$$

But it will be observed that this does *not* prove that no other solution exists under the given initial conditions: although it may be possible to infer, from general considerations of a physical kind, that the solution (if any) is unique.

This special type of solution gives a *normal mode of motion* (or *principal mode*) in the customary terminology.

It is then usual to assume that *every* motion possible can be obtained by the superposition of normal modes: that is, that we can satisfy the equations (1) of § 1 by taking

$$x_r = \Sigma A l_r e^{at}, \tag{22}$$

where the summation is extended to all roots a of $\Delta(\lambda) = 0$. To determine the $2n$ constants A we have the $2n$ equations

$$u_r = \Sigma A l_r, \qquad v_r = \Sigma A a l_r. \tag{23}$$

But it is by no means obvious that these $2n$ equations (23) are always algebraically capable of solution: it is conceivable that the $2n$ normal modes might not be algebraically independent.* The most direct proof

* As a matter of fact, it can be proved that they are algebraically independent when the roots of $\Delta(\lambda) = 0$ are *all different* (provided that the determinant A of § 1 is not zero). But failure may occur with equal roots: consider, for example, the equations

$$(2D^2 - 3)x - y = 0. \qquad x + (2D^2 - 1)y = 0.$$

Here the determinantal equation reduces to $(\lambda^2 - 1)^2 = 0$, so that $a = \pm 1$; and for all these

that the constants A *can* be found is to obtain formulæ giving them explicitly in terms of the initial values, u_r, v_r. Such formulæ have been given only, as far as I know, in certain special cases; and of these, the only simple case is that of symmetry for which

$$a_{rs} = a_{sr}, \quad b_{rs} = b_{sr}, \quad c_{rs} = c_{sr},$$

so that

$$\lambda_{rs} = \lambda_{sr}.$$

Under these conditions Routh* and Heaviside† have shewn that there is a conjugate relation between the normal solutions (l_r), (m_r) corresponding to any two unequal roots α, β of the determinantal equation $\Delta(\lambda) = 0$.

Following the notation used by Heaviside in the second paper quoted, the conjugate relation may be written‡

$$T_{12} = U_{12},$$

where

$$T_{12} = a_{11}(al_1)(\beta m_1) + a_{12}\{(al_1)(\beta m_2) + (al_2)(\beta m_1)\} + \ldots,$$

$$U_{12} = c_{11} l_1 m_1 + c_{12}(l_1 m_2 + l_2 m_1) + \ldots,$$

so that T_{12} is what may be called the relative kinetic energy of the two normal solutions, while U_{12} is their relative potential energy.

From this property it is easy to deduce that the value of the coefficient A is given by Heaviside's formula

$$A = \frac{T_{10} - U_{10}}{T_{11} - U_{11}}, \tag{24}$$

where T_{10}, U_{10} are constructed similarly to T_{12}, U_{12}, but with the initial velocities and displacements taking the place of those belonging to the

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roots the relations analogous to (20) reduce to $l + m = 0$. Thus the corresponding normal modes are subject to the relation $x + y = 0$, but the solution given by

$$x = te^t, \quad y = (4 - t)e^t$$

corresponds to the initial values $x_0 = 0$, $x_1 = 1$, $y_0 = 4$, $y_1 = 3$; and clearly this solution is not expressible as a sum of normal modes.

* Routh, *Rigid Dynamics*, Vol. 2, 1892, Arts. 383, 384; Routh published his method first in 1883.

† Heaviside, *Electrical Papers*, Vol. 1, pp. 520–531; and Vol. 2, p. 202; these papers were published in 1885 and 1886 respectively. It is clear that Heaviside was unaware of Routh's work; and his results are easier to state than Routh's.

‡ When there are no frictional terms ($b_{rs} = 0$), the conjugate property takes the form

$$T_{12} = 0, \quad U_{12} = 0,$$

which is a familiar result.

second normal solution. Thus

$$\left.\begin{aligned} T_{10} &= a_{11}(al_1)\,v_1+a_{12}\{(al_1)\,v_2+(al_2)\,v_1\}+\ldots \\ U_{10} &= c_{11}l_1u_1+c_{12}(l_1u_2+l_2u_1)+\ldots \\ \text{and}\qquad T_{11} &= a_{11}(al_1)^2+2a_{12}(al_1)(al_2)+\ldots \\ U_{11} &= c_{11}l_1^2+2c_{12}l_1l_2+\ldots \end{aligned}\right\}. \tag{25}$$

We proceed now to identify (24) with our former results. As a preliminary we note that in virtue of equations (20) we can write

$$\begin{aligned} &c_{11}l_1+c_{12}l_2+\ldots+c_{1n}l_n \\ &\quad=-\alpha^2(a_{11}l_1+a_{12}l_2+\ldots+a_{1n}l_n)-\alpha\,(b_{11}l_1+b_{12}l_2+\ldots+b_{1n}l_n), \end{aligned} \tag{26}$$

with similar results for the suffixes 2, 3, ..., n.

If we multiply equations (26) by $u_1, u_2, \ldots, u_n$ in order, and add, we see that

$$\begin{aligned} U_{10} = &-\alpha^2\{a_{11}l_1u_1+a_{12}(l_1u_2+l_2u_1)+\ldots\} \\ &-\alpha\,\{b_{11}l_1u_1+b_{12}(l_1u_2+l_2u_1)+\ldots\}. \end{aligned}$$

Thus the coefficient of l_1 in $T_{10}-U_{10}$ can be written

$$\begin{aligned} &\alpha\,(a_{11}v_1+a_{12}v_2+\ldots+a_{1n}v_n) \\ &+\alpha^2(a_{11}u_1+a_{12}u_2+\ldots+a_{1n}u_n) \\ &+\alpha\,(b_{11}u_1+b_{12}u_2+\ldots+b_{1n}u_n), \end{aligned}$$

which is equal to $\alpha\bar{p}_1$, by equations (9), where, as in (15), $\bar{p}_1$ denotes the result of writing $\lambda=\alpha$ in p_1.

Consequently we can write

$$T_{10}-U_{10}=\alpha(l_1\bar{p}_1+l_2\bar{p}_2+\ldots+l_n\bar{p}_n). \tag{27}$$

Again, if we take equations (26) and multiply them in order by $l_1, l_2, \ldots, l_n$ and add, we see that

$$U_{11}=-(a_{11}\alpha^2+b_{11}\alpha)\,l_1^2-2(a_{12}\alpha^2+b_{12}\alpha)\,l_1l_2-\ldots,$$

Thus we have the result

$$T_{11}-U_{11}=\alpha\{(2a_{11}\alpha+b_{11})\,l_1^2+2\,(2a_{12}\alpha+b_{12})\,l_1l_2+\ldots\}. \tag{28}$$

Again using equations (20), we see that, in the notation of § 1, we can write (for any suffix r)

$$\frac{l_1}{F_{1r}(\alpha)}=\frac{l_2}{F_{2r}(\alpha)}=\ldots=\frac{l_n}{F_{nr}(\alpha)}.$$

It follows that the minor $F_{rs}(a)$ is proportional to the product $l_r l_s$: and so we can write

$$F_{rs}(a) = k l_r l_s, \tag{29}$$

where k is the same for all pairs of suffixes r, s.

Now from the ordinary rule for differentiating a determinant, we see that

$$\Delta'(\lambda) = (2a_{11}\lambda + b_{11})\, F_{11}(\lambda) + 2(2a_{12}\lambda + b_{12})\, F_{12}(\lambda) + \ldots,$$

because now the determinant $\Delta(\lambda)$ is symmetrical.

Consequently, using (28) and (29), we have

$$a\Delta'(a) = k(T_{11} - U_{11}). \tag{30}$$

Also, from (27) and (29), we find that

$$k l_1 (T_{10} - U_{10}) = a \{F_{11}(a)\, \bar{p}_1 + F_{12}(a)\, \bar{p}_2 + \ldots + F_{1n}(a)\, \bar{p}_n\}. \tag{31}$$

Combining (30) and (31), we see that Heaviside's formula (24) gives

$$A l_1 = \frac{T_{10} - U_{10}}{T_{11} - U_{11}} l_1 = \frac{1}{\Delta'(a)} \{F_{11}(a)\, \bar{p}_1 + F_{12}(a)\, \bar{p}_2 + \ldots + F_{1n}(a)\, \bar{p}_n\},$$

or, comparing with (15), we now infer that $A l_1 e^{at}$ is equal to the residue at $\lambda = a$, calculated in § 1.

In other words, *we have identified Heaviside's formula* (24) *with the solution found in* § 1. But it should be borne in mind that (unless the determination of the normal modes is extremely easy) it is usually quicker to apply the process of § 1, rather than (24).*

There is only one other class of problems for which a rule similar to (24) has been devised; this is Routh's case of a dynamical system with gyrostatic forces, but no resistances. Thus we have the relations

$$a_{rs} = a_{sr}, \qquad b_{rs} = -b_{sr}, \qquad c_{rs} = c_{sr}.$$

It is not easy to translate Routh's rule (*l.c.*, p. 415, Art. 390) into a form similar to (24); and I have not constructed a general proof to shew that this rule is included in the general process given in § 1. I have, however, verified that in a special numerical example, worked out by Routh (*l.c.*, Art. 391), the equations given for finding the coefficients can be derived from the results given by the general process of § 1. But it should be noticed that Routh's equations are left in an unsolved form; one of the equations is a quadratic, and so the process of solution would be tedious. On the other hand, § 1 gives the coefficients at once (see Ex. 7 below).

* This fact will be observed in Heaviside's later writings, where the method of § 4 is used as a rule in preference to (24).

To sum up, it has been proved that *the method of* § 1 *includes all the results which can be derived by the use of normal coordinates: and, in general, it will give these results in a more convenient form.*

Ex. 7.—*Routh's Example of a Gyrostatic System* (*l.c.*, Art. 391),

$$\left.\begin{aligned}(D^2-8)\,x+\quad \surd 6\,Dy &= 0,\\ -\surd 6\,Dx+(D^2+2)\,y &= 0.\end{aligned}\right\}$$

Here
$$(\lambda^2-8)\,\xi+\quad \surd 6\,\lambda\eta = p = \lambda x_0+x_1+\surd 6\,y_0,$$
$$-\surd 6\,\lambda\xi+(\lambda^2+2)\,\eta = q = -\surd 6\,x_0+\lambda y_0+y_1.$$

Thus
$$(\lambda^4-16)\,\xi = (\lambda^2+2)\,p-\surd 6\,\lambda q = (\lambda^2+2)(\lambda x_0+x_1)+6\lambda x_0+\surd 6\,(2y_0-\lambda y_1),$$

with a similar formula for η.

The results given by Routh correspond to finding the residues of $\xi e^{\lambda t}$ for $\lambda=-2$ and for $\lambda=\pm 2\iota$. Take first the former, which gives

$$-\tfrac{1}{32}e^{-2t}\left\{6\,(-2x_0+x_1)-12x_0+2\surd 6\,(y_0+y_1)\right\} = -\tfrac{1}{16}e^{-2t}\left\{-12x_0+3x_1+\surd 6\,(y_0+y_1)\right\},$$

agreeing with one of Routh's results.

With regard to the residues for $\pm 2\iota$, suppose that they are expressed in the forms

$$\tfrac{1}{2}\,(X_1-\iota X_2)\,e^{2\iota t}+\tfrac{1}{2}\,(X_1+\iota X_2)\,e^{-2\iota t}.$$

Then their sum is $X_1\cos 2t+X_2\sin 2t$, which is Routh's form.

Now $\tfrac{1}{2}\,(X_1-\iota X_2)$ is the residue of ξ at $\lambda=+2\iota$, and so we find

$$-16\iota\,(X_1-\iota X_2) = -2\,(2\iota x_0+x_1)+12\iota x_0+2\surd 6\,(y_0-\iota y_1),$$

or
$$8\,(X_1-\iota X_2) = -4x_0-\iota x_1+\surd 6\,(\iota y_0+y_1).$$

The last equation gives X_1 and X_2 by direct inspection.

Routh's equations can be derived by multiplying up by $X_1+\iota X_2$, and then equating real and imaginary parts; clearly these equations must (in the long run) lead to the same results. But it is equally certain that the labour involved in deriving X_1, X_2 from Routh's equations is very great.

The final formula for x can be written

$$\tfrac{1}{8}\,(-4x_0+\surd 6\,y_1)\cos 2t\ +\tfrac{1}{8}\,(\ x_1-\surd 6\,y_0)\sin 2t$$
$$+\tfrac{1}{8}\,(\ 12x_0-\surd 6\,y_1)\cosh 2t+\tfrac{1}{8}\,(3x_1+\surd 6\,y_0)\sinh 2t.$$

4. *Comparison with Heaviside's Method of Resistance Operators.*

Heaviside's method is specially adapted for handling problems in which the impressed forces are steady, and are applied at $t=0$, to a system which is then entirely free from disturbance. Heaviside has used his method particularly in the theory of currents set up in a network of linear conductors* (allowing for inductances and capacities, as well as for resistances). The original equations of the network can then be reduced to a system of the type (16), given in § 2 above, with $\mu=0$; and with the special feature of symmetry. The coefficients a_{rs} involve

* See, in particular, *Electrical Papers*, Vol. 1, p. 412, and Vol. 2, p. 259.

the inductances, b_{rs} the resistances, and c_{rs} the reciprocals of capacities of condensers. Heaviside replaces the operator $D(=d/dt)$ by a symbol p, which is regarded as subject to the ordinary laws of algebra: and he then solves the resulting equations, as a system of linear algebraic equations in x_r. Following the notation adopted in § 1, we see that the result so obtained is*

$$\Delta(p)\, x_r = F_{r1}(p) P_1 + F_{r2}(p) P_2 + \ldots + F_{rn}(p) P_n;$$

or, supposing that one force only (P_s) is applied, we can write our result as

$$\Delta(p)\, x_r = F_{rs}(p) P_s. \tag{32}$$

Up to this stage the work is much the same as may be found in almost any elementary discussion of simultaneous linear differential equations; but we now encounter the essential novelty of Heaviside's process, which lies in the interpretation of the symbolic equation (32), or rather of the algebraically equivalent one

$$x_r = \frac{F_{rs}(p)}{\Delta(p)} P_s. \tag{33}$$

The result of Heaviside's interpretation is

$$x_r = P_s \left\{ \frac{F_{rs}(0)}{\Delta(0)} + \Sigma_\alpha \frac{F_{rs}(\alpha)}{\alpha \Delta'(\alpha)} e^{\alpha t} \right\}, \tag{34}$$

where α is any root of $\Delta(p) = 0$, and the summation extends to all such roots. The investigations leading up to the formula (34) are most instructive and will repay careful study:† but I am not sure that they have been sufficiently appreciated in the past. It is almost certain that few readers have fully grasped the complete and general character of the solution: and it is for this reason that I wish to call attention to it here.

To compare the results of § 2 with those of Heaviside's method, we note in the first place, that in (18a) all the P's are zero except P_s: and accordingly equation (10) gives

$$\xi_r = \frac{F_{rs}(\lambda)}{\Delta(\lambda)} \frac{P_s}{\lambda}. \tag{35}$$

Then, using the complex integrals of §§ 1, 2, the value of x_r is given by

$$x_r = \frac{1}{2\pi\iota} \int \frac{F_{rs}(\lambda) P_s}{\lambda \Delta(\lambda)} e^{\lambda t} d\lambda, \tag{36}$$

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* Heaviside, *Electrical Papers*, Vol. 2, p. 259; his symbol $F(p)$ corresponds to $\Delta(p)$, and $f_{rs}(p)$ to $F_{rs}(p)$.

† *Ibid.*, p. 226 (where the method in the footnote supersedes the text); and p. 373.

where the path is to enclose $\lambda = 0$ and all points $\lambda = \alpha$, where α is a root of $\Delta(\lambda) = 0$. It will be remembered that (as proved in § 2) the integral (36) then solves the problem for zero initial disturbances in *all* the coordinates $x_1, x_2, \ldots, x_n$.

On forming the sum of the residues of (36), it will be seen that the result agrees precisely with Heaviside's formula (34); and accordingly we have now provided an independent proof of the latter formula.

Ex. 8.—To discuss the effect of switching a steady electromotive force E into a primary coil in the presence of a secondary coil.

This is a familiar problem, but gives a simple illustration of Heaviside's method: to shorten the algebra, suppose the coils equal in all respects, having inductances L, and resistances R; and let their mutual inductance be M.

Then, if x, y denote the currents, we have the equations

$$L\frac{dx}{dt} + M\frac{dy}{dt} + Rx = E, \qquad M\frac{dx}{dt} + L\frac{dy}{dt} + Ry = 0.$$

According to Heaviside's process we take the auxiliary equations

$$\left.\begin{aligned} (Lp+R)\,x + Mpy &= E, \\ Mpx + (Lp+R)\,y &= 0. \end{aligned}\right\}$$

Hence
$$x = \frac{(Lp+R)\,E}{(Lp+R)^2 - M^2p^2}, \qquad y = \frac{-MpE}{(Lp+R)^2 - M^2p^2}.$$

Let α, β denote the roots of the denominator, so that

$$\alpha = -R/(L+M), \qquad \beta = -R/(L-M).$$

Then (34) gives the results

$$x = \frac{E}{R} + \frac{E\,(L\alpha+R)\,e^{\alpha t}}{\alpha\,(L^2-M^2)(\alpha-\beta)} + \frac{E\,(L\beta+R)\,e^{\beta t}}{\beta\,(L^2-M^2)(\beta-\alpha)},$$

$$y = 0 - \frac{EM\,e^{\alpha t}}{(L^2-M^2)(\alpha-\beta)} - \frac{EM\,e_{\beta t}}{(L^2-M^2)(\beta-\alpha)},$$

which simplify down to

$$x = (E/R)\left\{1 - \tfrac{1}{2}(e^{\alpha t} + e^{\beta t})\right\}, \qquad y = -\tfrac{1}{2}(E/R)(e^{\alpha t} - e^{\beta t}).$$

5. *Continuous Systems.*

When the equations of motion of a vibrating continuous system are formed, the equations corresponding to those of § 1 above will usually be differential equations (one or more in number) combined with certain boundary conditions. There is usually no particular difficulty in applying the same rules as have been given in §§ 1, 2 to such systems.

For instance, suppose that the differential equation takes the form

$$\rho\frac{\partial^2 u}{\partial t^2} + \sigma\frac{\partial u}{\partial t} - \Delta u = 0, \tag{37}$$

where Δ denotes a differential operator (with respect to the coordinates

employed), and ρ, σ will in general be functions of the coordinates appearing in Δ.

Then our general process suggests the following method:—

Write $$u = \frac{1}{2\pi\iota}\int v e^{\lambda t} d\lambda,$$

and then the equation for v will be

$$(\rho\lambda^2+\sigma\lambda)\, v - \Delta v = (\rho\lambda+\sigma)\, u_0 + \rho u_1, \tag{38}$$

where u_0 and u_1 are the initial values of u and $\partial u/\partial t$.

The boundary conditions may or may not involve differential coefficients with respect to the time: when they do, care must be taken to modify them (by introducing terms containing u_0 and u_1) in the same way as (38) is a modified form of (37).* This point may easily be overlooked; but unless the appropriate terms are inserted, errors will be found in the final solution which may not be easy to detect.

The determination of the path of integration is perhaps less obvious; with a continuous system, the number of poles of the integrand v will, as a rule, be infinite; and these poles usually tend towards infinity (in the λ-plane) in two directions. Very often these directions correspond to the two ends of the imaginary axis. Further, in all problems which have any physical interest, the poles are distributed either *on* the imaginary axis or on the negative side of that axis. The former arrangement represents undamped simple harmonic normal modes of oscillation: and the latter corresponds to damped oscillations.

It is evident, therefore, that the closed paths of integration used in §§ 1, 2 must now be replaced by an unclosed path. But we have seen in § 2 that the path from $c-\iota\infty$ to $c+\iota\infty$ (see Fig., p. 411) is equivalent to the paths of §§ 1, 2, and so will be the natural path to adopt here. For, just as in § 2, the poles of v all lie on the negative side of the path; and this is the property corresponding to the statement of § 1, that all the poles are to be inside the closed path. Accordingly the formula for u will be

$$u = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} e^{\lambda t} v d\lambda \quad (c > 0), \tag{39}$$

where v is obtained from (38) and from the boundary conditions.

It will be observed that the foregoing does not profess to give a complete proof of (39): all that we have done is to establish an analogy be-

* The reason is, of course, that *the boundary conditions really form part of the equations of motion:* and they must accordingly be put on the same footing as the other equations.

tween (39) and the formulæ which we have proved for discrete systems. Anything approaching a complete proof for a continuous system would necessarily be very long, on account of the great variety of geometrical figures to be considered: and it would further require very careful handling as a matter of pure mathematics.

Fortunately, in the problems involving only one geometrical coordinate, we can appeal to the results of a paper by Prof. A. C. Dixon;* we shall trace the connexion between his paper and the present method in § 8 below. His results justify the use of our method in almost all the problems which are capable of easy solution: and I feel no doubt that, in other problems, the final justification of our process will be found fairly simple when the solution v can be actually worked out.

We have spoken so far of the determination of u in terms of u_0 and u_1: but it is easy to see that a similar modification of the method of § 2 will enable us to deal with problems involving impressed forces. Thus, if the given equation is

$$\rho \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t} - \Delta u = Pe^{ut}, \tag{40}$$

the auxiliary equation for v becomes

$$(\rho\lambda^2 + \sigma\lambda)\, v - \Delta v = P/(\lambda - \mu). \tag{41}$$

The form (41) corresponds to the case in which the initial disturbance is zero, and the impressed force is applied at $t = 0$.

When the integral (39) has been determined, the expression of u by means of a sum of residues will lead *to the expansion of u in a series of normal functions.*

To illustrate the process, three examples are worked out below, taken from various problems in the Theory of Sound.

Ex. 9.—A uniform string is plucked aside at its centre and let go from rest, with the ends fixed.

If we choose the unit of velocity to be the velocity of transverse vibrations (u), the differential equation for u is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

Thus, from $x = 0$ to $x = \frac{1}{2}l$, we have the auxiliary equation for v,

$$\lambda^2 v - \frac{\partial^2 v}{\partial x^2} = \lambda k x,$$

because $u_0 = kx$, $u_1 = 0$; and the boundary conditions are

$$v = 0 \quad \text{at} \quad x = 0, \qquad \frac{\partial v}{\partial x} = 0 \quad \text{at} \quad x = \tfrac{1}{2}l.$$

* These *Proceedings*, Ser. 2, Vol. 3, 1905, p. 83.

It is easily verified that the solution v (from $x = 0$ to $x = \frac{1}{2}l$) is given by

$$v = \frac{kx}{\lambda} - \frac{k}{\lambda^2} \frac{\sinh \lambda x}{\cosh \frac{1}{2}\lambda l}.$$

In order to obtain a series of normal functions for u, we note that $\lambda = 0$ is not a pole of v: and that the poles are given by $\lambda l = (2n+1)\pi\iota$, where n is an integer, positive or negative. The corresponding residue of $ve^{\lambda t}$ is

$$(-1)^n \frac{kl^2}{(2n+1)^2 \pi^2 . \frac{1}{2}l} \sin\left\{(2n+1)\frac{\pi x}{l}\right\} \exp\left\{(2n+1)\frac{\pi\iota t}{l}\right\},$$

and so, on combining corresponding positive and negative terms, we find that

$$u = \frac{4kl}{\pi^2} \sum_0^\infty \frac{(-1)^n}{(2n+1)^2} \sin\left\{(2n+1)\frac{\pi x}{l}\right\} \cos\left\{(2n+1)\frac{\pi t}{l}\right\},$$

which is the formula derived by applying Fourier's theorem.*

For another method of treatment see below (§ 6, p. 429).

The problem of a string plucked at any point offers no fresh difficulty in calculation; but the results are rather longer to state, because two formulæ are needed for v, one for each of the two parts into which the string is divided by the plucking.

Ex. 10.—A pipe, open at one end, and closed at the other, is suddenly brought to rest at time $t = 0$, after having been for some time in motion, with uniform velocity u_1 parallel to the length of the pipe.†

Let u denote the displacement along the length of the tube; then taking the velocity of sound as unity (as in Ex. 9), we have

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

subject to $u = 0$ at $x = 0$, $\partial u/\partial x = 0$ at $x = l$. Thus the equation for v is

$$\lambda^2 v - \frac{\partial^2 v}{\partial x^2} = u_1,$$

because $u_0 = 0$; and so $$v = \frac{u_1}{\lambda^2}\left\{1 - \frac{\cosh \lambda (l-x)}{\cosh \lambda l}\right\}.$$

As in Ex. 9, $\lambda = 0$ is not a pole; the poles are given by $\lambda l = \pm(n+\frac{1}{2})\pi\iota$, or say $\lambda = \pm k\iota$, if $k = (n+\frac{1}{2})\pi/l$. The residue at the positive pole of $ve^{\lambda t}$ is then found to be

$$\frac{u_1}{\iota k^2 l} \sin(kx)\, e^{\iota k t}.$$

Combining the positive and negative poles, and summing, we find the formula

$$u = \frac{2u_1}{l} \sum \frac{1}{k^2} \sin(kx) \sin(kt).$$

To identify this with Lord Rayleigh's result for the velocity-potential ϕ, we note that

$$\frac{\partial u}{\partial t} = \frac{\partial \phi}{\partial x}.$$

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* Rayleigh, *Theory of Sound*, Vol. 1, § 127; to get the required formula, put $b = \frac{1}{2}l$, and $\gamma = kb$.

† *Ibid.*, Vol. 2, Art. 258.

Ex. 11.—As an example of a slightly more difficult type, suppose that a uniform force P per unit area is applied to the surface of a uniform circular membrane, which is initially undisturbed.

The differential equation for the displacement u is then*

$$\rho\left\{\frac{\partial^2 u}{\partial t^2}-c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)\right\}=P,$$

where r is the distance from the centre. Thus the equation for v becomes

$$\rho\left\{\lambda^2 v-c^2\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right)\right\}=P/\lambda.$$

Hence

$$v=\frac{P}{\rho\lambda^3}\left\{1-\frac{I_0(\lambda r/c)}{I_0(\lambda a/c)}\right\},$$

because $v=0$ at $r=a$. Here I_0 is the modified Bessel function, defined by

$$I_0(z)=1+\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}+\frac{z^6}{2^2.4^2.6^2}+\ldots.$$

The residue of $ve^{\lambda t}$ at $\lambda=0$ is thus

$$u_0=\frac{P}{\rho}\left(\frac{a^2}{4c^2}-\frac{r^2}{4c^2}\right),$$

which represents the statical displacement produced by the application of the force P.

The other poles of v are given by the roots of $I_0(\lambda a/c)=0$; these roots are all purely imaginary, and we can write them in the form

$$\lambda a/c=\pm\omega_n i,$$

where ω_n is the nth root of

$$J_0(\omega)=0.$$

The corresponding residue of $ve^{\lambda t}$ is then

$$-\frac{P}{\rho(\omega c/a)^3}\frac{J_0(\omega r/a)}{(a/c)J_1(\omega)}e^{\omega ct/a}.$$

And so, combining the positive and negative poles, we find

$$-\frac{2Pa^2}{\rho c^2\omega^3}\frac{J_0(\omega r/a)}{J_1(\omega)}\cos(\omega ct/a).$$

Accordingly the complete displacement produced is

$$u=\frac{P}{\rho c^2}\left\{\tfrac{1}{4}(a^2-r^2)-\Sigma\frac{2a^2}{\omega^3}\frac{J_0(\omega r/a)}{J_1(\omega)}\cos(\omega ct/a)\right\}.$$

This result agrees with one given by Lord Rayleigh† for the associated problem of the membrane held at rest in the statical position $u=u_0$, and then released. Thus Lord Rayleigh's formula will give the value of u_0-u, in the present problem.

To identify the formulæ, we note that in Lord Rayleigh's notation

$$p_{m0}=k_{m0}c=\omega c/a$$

in the above formula. Thus

$$k_{m0}p_{m0}^2\rho a=\omega^3\rho c^2/a^2;$$

also we have

$$J_0'(\omega)=-J_1(\omega),$$

and the identification becomes evident.

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* Rayleigh, *Theory of Sound*, Vol. 1, Art. 200: the vibration is clearly symmetrical about the centre, and so is independent of θ.

† *Ibid.*, Art. 204, (9)–(11).

It may seem, at a casual glance, that the present solution is longer than Lord Rayleigh's (who uses the method of normal coordinates): but to obtain a fair comparison of the two, account must be taken of the preliminary analysis, carried out by Lord Rayleigh in Art. 203. In the present solution, no such analysis is needed.

6. *Alternative method for interpreting the formulæ obtained as in* § 5.

It is often possible to replace the expression for v (found by the method of § 5) by a series of terms such as

$$e^{-x\lambda}/\lambda^{n+1},$$

where x is real and n is an integer.

The corresponding contribution to the value of the integral

$$\frac{1}{2\pi\iota}\int ve^{\lambda t}\,d\lambda$$

is then $$\frac{(t-x)^n}{n!} \qquad (t>x),$$

or $$0 \qquad (t<x).$$

To prove the accuracy of these statements we need only notice that when $t>x$ (so that the index in the exponential is a *positive* multiple of λ), the path PAQ of the diagram on p. 411 can be completed by the additional piece QBP, which then contributes zero. Thus the value of the integral along PAQ is equal to the residue at $\lambda=0$, which may be calculated in the usual way and gives the result just stated.

But, when $t<x$, we may add on the additional piece QCP; and since no pole of the integrand is contained within $PAQCP$, the result is zero.

When constructing the series to represent a given function, care must be taken to remember that *the real part of* λ *is positive* on the fundamental path PAQ; and this enables us to settle questions of convergence without trouble, as a rule. For example,

$$\frac{1}{\sinh\lambda}=\frac{2}{e^{\lambda}-e^{-\lambda}}=2e^{-\lambda}(1-e^{-2\lambda})^{-1},$$

and this function can accordingly be expanded as

$$2e^{-\lambda}(1+e^{-2\lambda}+e^{-4\lambda}+\ldots),$$

since $|e^{-\lambda}|$ is less than unity.

In order to bring out the points of this method of interpretation, we shall work out in some detail the solution of a known problem (Ex. 12) in the theory of extensional vibrations of a thin rod. The problems already treated in § 5 (Exs. 9, 10) will then be solved, more briefly, below; it will be noticed that in these cases, the results obtained by the present process

are equivalent to those derived by the familiar process of superposing positive and negative waves.*

EX. 12.—*Bar with one end fixed, struck at the other end by a particle moving in the direction of the length of the bar.*†

Without loss of generality we may suppose the velocity of propagation of extensional waves in the bar to be the unit of velocity; and the length of the bar to be the unit of length: thus the unit of time is the interval occupied by a wave in travelling along the bar.

Then if u denotes the displacement along the bar, and if y is the displacement of the particle, the equations of motion will reduce to‡

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 y}{\partial t^2} = -k\left(\frac{\partial u}{\partial x}\right)_{x=1},$$

and the end-conditions become

$$u = 0 \text{ at } x = 0, \qquad u = y \text{ at } x = 1.$$

The first steps in the solution are precisely similar to those adopted in Exs. 9–11 in § 5, and need only brief indications. We assume the solutions

$$u = \frac{1}{2\pi\iota}\int v e^{\lambda t}\, d\lambda, \qquad y = \frac{1}{2\pi\iota}\int \eta e^{\lambda t}\, d\lambda,$$

where v, η are functions of λ to be determined. The equations for v, η are found to be

$$\lambda^2 v - \frac{\partial^2 v}{\partial x^2} = 0, \qquad \lambda^2 \eta + k\left(\frac{\partial v}{\partial x}\right)_{x=1} = -V,$$

because the initial conditions are

$$u = 0, \quad \frac{\partial u}{\partial t} = 0, \quad \text{and} \quad y = 0, \quad \frac{\partial y}{\partial t} = -V.$$

Then using the end conditions

$$v = 0 \quad \text{at} \quad x = 0, \qquad v = \eta \quad \text{at} \quad x = 1,$$

we have $$v = \eta\,(\sinh \lambda x)/(\sinh \lambda),$$

which leads to the result $$(\lambda^2 + k\lambda \coth \lambda)\,\eta = -V.$$

Thus the values of v, η are now completely known; and we proceed to obtain the interpreta-

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* Rayleigh, *Theory of Sound*, Vol. 1, Art. 145; of course this simple process cannot easily be extended to harder problems, such as Ex. 12.

† The problem was considered first by Boussinesq and Saint-Venant; see Love's *Elasticity* (Art. 281, 2nd edition; Arts. 275–277, 1st edition).

‡ These equations follow at once from those given by Love, on writing $a = 1$, $l = 1$; it should be noticed that here k denotes the ratio of the mass of the bar to the mass of the particle, so that $k = 1/m$ in Love's notation.

tion of u by means of the method described above. For the sake of comparison with the results given by Love, we shall write u in the form

$$u = \frac{1}{2\pi\iota}\int \eta \frac{\sinh \lambda x}{\sinh \lambda} e^{\lambda t}\, d\lambda = \frac{1}{2\pi\iota}\int \frac{\eta}{2\sinh\lambda}\left\{e^{\lambda(t+x)} - e^{\lambda(t-x)}\right\} d\lambda.$$

Thus we can express our solution by the equation

$$u = f(t-x) - f(t+x),$$

where $$f(\theta) = \frac{1}{2\pi\iota}\int \frac{-\eta e^{\lambda\theta}\, d\lambda}{2\sinh\lambda} = \frac{1}{2\pi\iota}\int \frac{V e^{\lambda\theta}\, d\lambda}{2\lambda(\lambda\sinh\lambda + k\cosh\lambda)}.$$

In accordance with our general process we now proceed to expand the integrand according to powers of $e^{-\lambda}$, using the identity*

$$\frac{1}{2(\lambda\sinh\lambda + k\cosh\lambda)} = \frac{e^{-\lambda}}{(\lambda+k)-(\lambda-k)e^{-2\lambda}}$$

$$= \frac{e^{-\lambda}}{\lambda+k}\left\{1 + \frac{\lambda-k}{\lambda+k}e^{-2\lambda} + \left(\frac{\lambda-k}{\lambda+k}\right)^2 e^{-4\lambda} + \ldots\right\}.$$

Thus our formula for $f(\theta)$ takes the shape

$$f(\theta) = \frac{1}{2\pi\iota}\int \frac{V e^{\lambda(\theta-1)} d\lambda}{\lambda(\lambda+k)}\left\{1 + \frac{\lambda-k}{\lambda+k}e^{-2\lambda} + \left(\frac{\lambda-k}{\lambda+k}\right)^2 e^{-4\lambda} + \ldots\right\}.$$

Now, as explained in our general remarks on p. 425, it is only necessary to retain the exponentials with *positive* indices: those with negative indices simply contribute zero to the result.

It is therefore clear that

$$f(\theta) = 0 \quad (\theta < 1),$$

and that $$f(\theta) = \frac{1}{2\pi\iota}\int \frac{V e^{\lambda(\theta-1)}\, d\lambda}{\lambda(\lambda+k)} \quad (1 < \theta < 3).$$

Now, calculating the residues at $\lambda = 0$ and $\lambda = -k$, we see at once that

$$f(\theta) = (V/k)\left\{1 - e^{-k(\theta-1)}\right\} \quad (1 < \theta < 3).$$

Similarly, when $3 < \theta < 5$, we take

$$f(\theta) = \frac{1}{2\pi\iota}\int \frac{V e^{\lambda(\theta-1)}\, d\lambda}{\lambda(\lambda+k)}\left(1 + \frac{\lambda-k}{\lambda+k}e^{-2\lambda}\right),$$

---

* Convergence is assured because the real part of λ is positive, and so

$$|e^{-2\lambda}| < 1, \quad |\lambda-k| < |\lambda+k|.$$

and here the residue for $\lambda = 0$ is zero, and so, calculating the residue for $\lambda = -k$, we find

$$f(\theta) = -(V/k)e^{-k(\theta-1)} + (V/k)\{1+2k(\theta-3)\}e^{-k(\theta-3)} \quad (3 < \theta < 5).$$

As a last example, suppose that $5 < \theta < 7$, so that

$$f(\theta) = \frac{1}{2\pi\iota}\int \frac{Ve^{\lambda(\theta-1)}\,d\lambda}{\lambda(\lambda+k)}\left\{1+\frac{\lambda-k}{\lambda+k}e^{-2\lambda}+\left(\frac{\lambda-k}{\lambda+k}\right)^2 e^{-4\lambda}\right\}.$$

Then, on calculating the residues for $\lambda = 0$ and $\lambda = -k$, we find the result

$$\begin{aligned} f(\theta) = {} & (V/k)\{1-e^{-k(\theta-1)}\} + (V/k)\{1+2k(\theta-3)\}e^{-k(\theta-3)} \\ & -(V/k)\{1+2k^2(\theta-5)^2\}e^{-k(\theta-5)} \qquad (5 < \theta < 7). \end{aligned}$$

These three formulæ will be seen to agree with those given by Love.*

It is sometimes of special interest, in problems of this type, to determine the displacement of one particular point: for instance, here we might wish to find the value of y (the displacement of the particle). When we have already found the forms of $f(\theta)$, this displacement is given by

$$y = f(t-1) - f(t+1).$$

But supposing that $f(\theta)$ has not been already worked out, we can find y by expanding η in the same way, and writing

$$\begin{aligned} -\eta &= \frac{V\sinh\lambda}{\lambda(\lambda\sinh\lambda + k\cosh\lambda)} \\ &= \frac{V(1-e^{-2\lambda})}{\lambda\{(\lambda+k)-(\lambda-k)e^{-2\lambda}\}} \\ &= \frac{V}{\lambda(\lambda+k)}\left\{1-\frac{2k}{\lambda+k}e^{-2\lambda}-\frac{2k(\lambda-k)}{(\lambda+k)^2}e^{-4\lambda}-\ldots\right\}. \end{aligned}$$

From this series the values of y can be determined in exactly the way in which we found the values of $f(\theta)$: but instead of writing out the results, we shall deal similarly with the problem of finding $(\partial u/\partial x)_{x=1}$, from which the duration of the impact may be found.

It will be seen that

$$\left(\frac{\partial u}{\partial x}\right)_{x=1} = \frac{1}{2\pi\iota}\int \lambda\eta\coth\lambda\, e^{\lambda t}\,d\lambda,$$

* See the table on p. 414 of the second edition: the corresponding table in the first edition is arranged a little differently, and gives the values of $f(1+\theta)$. It should be remembered that, in the units used here, $a = 1$, $l = 1$, and that $k = 1/m$.

and so we write

$$-\lambda\eta\coth\lambda = \frac{V\cosh\lambda}{\lambda\sinh\lambda+k\cosh\lambda}$$
$$= \frac{V}{\lambda+k}\left\{1+\frac{2\lambda}{\lambda+k}e^{-2\lambda}+\frac{2\lambda(\lambda-k)}{(\lambda+k)^2}e^{-4\lambda}+\ldots\right\}.$$

We have now to write down the residues for $\lambda=-k$, and it will be found that

$$-\left(\frac{\partial u}{\partial x}\right)_{x=1} = Ve^{-kt} \qquad (0<t<2),$$
$$= Ve^{-kt}+2V\{1-k(t-2)\}e^{-k(t-2)} \qquad (2<t<4),$$
$$= Ve^{-kt}+2V\{1-k(t-2)\}e^{-k(t-2)}$$
$$+2V\{1-3k(t-4)+k^2(t-4)^2\}e^{-k(t-4)} \qquad (4<t<6).$$

The first and second of these results agree with those given by Love; the third is not actually given by Love, but can be verified from results given there.

Before leaving this problem it may be useful to remark that any of the results can be expanded in terms of normal functions, if desired; suppose, for instance, that we want a series for y. The poles of η are given by $\lambda=\pm\iota\nu$, where ν is a positive root of

$$\nu-k\cot\nu=0.$$

The residue of $\eta e^{\lambda t}$ at $\lambda=\iota\nu$ is then seen to be

$$-Ve^{\iota\nu t}/\{\iota\nu(1+k\operatorname{cosec}^2\nu)\};$$

and so we find that $$y=-2V\Sigma\frac{\sin\nu t}{\nu(1+k\operatorname{cosec}^2\nu)},$$

where $$\nu-k\cot\nu=0.$$

This result may be confirmed by Lord Rayleigh's process.*

We pass now to a brief consideration of Exs. 9, 10 of § 5.

Ex. 9 (*bis*).—It was proved above that, when $0\leqslant x\leqslant\frac{1}{2}l$,

$$v=\frac{kx}{\lambda}-\frac{k}{\lambda^2}\frac{\sinh\lambda x}{\cosh\frac{1}{2}\lambda l}=\frac{kx}{\lambda}-\frac{k}{\lambda^2}\left\{e^{-(\frac{1}{2}l-x)\lambda}-e^{-(\frac{1}{2}l+x)\lambda}\right\}(1-e^{-l\lambda}+e^{-2l\lambda}-\ldots),$$

the convergence being assured because $|e^{-l\lambda}|<1$, on account of the real part of λ being positive.

In applying the formulæ given above to find the residues of $ve^{\lambda t}$, it may be noticed that each successive result is derived from the preceding by adding on an extra term—corresponding to a new index of the exponential which becomes positive as t increases.

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* *Theory of Sound*, Vol. 1, Arts. 93, 94.

Thus we find

(i) $u = kx$ $(0 < t < \frac{1}{2}l-x)$.

(ii) $u = kx \quad -k(t-\frac{1}{2}l+x) = k(\frac{1}{2}l-t)$ $(\frac{1}{2}l-x < t < \frac{1}{2}l+x)$.

(iii) $u = k(\frac{1}{2}l-t)+k(t-\frac{1}{2}l-x) = -kx$ $(\frac{1}{2}l+x < t < \frac{3}{2}l-x)$.

(iv) $u = -kx \quad +k(t-\frac{3}{2}l+x) = k(t-\frac{3}{2}l)$ $(\frac{3}{2}l-x < t < \frac{3}{2}l+x)$.

(v) $u = k(t-\frac{3}{2}l)-k(t-\frac{3}{2}l-x) = kx$ $(\frac{3}{2}l+x < t < \frac{5}{2}l-x)$,

and so on.

These results are calculated for a given value of x, corresponding to different values of t: but it is rather easier to appreciate the motion of the string by classifying the results according to the values of x, for a given t. We then arrive at the formulæ :—

(*a*) $0 < t < \frac{1}{2}l$, $u = kx$, if $x < \frac{1}{2}l-t$,
$u = k(\frac{1}{2}l-t)$, if $x > \frac{1}{2}l-t$.

(*b*) $\frac{1}{2}l < t < l$, $u = -kx$, if $x < t-\frac{1}{2}l$,
$u = -k(t-\frac{1}{2}l)$, if $x > t-\frac{1}{2}l$.

(*c*) $l < t < \frac{3}{2}l$, $u = -kx$, if $x < \frac{3}{2}l-t$,
$u = -k(\frac{3}{2}l-t)$, if $x > \frac{3}{2}l-t$.

(*d*) $\frac{3}{2}l < t < 2l$, $u = kx$, if $x < t-\frac{3}{2}l$,
$u = k(t-\frac{3}{2}l)$, if $x > t-\frac{3}{2}l$.

(*e*) $2l < t < \frac{5}{2}l$, $u = kx$, if $x < \frac{5}{2}l-t$,
$u = k(\frac{5}{2}l-t)$, if $x > \frac{5}{2}l-t$,

and so on; where it must be borne in mind that $0 \leqslant x \leqslant \frac{1}{2}l$. It is not necessary to write out the formulæ which refer to the second half of the string; for the shape of the string is always symmetrical about its centre.

From these formulæ it is easy to construct a diagram to shew the motion of the string: the string in fact consists always of three straight pieces.*

Ex. 10 (*bis*).—We found that

$$v = \frac{u_1}{\lambda^2}\left\{1-\frac{\cosh\lambda(l-r)}{\cosh\lambda l}\right\} = \frac{u_1}{\lambda^2}\left[1-\left\{e^{-x}+e^{-(2l-x)}\right\}(1-e^{-2l\lambda}+e^{-4l\lambda}-\ldots)\right].$$

Proceeding, as before, we find for the displacements

(i) $u_1 t$ $(0 < t < x)$.

(ii) $u_1 t \quad -u_1(t-x) \quad = u_1 x$ $(x < t < 2l-x)$.

(iii) $u_1 x \quad -u_1(t-2l+x) = u_1(2l-t)$ $(2l-x < t < 2l+x)$.

(iv) $u_1(2l-t)+u_1(t-2l-x) = -u_1 x$ $(2l+x < t < 4l-x)$.

(v) $-u_1 x \quad +u_1(t-4l+x) = u_1(t-4l)$ $(4l-x < t < 4l+x)$,

and so on.

These conclusions may be summed up as follows:—A wave-front may be regarded as travelling to and fro, along the tube. The velocity of the air is zero in the part between the fixed end and the wave-front; and in the part between the wave-front and the open end, the velocity is $\pm u_1$. The latter changes sign when the wave-front is reflected at the open end.

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* See Helmholtz's diagram for the plucked string: Rayleigh's *Theory of Sound*, Vol. 1, Art. 146. In the special problem considered here, however, the central portion is always parallel to the line joining the ends.

7. *Waves with an Advancing Wave-Front.*

As another group of problems which can be readily solved by the method of § 6, we may refer to those given by Prof. Love* on wave-systems with an advancing wave-front. To illustrate the process we shall work out the simplest case: some of the other problems are rather more interesting, but the algebra involved is heavier.

We consider, then, the problem of *waves communicated to air by a sphere which vibrates radially under the action of some internal mechanism.* Let ξ denote the radial displacement of the surface of the sphere, and ϕ the velocity-potential in the air, while $2\pi/n$ is the period of oscillation of the sphere under the action of the mechanism alone. Then the equations of motion are†

$$\left.\begin{aligned} \sigma\left(\frac{\partial^2\xi}{\partial t^2}+n^2\xi\right)-\rho\frac{\partial\phi}{\partial t}=0 \\ \frac{\partial\xi}{\partial t}-\frac{\partial\phi}{\partial r}=0 \end{aligned}\right\}, \text{ at } r=b, \tag{42}$$

and

$$\frac{\partial^2}{\partial t^2}(r\phi)=\frac{\partial^2}{\partial r^2}(r\phi), \quad \text{if} \quad r>b, \tag{43}$$

where the velocity of sound is taken as unity, and σ is the surface-density of the sphere, ρ the density of air, and b the mean radius of the sphere.

In the problem considered the air is supposed initially undisturbed, while the sphere may have both displacement and velocity initially, say ξ_0 and ξ_1.

If we solve by writing

$$\xi=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\eta\, e^{\lambda t}d\lambda, \qquad \phi=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\psi\, e^{\lambda t}d\lambda,$$

the previous methods lead to the equations, derived from (42), (43),

$$\left.\begin{aligned} \sigma(\lambda^2+n^2)\eta-\rho\lambda\psi=\sigma(\lambda\xi_0+\xi_1) \\ \lambda\eta-\frac{\partial\psi}{\partial r}=\xi_0 \end{aligned}\right\}, \text{ at } r=b, \tag{44}$$

and

$$\lambda^2(r\psi)-\frac{\partial^2}{\partial r^2}(r\psi)=0, \quad r>b. \tag{45}$$

* These *Proceedings*, Ser. 2, Vol. 2, 1904, p. 88.

† *L.c.*, p. 94, equations (9) and (10).

Since the real part of λ is positive, and the last equation (45) is to hold for all values of r greater than b, we must take

$$r\psi = Ae^{-\lambda r}.$$

Then, substituting in the previous equations, we find that

$$\{(\lambda^2+n^2)(\lambda b+1)+(\rho/\sigma)\lambda^2 b\}A = b^2 e^{\lambda b}(n^2\xi_0-\lambda\xi_1). \tag{46}$$

Hence the value of ϕ at any point is given by the integral

$$r\phi = \frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} \frac{e^{\lambda(b+t-r)}\, b^2(n^2\xi_0-\lambda\xi_1)}{(\lambda^2+n^2)(\lambda b+1)+(\rho/\sigma)\lambda^2 b}\, d\lambda. \tag{47}$$

By what has been explained in § 6, the value of ϕ is zero so long as $b+t<r$; and when $b+t>r$, the value is

$$r\phi = \Sigma \frac{e^{\lambda_1(b+t-r)}\, b(n^2\xi_0-\lambda_1\xi_1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \tag{48}$$

where λ_1, λ_2, λ_3 are the roots of the denominator in (47). This result (48) agrees with Prof. Love's formula (19) for the special case $\xi_0 = 0$.

If we neglect the effect of the term in ρ/σ in the denominator of (47), the roots are $\lambda = \pm\iota n$ and $1/b$. It is easy to calculate the residues, and we obtain the formula

$$r\phi = \frac{b^2}{1+n^2b^2}[(n^2b\xi_0+\xi_1)\, e^{-(b+t-r)/b} - (n^2b\xi_0+\xi_1)\cos\{n(b+t-r)\} + n(\xi_0-b\xi_1)\sin\{n(b+t-r)\}]. \tag{49}$$

The formula (49) agrees with that given on p. 97 of Prof. Love's paper, except that in the latter the sign of ϕ is left ambiguous. It is easy to confirm the sign of (49) by observing that, when $t=0$ and $r=b$, it leads to the correct result $\partial\phi/\partial r = \xi_1$; moreover, in the special case $\xi_0 = 0$, (49) reduces to Prof. Love's formula (20).

As remarked by Sir Joseph Larmor (see Prof. Love's footnote on p. 97), the exponential pulse in (49) disappears if

$$n^2b\xi_0+\xi_1 = 0, \tag{50}$$

and can only disappear under this special condition. This conclusion can be foreseen from the integral (47) for ϕ: for the exponential term disappears if (and only if) the factor $\lambda b+1$ cancels out of the numerator and denominator in (47); which gives the same condition (50) again. The harmonic wave is then given by

$$r\phi = b^2 n\xi_0 \sin n(b+t-r),$$

agreeing with Prof. Love's formula (21a).

It is now an easy matter to deal with the problem of *the waves sent out by the sphere when acted on by a periodic force for a definite time.* Prof. Love has given materials for the solution of this problem (*l.c.*, pp. 98–100), but has not completed the details.

To obtain a definite result, suppose that the force applied is represented by $\sigma F \sin pt$, and acts for the half period $t = 0$ to $t = \pi/p$. Then (see Ex. 6, p. 413 above) we replace equations (44) by

$$\left.\begin{aligned} \sigma(\lambda^2+n^2)\eta - \rho\lambda\psi &= \frac{pF\sigma}{\lambda^2+p^2}(1+e^{-\lambda\pi/p}) \\ \lambda\eta - \frac{\partial\psi}{\partial r} &= 0 \end{aligned}\right\}, \text{ at } r = b. \qquad (51)$$

From (45) and (51) we deduce the formulæ

$$r\psi = A_1 e^{-\lambda r},$$

$$\text{where} \quad \{(\lambda^2+n^2)(\lambda b+1)+(\rho/\sigma)\lambda^2 b\}\, A_1 = -\frac{pF\lambda b^2 e^{\lambda b}}{\lambda^2+p^2}(1+e^{-\lambda\pi/p}). \qquad (52)$$

Hence the value of ϕ is given by

$$r\phi = -\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty} \frac{e^{\lambda(b+t-r)}(1+e^{-\lambda\pi/p})}{\lambda^2+p^2}\,\frac{pF\lambda b^2 d\lambda}{(\lambda^2+n^2)(\lambda b+1)+(\rho/\sigma)\lambda^2 b}, \qquad (53)$$

from which the residues can be written down as before. It should be noticed that (as remarked by Prof. Love) the waves of period $2\pi/p$ disappear as soon as $b+t-r$ exceeds π/p; for then both exponentials in the numerator of (53) must be used in forming the residues, and their contribution at the points $\lambda = \pm \iota p$ will cancel each other.*

One further point may be noticed: since the numerator in (53) cannot be made to vanish for a real value of λ (other than $\lambda = 0$), it follows that *the exponential pulse can never disappear;* this, of course, is in contrast to the special case represented by (50) above.

Ex. 13.—The corresponding problem for *a rigid sphere set moving in a straight line* has been proposed by Prof. Love, and the solution was completed by Prof. Lamb.†

It may suffice to state the results obtained by the present method, supposing the initial velocity of the sphere to be U. It will be found that the velocity potential is given by

$$\phi = \frac{\partial\chi}{\partial r}\cos\theta,$$

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* Because $(1+e^{-\lambda\pi/p})$ is zero for these values of λ.

† Love, *l.c.*, pp. 100–102: Lamb, *Hydrodynamics*, 1906, p. 497.

where $$\chi = \frac{(Ub^3/r)}{2\pi\iota} \int_{c-\iota\infty}^{c+\iota\infty} \frac{e^{\lambda(t+b-r)}\lambda\, d\lambda}{(\lambda^2+n^2)(2+\lambda b+\lambda^2 b^2)+\beta\lambda^2(1+\lambda b)},$$

where β is the ratio of the mass of air displaced to the mass of the sphere.

It will be seen at a glance that this leads to a formula of the type given in Lamb's equation (22): on taking $\beta = 0$ and completing the calculation of the residues, it will be found that we obtain Lamb's result (25), except for some small numerical discrepancies.*

8. *Further Details of the Theory for One Type of Continuous Systems.*

We confine the following to problems in which one geometrical coordinate only is required to specify both the dependent and the independent variables. We denote the dependent variable by u, and the independent by x. We shall suppose the system to be a dynamical one specified by a system of energy functions

$$2T = \int_0^l \rho \left(\frac{\partial u}{\partial t}\right)^2 dx, \qquad 2V = \int_0^l p \left(\frac{\partial u}{\partial x}\right)^2 dx,$$

and a dissipation function

$$F = \int_0^l \sigma \left(\frac{\partial u}{\partial t}\right)^2 dx,$$

where ρ, σ, p may be functions of x, but will be supposed to be essentially positive. Thus the system will correspond to the type of *symmetrical* systems considered in § 1.

With regard to the mechanism at the ends $x = 0$, $x = l$, the most natural assumption is to include terms representing the ends in each of the functions T, V, F. Thus, if the end-displacements are denoted by y_0, y_1 respectively,† we might include the terms

$$m_0 \left(\frac{\partial y_0}{\partial t}\right)^2 + m_1 \left(\frac{\partial y_1}{\partial t}\right)^2, \qquad q_0 y_0^2 + q_1 y_1^2, \qquad k_0 \left(\frac{\partial y_0}{\partial t}\right)^2 + k_1 \left(\frac{\partial y_1}{\partial t}\right)^2,$$

in $2T$, $2V$, and F, respectively. The first and third of these can, however, be regarded as included in the corresponding integrals by supposing that ρ and σ tend to infinity at the ends $x = 0$, $x = l$: and, to avoid complicated statements in reference to the end-conditions, we shall make this hypothesis in our general theory, instead of using the terms containing m_0, m_1, k_0, k_1. Of course we cannot use this device for the solution of

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* It appears to me that these have arisen in the course of Lamb's work, as I have checked the results obtained from the complex integral, by using the older method.

† Note that the symbols u_0, u_1 are used below to denote the *initial values* of u and $\partial u/\partial t$ (as in § 5).

practical problems ; instead, we should first write down the actual end-conditions and then modify them according to our regular process ; for an illustration, see Ex. 12 in § 6 (in particular, the small type on p. 426).

Thus in future we shall adopt the specification

$$\left.\begin{aligned} 2T &= \int_0^l \rho \left(\frac{\partial u}{\partial t}\right)^2 dx, \qquad F = \int_0^l \sigma \left(\frac{\partial u}{\partial t}\right)^2 dx \\ 2V &= \int_0^l p \left(\frac{\partial u}{\partial x}\right)^2 dx + q_0 y_0^2 + q_1 y_1^2 \end{aligned}\right\}. \tag{54}$$

Forming the variational equation of motion in the usual way from (54), we deduce the equation

$$\rho \frac{\partial^2 u}{\partial t^2} + \sigma \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) = 0, \tag{55}$$

with the end-conditions

$$q_0 y_0 - \left(p \frac{\partial u}{\partial x}\right)_0 = 0, \tag{56}$$

$$q_1 y_1 + \left(p \frac{\partial u}{\partial x}\right)_1 = 0. \tag{57}$$

Proceeding as in § 5, we obtain, as a conjectural solution of (55),

$$u = \frac{1}{2\pi\iota} \int_{c-\iota\infty}^{c+\iota\infty} v e^{\lambda t} d\lambda, \tag{58}$$

where* $$(\rho\lambda^2 + \sigma\lambda) v - \frac{\partial}{\partial x}\left(p \frac{\partial v}{\partial x}\right) = (\rho\lambda + \sigma) u_0 + \rho u_1, \tag{59}$$

and v is to satisfy conditions of the same form† as (56), (57).

In practical work it is usually easiest to solve the equation (59) *directly*, just as we have done in the various examples considered above : because the functions selected for the initial values u_0, u_1 are ordinarily of a simple character. But *in theory* it is better to reduce the solution of (59) to a quadrature. Suppose, then, that we consider the associated equation

$$(\rho\lambda^2 + \sigma\lambda) w - \frac{\partial}{\partial x}\left(p \frac{\partial w}{\partial x}\right) = 0, \tag{60}$$

* Here u_0, u_1 denote the *initial values* of u and $\partial u/\partial t$.

† When inertia and frictional terms occur in the end-conditions, care must be taken to introduce the proper terms on the right in forming the equations for v.

and let $w = \phi(x)$ be a solution of (60) which satisfies the end-condition corresponding to (56), while $w = \psi(x)$ is a second solution satisfying the end-condition corresponding to (57).

Then we can solve (59) by the method of "variation of parameters," assuming that

$$v = A\phi(x) + B\psi(x), \tag{61}$$

where A, B are functions of x. If we assume further that*

$$0 = A'\phi(x) + B'\psi(x), \tag{62}$$

it is easy to see that (59) becomes

$$-p\{A'\phi'(x) + B'\psi'(x)\} = (\rho\lambda + \sigma)u_0 + \rho u_1. \tag{63}$$

Now, since $\phi(x)$, $\psi(x)$ both satisfy the same equation (60), we have

$$p\{\phi'(x)\psi(x) - \psi'(x)\phi(x)\} = \text{const.} = \Delta(\lambda), \tag{64}$$

say.† Using (62), (63) and (64), we now find

$$\frac{A'}{-\psi(x)} = \frac{B'}{\phi(x)} = \frac{(\rho\lambda + \sigma)u_0 + \rho u_1}{\Delta(\lambda)}. \tag{65}$$

We have next to consider the effect of the end-conditions: since $\phi(x)$ satisfies the condition at $x = 0$, which is also to be satisfied by v, it follows‡ that B must vanish at $x = 0$; and similarly A must vanish at $x = l$. Thus, in consequence of (65), we can write

$$A = \frac{1}{\Delta(\lambda)} \int_x^l \{(\rho\lambda + \sigma)u_0(\xi) + \rho u_1(\xi)\}\psi(\xi)\,d\xi,$$

and

$$B = \frac{1}{\Delta(\lambda)} \int_0^x \{(\rho\lambda + \sigma)u_0(\xi) + \rho u_1(\xi)\}\phi(\xi)\,d\xi.$$

Hence we can reduce v to the form

$$v = \int_0^l \{(\rho\lambda + \sigma)u_0(\xi) + \rho u_1(\xi)\}\, G(x, \xi)\,d\xi, \tag{66}$$

---

* Accents denote differentiation with regard to x.

† Since λ occurs in both $\phi(x)$ and $\psi(x)$, it is clear that (64) is a function of λ. The notation $\Delta(\lambda)$ is used because this function now plays a part corresponding to the determinant (11) used in §§ 1 and 3 (for details see § 9).

‡ Note that $v' = A\phi'(x) + B\psi'(x)$, in consequence of (62).

where we have written

$$\left.\begin{aligned} G(x,\xi) &= \frac{\phi(x)\,\psi(\xi)}{\Delta(\lambda)} \quad (0<x<\xi<l) \\ \text{and} \qquad &= \frac{\phi(\xi)\,\psi(x)}{\Delta(\lambda)} \quad (0<\xi<x<l) \end{aligned}\right\}. \tag{67}$$

The function $G(x, \xi)$ is equivalent to the function often called the *Green's function** of the equation (60); this function is continuous as x varies from 0 to l, but its first differential coefficient has a discontinuity at $x = \xi$, given by

$$p\left(\frac{\partial G}{\partial x}\right)_{\xi-0} - p\left(\frac{\partial G}{\partial x}\right)_{\xi+0} = \frac{p}{\Delta(\lambda)}\{\phi'(\xi)\,\psi(\xi) - \phi(\xi)\,\psi'(\xi)\} = 1. \tag{68}$$

Further $G(x, \xi)$ is symmetrical in the two arguments x, ξ; and satisfies equation (60) with the end-conditions analogous to (56), (57).

It may be worth while to call attention to the fact that equation (66) is really analogous to equation (10) of § 1, which gives the solution of a set of linear equations by means of a sum: here we get an integral as the solution of a differential equation.

It may be convenient to mention some of the applications of the function $G(x, \xi)$ which present themselves naturally in the present theory :—

1. *If an impulse J is applied at the point $x = \xi$,* the solution is given by

$$v = J.G(x, \xi).$$

For then u_0 is everywhere zero, and u_1 is zero except near $x = \xi$: further we have the relation

$$\int_0^l \rho u_1 d\xi = \text{the total momentum of the blow} = J;$$

and accordingly (66) gives the required result.

2. *For a constant force P steadily maintained at the point $x = \xi$,* we may take

$$v = P.G(x, \xi)/\lambda.$$

This follows at once from the last result, using the method by which the solution of § 2 is derived from that of § 1.

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* Picard, *Traité d'Analyse*, t. 3 ; Burkhardt, *Bull. Soc. Math. de France*, t. 22, 1894 ; Hilbert, *Integralgleichungen*, 1912, pp. 40–42.

3. *For a force $Pe^{\mu t}$ applied at $x = \xi$*, we take similarly

$$v = P.G(x, \xi)/(\lambda - \mu).$$

EXAMPLES.—It will perhaps tend to clear ideas if the forms taken by $G(x, \xi)$ are stated in a few simple cases (corresponding to $\sigma = 0$, $p = \rho a^2$).

1. A string with fixed ends $(x = 0, x = l)$:

$$G(x, \xi) = \frac{\sinh(\lambda x/a) \sinh\{\lambda(l-\xi)/a\}}{\rho a \lambda \sinh(\lambda l/a)}.$$

2. A pipe with two open ends $(x = 0, x = l)$:

$$G(x, \xi) = \frac{\cosh(\lambda x/a) \cosh\{\lambda(l-\xi)/a\}}{\rho a \lambda \sinh(\lambda l/a)}.$$

3. A pipe with one end closed $(x = 0)$ and one end open $(x = l)$:

$$G(x, \xi) = \frac{\sinh(\lambda x/a) \cosh\{\lambda(l-\xi)/a\}}{\rho a \lambda \cosh(\lambda l/a)}.$$

4. A string with both ends attached to springs (contributing potential energy but not kinetic energy):

$$G(x, \xi) = \frac{\{\theta_0 \sinh(\lambda x/a) + \lambda \cosh(\lambda x/a)\}\left[\theta_1 \sinh\{\lambda(l-\xi)/a\} + \lambda \cosh\{\lambda(l-\xi)/a\}\right]}{\rho a \lambda\{(\theta_0\theta_1 + \lambda^2) \sinh(\lambda l/a) + (\theta_0 + \theta_1)\lambda \cosh(\lambda l/a)\}},$$

where $\rho a \theta_0 = q_0$, $\rho a \theta_1 = q_1$ denote the strengths of the springs.

In each of these examples one only of the two forms for $G(x, \xi)$ has been written out (corresponding to $0 < x < \xi < l$). The other form is obtained by interchanging x and ξ.

5. To illustrate the application of these forms to physical cases, we might consider *the effect of an impulse J applied to a string with fixed ends.*

We have here to use the first of the foregoing functions, and we multiply by J in accordance with (1) of the previous page. Thus we take

$$v = \frac{J}{\rho a \lambda} \frac{\sinh(\lambda x/a) \sinh\{\lambda(l-\xi)/a\}}{\sinh(\lambda l/a)}.$$

It is obvious that $\lambda = 0$ is not a pole of $ve^{\lambda t}$; and that the poles correspond to

$$\lambda l/a = n\pi\iota$$

where n is a positive or negative integer. The residue of $ve^{\lambda t}$ is thus

$$-\frac{Je^{\lambda t}}{\rho a \lambda} \frac{\sinh(\lambda x/a) \sinh(\lambda \xi/a)}{l/a} = \frac{J}{l\rho} \sin\frac{n\pi x}{l} \sin\frac{n\pi\xi}{l} \frac{e^{\iota(n\pi a t/l)}}{n\pi a\iota/l}.$$

Thus, on combining together the positive and negative terms, we obtain

$$\frac{2J}{l\rho} \Sigma \sin\frac{n\pi x}{l} \sin\frac{n\pi\xi}{l} \frac{\sin(n\pi a t/l)}{n\pi a/l},$$

as in Rayleigh's *Theory of Sound*, Vol. 1, Art. 129.

We have still to examine the question as to whether the solution derived from (58) and (66) really does satisfy the prescribed initial conditions. As we have already remarked, anything approaching a complete

proof is out of the question here ;* but, by reference to some of the results contained in Prof. A. C. Dixon's paper on expansions in series of oscillating functions,† we can at least sketch some outlines of a proof.

In the first place it must be remembered that in calculating the initial values of u and of $\partial u/\partial t$ from the integrals

$$u = \frac{1}{2\pi\iota}\int v e^{\lambda t}\, d\lambda, \qquad \frac{\partial u}{\partial t} = \frac{1}{2\pi\iota}\int \lambda v e^{\lambda t}\, d\lambda,$$

we must (as explained on p. 412) regard the initial values as obtained by making t tend to zero through positive values. We can accordingly (for this purpose) replace our standard path by the limit of a closed path such as $PAQBP$ (in the figure, p. 411) : and so the initial values are given by

$$u = \frac{1}{2\pi\iota}\int v\, d\lambda, \qquad \frac{\partial u}{\partial t} = \frac{1}{2\pi\iota}\int \lambda v\, d\lambda, \tag{69}$$

each being evaluated for a closed path such as $PAQBP$, which afterwards must be supposed to extend to infinity.

It will appear from the formulæ to be obtained for v that these integrals remain convergent when taken round the arc PCQ ;‡ and so by Cauchy's theorem (since v has no poles inside the space $PCQAP$), we can regard the integrals (69) as taken round the path which is the limit of the circular path $PCQBP$.

The advantage of this preliminary transformation of the path of integration lies in the fact that we can now restrict our discussion of v to those values of λ, for which $|\lambda|$ is large. And, for such values of λ, Prof. Dixon has proved that (at least when $\sigma = 0$) the order of magnitude of the functions can be estimated by treating ρ, p as having constant values (averaged over their actual range of values).§ It is not very easy to quote conveniently Prof. Dixon's actual estimates, on account of differences of

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* To indicate the nature of the difficulties, it is perhaps enough to remark that the simplest case of all (with ρ, p constant, and $\sigma = 0$) leads to Fourier's theorem.

† These *Proceedings*, Ser. 2, Vol. 3, 1905, p. 83.

‡ It should be carefully noticed that this statement is *not* true as long as the factor $e^{\lambda t}$ forms part of the integrand.

§ As will be seen from Prof. Dixon's work, the proof of the sufficiency of this estimate is one of the chief difficulties of the discussion.

notation:* but it is easy to apply the same principle to our problem.†

Now, if ρ, σ, p are constants, the two functions $\phi(x)$, $\psi(x)$ will satisfy the differential equation

$$\frac{d^2v}{dx^2} - \mu^2 v = 0,$$

where $$\mu^2 = (\rho\lambda^2 + \sigma\lambda)/p.$$

Thus, when $|\lambda|$ is large, we can express μ in the form

$$\mu = \frac{\lambda}{a}\left(1 + \frac{b}{\lambda} + \frac{c}{\lambda^2} + \ldots\right),$$

where a is positive; and the functions $\phi(x)$, $\psi(x)$ will be of the form

$$A\cosh\mu x + B\sinh\mu x,$$

the values of the coefficients depending on the end conditions of the problem. A reference to the examples on p. 438 will show the various forms of $G(x, \xi)$; it is only necessary to replace $\lambda x/a$ by μx, and so on.

Now when the real part of μ is large and positive (so that the same is true of the real part of λ), the most important term both in $\cosh\mu x$ and in $\sinh\mu x$ will be

$$\tfrac{1}{2}e^{\mu x},$$

because x is positive. If we make this and corresponding substitutions in the formulæ of p. 438 for $G(x, \xi)$, we readily see that the leading term in

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* Prof. Dixon's value of λ is equivalent to our $-\lambda^2$: and his differential equation is expressed in a more precise standard form than our (60), involving changes of both dependent and independent variables.

† To indicate (as far as we can) the corresponding functions, the following table may be useful:—

T. J. I'A. B.	A. C. D.
$\phi(x)$	$\theta_0\phi + \psi$, with $\theta_0 = q_0/p_0$
$\psi(x)$	$\theta_1\chi - \omega$, with $\theta_1 = q_1/p_1$
$G(x, \xi)$	$\Omega(x, \xi, \lambda)$ (see p. 89), with $E = \theta_0\theta_1$, $G = \theta_1$, $H = \theta_0$, $K = 1$, $L = 0$

$G(x, \xi)$ is given by*

$$\frac{\frac{1}{2}e^{\mu x} \cdot \frac{1}{2}e^{\mu(l-\xi)}}{\rho a\lambda \cdot \frac{1}{2}e^{\mu l}} = \frac{1}{2\rho a\lambda} e^{-\mu(\xi-x)},$$

when the real part of μ is positive, and $0 < x < \xi < l$.

When the real part of μ is negative, the sign of the exponential in the last formula must be reversed; and similarly when $\xi < x$. Now, differentiating, and remembering the relation between μ and λ, we see that the leading term in $\partial G/\partial \xi$ is

$$\frac{1}{2\rho a^2} e^{\mp\mu(\xi-x)} = \frac{1}{2p} e^{\mp\mu(\xi-x)}.$$

Hence our final estimate of the order of magnitude, when $|\lambda|$ is large, is expressed by the equation

$$\left| p \frac{\partial G}{\partial \xi} \right| = \tfrac{1}{2}e^{-\Lambda}, \tag{70}$$

where $$\Lambda = \kappa|\xi - x|,$$

and κ is the real part of λ, taken with a positive sign.

Again, since $G(x, \xi)$ is a solution of equation (60), it follows that†

$$\int (\rho\lambda^2 + \sigma\lambda) G \, d\xi = p \frac{\partial G}{\partial \xi};$$

and if we take the integral between limits a, b, such that

$$0 \leqslant a < x < b \leqslant l,$$

we find
$$\int_a^b (\rho\lambda^2 + \sigma\lambda) \, G \, d\xi = \left[p \frac{\partial G}{\partial \xi} \right]_a^{x-0} + \left[p \frac{\partial G}{\partial \xi} \right]_{x+0}^b$$
$$= 1 - \left[p \frac{\partial G}{\partial \xi} \right]_{\xi=a} + \left[p \frac{\partial G}{\partial \xi} \right]_{\xi=b},$$

where we have made use of (68) in the form obtained by interchanging x and ξ.

Now, referring to (70), it is clear that the terms given by $\xi = a$,

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* A little consideration will shew that this conclusion remains correct whatever combination of end-conditions is required.

† Since $G(x, \xi)$ is symmetrical in the two variables, the parts played by x and ξ may be interchanged.

$\xi = b$ in the last equation must tend to zero exponentially as κ tends to infinity; we shall indicate this by writing the equation in the form

$$\int_a^b (\rho\lambda^2+\sigma\lambda) G\, d\xi = 1+O(e^{-\Lambda}), \tag{71}$$

where Λ denotes any positive multiple of κ, such as $\kappa(x-a)$ or $\kappa(b-x)$.

Similarly we prove that

$$\int_0^a (\rho\lambda^2+\sigma\lambda) G\, d\xi = O(e^{-\Lambda}), \qquad \int_b^l (\rho\lambda^2+\sigma\lambda) G\, d\xi = O(e^{-\Lambda}), \tag{72}$$

We can now estimate the order of v as derived from (66): assuming that the functions $u_0(\xi)$ and $u_1(\xi)$ satisfy Dirichlet's conditions (as laid down for Fourier's theorem), we can apply the second theorem of the mean* (first dividing the range of integration at the point $\xi = x$). The result is

$$\begin{aligned} v = \quad & u_0(0) \int_0^a (\rho\lambda+\sigma) G\, d\xi + u_0(x) \int_a^x (\rho\lambda+\sigma) G\, d\xi \\ & + u_0(x) \int_x^b (\rho\lambda+\sigma) G\, d\xi + u_0(l) \int_b^l (\rho\lambda+\sigma) G\, d\xi \\ & + \text{similar terms arising from } u_1. \end{aligned}$$

Applying (71) and (72), we deduce that

$$v = \frac{1}{\lambda} \{u_0(x)+O(e^{-\Lambda})\} + \frac{1}{\lambda^2} \left\{ u_1(x)+O(e^{-\Lambda})+O\left(\frac{1}{\lambda}\right)\right\}. \tag{73}$$

If we substitute from (73) in (69), and then integrate round a circle whose radius afterwards tends to infinity, we now see that the initial values of u and $\partial u/\partial t$ are respectively equal to $u_0(x)$ and $u_1(x)$. This justifies the claims made for the solution given by (58) and (66).

One further point should be mentioned in which this discussion is not quite complete. The terms which have been written as $O(e^{-\Lambda})$ tend to zero when κ tends to infinity: but there are certain minor arcs† of the circle on which these terms need not tend to zero: these arcs are (comparatively speaking) short, but their effect ought not to be overlooked. Prof. Dixon (*l.c.*, §§ 14–19) has shewn that in his problem the contributions of the minor arcs tend to zero: and I anticipate that a similar proof could be made here,

* Strictly speaking the functions u_0 and u_1 must be monotonic from $x = 0$ to $x = l$, to allow the use of the second mean value theorem; but under Dirichlet's conditions u_0 and u_1 can each be expressed as the difference of two monotonic functions. Accordingly the final conclusion (73) will remain valid.

† In the figure of p. 411, these arcs surround the points P and Q.

but I have not actually gone into the details of a general proof—although in special examples the discussion of the minor arcs can always be carried out by reasoning of a type which is of constant occurrence in elementary work with complex integrals.*

9. *Comparison of the Results obtained in* § 8 *with those of the Method of Normal Functions.*

In view of what has been proved already (in § 3) for the case of discrete systems, it is natural to expect that the formulæ of § 8 will again lead to the same results as are found by the use of normal functions. It will be instructive, however, to trace the corresponding steps in the proof.

We begin then with the roots $\alpha_1, \alpha_2, \alpha_3, \ldots$ of the equation $\Delta(\lambda) = 0$, where $\Delta(\lambda)$ is defined by equation (64). For each such root we have special values of the functions ϕ, ψ: denote them in order by $\phi_1, \phi_2, \phi_3, \ldots$ and $\psi_1, \psi_2, \psi_3, \ldots$. Then, in virtue of equation (64), we have

$$\phi_1/\psi_1 = \text{const.} = c_1, \qquad \phi_2/\psi_2 = \text{const.} = c_2, \quad \&c. \tag{74}$$

Accordingly the special functions $\phi_1, \phi_2, \phi_3, \ldots$ (or $\psi_1, \psi_2, \psi_3, \ldots$) satisfy equation (60) with $\lambda = \alpha_1, \alpha_2, \alpha_3, \ldots$, and moreover satisfy *both* the end-conditions (56) and (57). *These functions are the normal functions of the problem.*

We shall now obtain *the conjugate property of the normal functions.* In fact we have

$$\left.\begin{aligned}(\rho\alpha_1^2+\sigma\alpha_1)\,\phi_1-\frac{\partial}{\partial x}\left(p\,\frac{\partial\phi_1}{\partial x}\right) &= 0,\\ (\rho\alpha_2^2+\sigma\alpha_2)\,\phi_2-\frac{\partial}{\partial x}\left(p\,\frac{\partial\phi_2}{\partial x}\right) &= 0,\end{aligned}\right\}$$

and so $$(\alpha_1^2-\alpha_2^2)\,\rho\phi_1\phi_2+(\alpha_1-\alpha_2)\sigma\phi_1\phi_2 = \frac{\partial}{\partial x}\left\{p\left(\phi_2\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_2}{\partial x}\right)\right\}.$$

Now integrate from 0 to l, and we have

$$(\alpha_1^2-\alpha_2^2)\int_0^l \rho\phi_1\phi_2\,dx+(\alpha_1-\alpha_2)\int_0^l \sigma\phi_1\phi_2\,dx = \left[p\left(\phi_2\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_2}{\partial x}\right)\right]_0^l.$$

* Such as the deduction of $\int_0^\infty \frac{\sin x}{x}\,dx$ from the complex integral $\int \frac{e^{iz}}{z}\,dz$; the minor arcs here would be two arcs of the circle close to the real axis.

In virtue of the fact that ϕ_1, ϕ_2 satisfy *both* the conditions (56), (57), at $x = 0$ and $x = l$, it follows that

$$\phi_2 \frac{\partial \phi_1}{\partial x} - \phi_1 \frac{\partial \phi_2}{\partial x} = 0,$$

both at $x = 0$ and at $x = l$. Using this fact, we see (after division by $\alpha_1 - \alpha_2$) that

$$(\alpha_1 + \alpha_2) \int_0^l \rho \phi_1 \phi_2 dx + \int_0^l \sigma \phi_1 \phi_2 dx = 0, \tag{75}$$

care being taken to include contributions from the terminal mechanism, if any.

The equation (75) constitutes *the conjugate property of normal functions* in the form most convenient for our present purpose; it can also be transformed into the form

$$\int_0^l \rho (\alpha_1 \phi_1)(\alpha_2 \phi_2)\, dx = \int_0^l p \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{dx} dx + [q \phi_1 \phi_2]_{x=0} + [q \phi_1 \phi_2]_{x=l},$$

which corresponds to Heaviside's relation $T_{12} = U_{12}$, given in § 3 above.

In the special case $\sigma = 0$, (75) reduces to Lord Rayleigh's form*

$$\int_0^l \rho \phi_1 \phi_2 dx = 0. \tag{76}$$

The equation (76) also holds when σ/ρ is *constant*: but care must be taken to notice that *the same constant must apply to the terminal mechanism as to the main parts of the system.* Various examples of the type (75) will be found amongst Heaviside's earlier investigations on cables;† a simple example is given by supposing a constant resistance in the cable, with an inductance coil at an end. Here $\rho = 0$ in general, and σ is constant: but to get the correct result, the terminal inductance must be allowed for, and thus the equation $\int_0^l \phi_1 \phi_2 dx = 0$ is no longer correct.

Suppose next that α_1, α_2 are conjugate complex roots of $\Delta(\lambda) = 0$: then $\int_0^l \rho \phi_1 \phi_2 dx$ and $\int_0^l \sigma \phi_1 \phi_2 dx$ are essentially *positive*;‡ and so (75) shews that $\alpha_1 + \alpha_2$ is *negative.* Thus *the real part of any complex root of* $\Delta(\lambda) = 0$ *is negative*: this is a property which has been constantly

* *Theory of Sound*, Vol. 1, Arts. 93, 94.

† See, for instance, *Electrical Papers*, Vol. 1, pp. 71 (see p. 81); p. 123; p. 141, &c.

‡ Because ρ, σ are positive, and ϕ_1, ϕ_2 are conjugate complexes.

assumed in the foregoing work, and may, perhaps, be regarded as obvious. Of course, *when* $\sigma = 0$, *the real parts are all zero*, a well-known result.

We shall need one more general formula to complete our preliminary work. We have defined $\phi(x)$ by the equation (60),

$$(\rho\lambda^2+\sigma\lambda)\phi - \frac{\partial}{\partial x}\left(p\frac{\partial\phi}{\partial x}\right) = 0,$$

along with the end-condition (56). Differentiate with regard to λ, and write

$$\frac{\partial\phi}{\partial\lambda} = \omega;$$

then we have $$(2\rho\lambda+\sigma)\phi+(\rho\lambda^2+\sigma\lambda)\omega-\frac{\partial}{\partial x}\left(p\frac{\partial\omega}{\partial x}\right) = 0.$$

Thus, since ψ also satisfies (60), we have

$$(2\rho\lambda+\sigma)\phi\psi = \frac{\partial}{\partial x}\left\{p\left(\psi\frac{\partial\omega}{\partial x}-\omega\frac{\partial\psi}{\partial x}\right)\right\},$$

or
$$\int_0^l (2\rho\lambda+\sigma)\phi\psi\,dx = \left[p\left(\psi\frac{\partial\omega}{\partial x}-\omega\frac{\partial\psi}{\partial x}\right)\right]_0^l. \tag{77}$$

Now since ϕ satisfies (56) *for all values of* λ, the same condition will be satisfied by ω; and for the special values $\lambda = \alpha_1, \alpha_2, \ldots$ this condition is also satisfied by ψ. Accordingly we find that, for these special values of λ, $\psi\frac{\partial\omega}{\partial x}-\omega\frac{\partial\psi}{\partial x} = 0$ at $x = 0$; and so (77) yields

$$\int_0^l (2\rho\alpha_1+\sigma)\phi_1\psi_1\,dx = \left[p\left(\psi_1\frac{\partial\omega_1}{\partial x}-\omega_1\frac{\partial\psi_1}{\partial x}\right)\right]_{x=l}. \tag{78}$$

Now we have, from (64),

$$\Delta(\lambda) = \left[p\left(\psi\frac{\partial\phi}{\partial x}-\phi\frac{\partial\psi}{\partial x}\right)\right]_{x=l},$$

and so, if we differentiate with respect to λ, and then put $\lambda = \alpha_1$, we get*

$$\Delta'(\alpha_1) = \left[p\left(\psi_1\frac{\partial\omega_1}{\partial x}-\omega_1\frac{\partial\psi_1}{\partial x}\right)\right]_{x=l}. \tag{79}$$

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* Observe that since ψ satisfies (57) for all values of λ, the same is true of $\partial\psi/\partial\lambda$: and so

$$\frac{\partial\phi_1}{\partial x}\frac{\partial\psi_1}{\partial\lambda}-\phi_1\frac{\partial}{\partial x}\left(\frac{\partial\psi_1}{\partial\lambda}\right) = 0,$$

when $\lambda = \alpha_1$ and $x = l$.

Combining (78) and (79), we have the result

$$\int_0^l (2\rho a_1+\sigma)\phi_1\psi_1\,dx = \Delta'(a_1)\,; \tag{80}$$

it will be noticed that equation (80) corresponds to equations (28) and (30) of § 3.

It is now easy to find formulæ for the residues of v, as defined in § 8; in fact, from (66) and (67), we deduce that *the residue of v corresponding to $\lambda = a_1$ is*

$$\frac{\phi_1(x)}{c_1\Delta'(a_1)}\int_0^l \{(\rho a_1+\sigma)\,u_0(\xi)+\rho u_1(\xi)\}\,\phi_1(\xi)\,d\xi,$$

where c_1 has the value given in (74). Now, substituting from (80), we obtain the residue of v, namely,

$$\phi_1(x)\,\frac{\int_0^l \{(\rho a_1+\sigma)\,u_0(\xi)+\rho u_1(\xi)\}\,\phi_1(\xi)\,d\xi}{\int_0^l (2\rho a_1+\sigma)\{\phi_1(\xi)\}^2\,d\xi}.$$

Thus, finally, *the solution u corresponding to the given initial values can be written in the form*

$$u = \Sigma A_1\phi_1(x)e^{a_1 t},$$

where

$$A_1\int_0^l (2\rho a_1+\sigma)\{\phi_1(x)\}^2\,dx = \int_0^l \{(\rho a_1+\sigma)\,u_0(x)+\rho u_1(x)\}\,\phi_1(x)\,dx. \tag{81}$$

It is now to be proved that the solution (81) agrees with that which would be obtained by the method of normal functions. According to that method we *assume* that the solution must take the form (81), and then find the constants A from the two initial conditions

$$\left.\begin{aligned} u_0(x) &= \Sigma A_1\phi_1(x) \\ u_1(x) &= \Sigma A_1 a_1\phi_1(x) \end{aligned}\right\}. \tag{82}$$

If we thus assume the possibility of expanding $u_0(x)$, $u_1(x)$, in terms of normal functions, we can obtain the coefficients fairly quickly, as follows. Multiply the first of equations (82) by $(\rho a_1+\sigma)$, and the second by ρ, and add: this gives

$$(\rho a_1+\sigma)\,u_0(x)+\rho u_1(x) = A_1\phi_1(x)(2\rho a_1+\sigma)+A_2\phi_2(x)\{\rho(a_1+a_2)+\sigma\}+\ldots. \tag{83}$$

Now multiply (83) by $\phi_1(x)$ and integrate: then, in virtue of (75), all the coefficients $A_2, A_3, A_4, \ldots$ disappear from the equation: and so we obtain

the formula

$$\int_0^l \{(\rho a_1+\sigma)\, u_0(x)+\rho u_1(x)\}\, \phi_1(x)\, dx = A_1 \int_0^l (2\rho a_1+\sigma)\{\phi_1(x)\}^2 dx,$$

which is the same result as we obtained in (81) by the method of residues.

We have accordingly verified the conclusion that *the method of normal functions is equivalent to the method of* § 8, *as regards results.* But, from the point of view of theory, the method of § 8 gives some indication as to the necessary restrictions on the functions $u_0(x)$, $u_1(x)$, in order that the results shall be correct; whereas the method of normal functions gives no such indication. Moreover, as we have seen in some special examples, it may be possible to obtain more precise information from the complex integrals than is possible from the series of normal functions; and the problems solved in §§ 6 and 7 indicate that our general process is successful in cases which have hitherto proved insoluble by the usual forms of the method of normal functions.

[*Added November* 14*th*, 1916.

While the preceding pages were in the press, I have been reminded that various authors have applied complex integrals, as a means for the summation of series of normal functions: the whole problem being regarded mainly from the point of view of Pure Mathematics. This method goes back to Cauchy;* it was afterwards applied to series of more general types of normal functions by Dini,† and Dini's work has been carried further by W. B. Ford.‡

In all these discussions, it will be seen that the complex integrals are developed from the other end of the problem. Thus, starting from a series such as (81) above, a complex integral is devised which yields the terms of the series as its residues; but the series is found first, and not the integral. I believe that it will be found that the integrals so invented are more complicated than those derived directly from the formulæ (58), (66), (67), and (69), of § 8 above.

When the functions $u_0(x)$, $u_1(x)$ do not satisfy Dirichlet's conditions,

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* *Œuvres Complètes*, t. 7 (2 sér.), p. 393; see also Picard, *Traité d'Analyse*, t. 2, pp. 167–183.

† *Serie di Fourier, &c.*, §§ 61–64, 90–109.

‡ *Studies on Divergent Series and Summability*, New York, 1916, pp. 123–183. It was through the study of this monograph that I was reminded of the earlier use of complex integrals as a means of summation.

it is possible (at least in certain special series) to establish the fact that the corresponding series are summable (instead of convergent), provided that the functions do satisfy certain less restrictive conditions; and Prof. Ford has shewn (*l.c.*, pp. 134–183) that his complex integrals can be used for this purpose. It seems likely that a corresponding investigation could be carried out, starting from the complex integrals of § 8 above; but I have made no attempt to consider this problem—which is mainly of interest from the side of Pure Mathematics, because Dirichlet's conditions are satisfied by almost all functions which are used in Applied Mathematics.—T. J. I'A. B.]

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